PORTUGALIAE MATHEMATICA
Vol. 57 Fasc. 2 – 2000

INVARIANCE PRINCIPLES IN HÖLDER SPACES

D. Hamadouche

Abstract: We study the weak convergence of random elements in the space of Hölder functions $\mathrm{H}_\alpha[0,1]$. Using this space instead of $C[0,1]$ enables us to obtain functional limit theorems of a wider scope. Some examples of Hölder continous functionals of the paths are proposed to illustrate this improvement. A new tightness condition is established. We obtain an Hölderian version of Donsker–Prohorov's invariance principle about the polygonal interpolation of the partial sums process, generalizing Lamperti's i.i.d. invariance principle to the case of strong mixing or associated random variables. Similar results are proved for the convolution smoothing of partial sums process.

1 – Introduction

Let $(X_j)_{j\geq 1}$ be a sequence of independent identically distributed random variables with $\mathbb{E}X_j = 0$ and $\mathbb{E}X_j^2 = 1$. Write ξ_n for the random polygonal lines obtained by linear interpolation between the points $(j/n,\, S_j/\sqrt{n})$, where $S_j = \sum_{k=1}^{j} X_k$.

The Donsker Prokhorov's invariance principle establishes then the $C[0,1]$ weak convergence of ξ_n to the Brownian motion W. This gives the weak convergence of continuous functionals on $C[0,1]$ for example: $\|\xi_n\|_\infty = \sup\limits_{t\in[0,1]} |\xi_n(t)|$.

It is well known that the paths of W are (with probability one) of Hölder regularity α for any $\alpha < 1/2$ and those of ξ_n are of Hölder regularity 1. It is then natural to study for $\alpha < 1/2$, the weak convergence of ξ_n as random elements in the Banach space $\mathrm{H}_\alpha[0,1]$ of α-Hölder functions. Such a convergence gives

Received: January 22, 1998.
AMS Subject Classification: 60B10, 60F05, 60F17, 62G30.
Keywords: Tightness; Hölder space; Triangular functions; Invariance principles; Strong mixing; Association; Brownian motion.

more applications to continuous functionals than in the $C[0,1]$ framework. The invariance principle in $\mathrm{H}_\alpha[0,1]$ has been established by Lamperti [19] and derived again recently by Kerkyacharian and Roynette [18] using the Faber–Schauder basis of triangular functions.

Theorem 1 (Lamperti [19]). *Let $(X_j)_{j\geq 1}$ be a sequence of i.i.d. random variables with $\mathbb{E}X_j = 0$ and $\mathbb{E}X_j^2 = \sigma^2$. Suppose that for some constant $\gamma > 2$, $\mathbb{E}|X_j|^\gamma < \infty$. For all $n \in \mathbb{N}^*$, $0 \leq j < n$, define*

$$(1) \qquad \xi_n(t,\omega) = \frac{1}{\sigma\sqrt{n}} \Big[\sum_{0<k\leq j} X_k(\omega) + (n\,t - j)\, X_{j+1}(\omega) \Big], \qquad \frac{j}{n} \leq t < \frac{j+1}{n}\,.$$

Then the sequence $(\xi_n)_{n\geq 1}$ converges weakly to the Brownian motion W in H_α^0 for all $\alpha < 1/2 - 1/\gamma$. ∎

The present contribution is devoted to some extensions of this result. First, we recall in Section 2 the Ciesielski's study of the Banach Hölder space $\mathrm{H}_\alpha[0,1]$. For separability convenience, we consider its closed subspace $\mathrm{H}_\alpha^0[0,1]$. Using the Ciesielski's characterization of the dual of H_α^0, we give an intrinsic representation of an element of $(\mathrm{H}_\alpha^0)'$ by a pair of signed measures and a list of examples of continuous functionals on H_α^0. In Section 3, we consider stochastic processes with paths in H_α^0. We treat them as random elements of H_α^0. Their weak convergence is equivalent to the tightness on H_α^0 and the convergence of the finite dimensional distributions. For the tightness, a basic tool available in the literature is the condition of Lamperti [19] based on the moment inequality

$$\mathbb{E}|\xi_n(t) - \xi_n(s)|^\gamma < C\,|t-s|^{1+\delta}, \qquad s,t \in [0,1]\,.$$

We prove that it is sufficient to verify this inequality for $|t-s| \geq a_n$, where a_n decreases to zero, together with convergence in probability to zero of the Hölder modulus of continuity $w_\alpha(\xi_n, a_n)$. We propose to extend the result of Lamperti to dependent random variables, namely we consider the cases of α-mixing and association. These results rely on a moment inequality for sums of dependent random variables and some central limit theorems. These dependence tools are recalled in Section 4. Our invariance principles under dependence are presented in Section 5. Next, we consider convolution smoothing of the process of normalized partial sums of Donsker–Prokhorov for independent random variables and we prove the weak convergence in H_α^0 of this smoothed process to the Brownian motion. This last result is extended to α-mixing or associated random variables.

2 – The functional framework

2.1. The Banach spaces $\mathrm{H}_\alpha[0,1]$ and $\mathrm{H}^0_\alpha[0,1]$

2.1.1. Definitions

We use the notations and results of Ciesielski [6] about the spaces of Hölder functions on $[0,1]$. We define the Hölder space $\mathrm{H}_\alpha[0,1]$ $(0<\alpha\le 1)$ as the space of functions f vanishing at 0 such that

$$\|f\|_\alpha = \sup_{0<|t-s|\le 1} \frac{|f(t)-f(s)|}{|t-s|^\alpha} < \infty \ .$$

Define the Hölderian modulus of continuity of f by

$$w_\alpha(f,\delta) = \sup_{0<|t-s|<\delta} \frac{|f(t)-f(s)|}{|t-s|^\alpha}$$

and the subspace $\mathrm{H}^0_\alpha[0,1]$ of $\mathrm{H}_\alpha[0,1]$ by

$$f\in \mathrm{H}^0_\alpha \quad\Longleftrightarrow\quad f\in \mathrm{H}_\alpha \ \text{ and } \ \lim_{\delta\to 0} w_\alpha(f,\delta)=0 \ .$$

$(\mathrm{H}_\alpha, \|\cdot\|_\alpha)$ is a non-separable Banach space. $(\mathrm{H}^0_\alpha, \|\cdot\|_\alpha)$ is a separable closed subspace. $(\mathrm{H}_\alpha, \|\cdot\|_\alpha)$ is separable for the norm $\|\cdot\|_\beta$, for any $0<\beta<\alpha$ and is topologically embedded in H_β.

2.1.2. Analysis by triangular functions

To obtain an isomorphism of the spaces $\mathrm{H}_\alpha[0,1]$ and $\mathrm{H}^0_\alpha[0,1]$ with some Banach sequence spaces, Ciesielski used the Faber–Schauder basis, obtained by translations and dyadic changes of scales from the triangular function

$$\Delta(t) = \begin{cases} 2\,t & \text{if } 0\le t\le 1/2\,, \\ 2\,(1-t) & \text{if } 1/2\le t\le 1\,, \\ 0 & \text{elsewhere}\ . \end{cases}$$

Putting for $n=2^j+k$, $j\ge 0$, $0\le k<2^j$, $t\in[0,1]$

$$\Delta_n(t)=\Delta_{j,k}(t)=\Delta(2^j t-k) \quad \text{and} \quad \Delta_0 = t\,\mathbf{1}_{[0,1]}(t) \ .$$

The addition of the function Δ_{-1} defined by $\Delta_{-1}(t) = \mathbf{1}_{[0,1]}(t)$ to the scale $\{\Delta_n,\, n \in \mathbb{N}\}$ gives a Schauder basis of $(\mathcal{C}[0,1], \|\cdot\|_\infty)$ the space of continuous functions equipped with the supremum norm. $\{\Delta_n,\, n \in \mathbb{N}\}$ is a Schauder basis of the closed subspace $\mathcal{C}_0[0,1]$ of functions vanishing at 0. More precisely we have

Lemma 1 (Faber–Schauder). *For any function f of $\mathcal{C}_0[0,1]$,*

$$f(t) \;=\; \sum_{n=0}^{\infty} \lambda_n(f)\, \Delta_n(t)\ , \tag{2}$$

where $\lambda_0(f) = f(1)$ and for $n = 2^j + k$ $(j \geq 0,\ 0 \leq k < 2^j)$

$$\lambda_n(f) \;=\; \lambda_{j,k}(f) \;=\; f\Big(\frac{k+1/2}{2^j}\Big) - \frac{1}{2}\Big\{ f\Big(\frac{k}{2^j}\Big) + f\Big(\frac{k+1}{2^j}\Big)\Big\}\ . \tag{3}$$

The series (2) converges in the sense of the norm of $\mathcal{C}_0[0,1]$ (i.e. uniformly on $[0,1]$). ∎

We adopt the classical notation ℓ^∞ for the Banach space of bounded sequences $u = (u_n)_{n\in\mathbb{N}}$ equipped with the norm $\|u\|_\infty = \sup_{n\geq 0} |u_n|$ and c_0 for the closed subspace of sequences vanishing at infinity. Since any function f of $\mathrm{H}_\alpha[0,1]$ is in $\mathcal{C}_0[0,1]$, it has also the decomposition (2) and the series converges at least in the $\mathcal{C}_0[0,1]$ sense.

Theorem 2 (Ciesielski [6]). *For any function f of H^0_α, the series*

$$f(t) \;=\; \sum_{n=0}^{\infty} \lambda_n(f)\, \Delta_n(t)$$

converges in H^0_α. $\{\Delta_n,\, n \geq 1)\}$ is a Schauder basis of $(\mathrm{H}^0_\alpha, \|\cdot\|_\alpha)$. ∎

Theorem 3 (Ciesielski [6]). *For $n = 2^j + k$ $(j \geq 0,\ 0 \leq k < 2^j)$, write $\Delta_n^{(\alpha)} = 2^{-(j+1)\alpha}\Delta_n$ and $\Delta_0^{(\alpha)} = \Delta_0$. The spaces $(\mathrm{H}_\alpha, \|\cdot\|_\alpha)$ and $(\ell^\infty, \|\cdot\|_\infty)$ are isomorphic by the operators S_α and $T_\alpha = S_\alpha^{-1}$ defined as follows:*

$$\begin{aligned} S_\alpha\colon\ \mathrm{H}_\alpha &\longrightarrow \ell^\infty \\ f &\longmapsto u = (u_n)_{n\geq 0} \end{aligned}$$

with $u_n = 2^{(j+1)\alpha}\lambda_n(f)$, $n \geq 1$ and $u_0 = \lambda_0(f)$.

$$\begin{aligned} T_\alpha\colon\ \ell^\infty &\longrightarrow H^\alpha \\ u = (u_n)_{n\geq 0} &\longmapsto f = \sum_{n=0}^{\infty} u_n\, \Delta_n^{(\alpha)}\ . \end{aligned}$$

Moreover $\|S_\alpha\| = 1$ *and*

$$\frac{2}{3\,(2^\alpha-1)\,(2^{1-\alpha}-1)} \;\le\; \|T_\alpha\| \;\le\; \frac{2}{(2^\alpha-1)\,(2^{1-\alpha}-1)}\;. \;\blacksquare$$

Theorem 4 (Ciesielski [6]). $\mathrm{H}^0_\alpha[0,1]$ *is isomorphic by* S_α *to the subspace* $c_{o,\alpha}$ *of sequences* $(u_n)_{n\ge 0}$ *of* ℓ^∞ *such that* $\lim_{j\to\infty} 2^{(j+1)\alpha} \sup_{0\le k<2^j} |u_{j,k}| = 0$, *where* $u_{j,k} = u_n$ *for* $n = 2^j + k$. $\blacksquare$

Finally we have

$$f \in \mathrm{H}_\alpha[0,1] \quad\Longleftrightarrow\quad \sup_{j\ge 0} 2^{(j+1)\alpha} \sup_{0\le k<2^j} |\lambda_{j,k}(f)| < \infty, \quad |\lambda_0(f)| < \infty$$

and

$$f \in \mathrm{H}^0_\alpha[0,1] \quad\Longleftrightarrow\quad f \in \mathrm{H}_\alpha[0,1] \;\;\text{and}\;\; \lim_{j\to\infty} 2^{(j+1)\alpha} \sup_{0\le k<2^j} |\lambda_{j,k}(f)| = 0\;.$$

2.2. Functionals and operators on $\mathrm{H}^0_\alpha[0,1]$

Some operators and functionals useful in statistics and important operators in analysis are continuous on H^0_α. Since weak convergence is preserved by continuous mappings, the weak convergence in H_α provides weak convergence results for H^0_α-continuous functionals of paths and for some image process. Moreover, since H_α is topologically embedded in $\mathcal{C}[0,1]$, there are more continuous functionals on $\mathrm{H}_\alpha[0,1]$ than on $\mathcal{C}[0,1]$. In this section are presented some examples of continuous functionals and operators on H_α. We begin by the dual of H^0_α.

2.2.1. Dual of $\mathrm{H}^0_\alpha[0,1]$

Denote by $(\ell^1, \|\cdot\|_{\ell^1})$ the space of real sequences $a = (a_0, a_1, \ldots)$ such that $\|a\|_{\ell^1} = \sum_{n\ge 0} |a_n| < \infty$. The dual of $\mathrm{H}^0_\alpha[0,1]$ is given by the following results.

Theorem 5 (Ciesielski [6]). *Any continuous linear functional* φ *on* $(\mathrm{H}^0_\alpha, \|\cdot\|_\alpha)$ *has the form*

$$\varphi(f) \;=\; \sum_{n=0}^{\infty} a_n\, u_n \tag{4}$$

where $u_0 = \lambda_0(f)$, $u_n = 2^{(j+1)\alpha}\lambda_n(f)$, $n = 2^j + k$ $(j \geq 0,\ 0 \leq k < 2^j)$ *and* $a = (a_0, a_1, \ldots) \in \ell^1$. *Moreover* $\|\varphi\|_{(\mathrm{H}^0_\alpha)'} \leq \|S_\alpha\| \cdot \|a\|_{\ell^1}$, $\|a\|_{\ell^1} \leq \|T_\alpha\| \cdot \|\varphi\|_{(\mathrm{H}^0_\alpha)'}$ *and the constants* $\|S_\alpha\|$ *and* $\|T_\alpha\|$ *are optimal.* ∎

This theorem allows us to propose a more intrinsic characterization of $(\mathrm{H}^0_\alpha)'$.

Theorem 6. φ *is a continuous linear functional on* H^0_α *if and only if there exists a signed measure* μ *on* $[0,1]$ *and a signed measure* ν *on* $[0,1]^2$ *such that*

$$\varphi(f) = \int_{[0,1]} f(t)\,\mu(dt) + \int_{[0,1]^2} \frac{2\,f(t) - f(t+u) - f(t-u)}{u^\alpha}\,\nu(dt, du) \tag{5}$$

where the second integrand vanishes at $u = 0$, *which amounts to extend it by continuity since* $f \in \mathrm{H}^0_\alpha[0,1]$.

Proof: Recall that a signed measure is a difference of two positive measures each with finite mass. Clearly φ defined by (5) is a linear functional and

$$|\varphi(f)| \leq \|f\|_\infty\,|\mu|\,([0,1]) + 2\,\|f\|_\alpha\,|\nu|\,([0,1]^2)\ .$$

Thus $|\varphi(f)| \leq c\,\|f\|_\alpha$ where $c = |\mu|\,([0,1]) + 2\,|\nu|\,([0,1]^2)$ and φ is continuous.

Conversely, if φ is a continuous linear functional on $\mathrm{H}^0_\alpha[0,1]$, by theorem 5, there exists $a = (a_n)_{n\geq 0} \in \ell^1$ such that

$$\varphi(f) = \sum_{n\geq 0} a_n\,u_n \quad \text{where} \quad u_0 = \lambda_0(f) \quad \text{and} \quad u_n = 2^{(j+1)\alpha}\lambda_n(f), \quad n \geq 1\ .$$

Writing $\mu = a_0\,\delta_1$ (δ is a Dirac measure) and

$$\nu = \sum_{j\geq 0}\sum_{k=0}^{2^j-1} \frac{1}{2}\,a_{2^j+k}\,\delta_{t_{j,k}} \otimes \delta_{2^{-j-1}}, \quad \text{where} \quad t_{j,k} = \frac{k+1/2}{2^j}\ ,$$

we have

$$\begin{aligned}
\varphi(f) &= \sum_{n\geq 0} a_n\,u_n \\
&= a_0\,f(1) + \sum_{j\geq 0}\sum_{k=0}^{2^j-1} \frac{1}{2}\,a_{2^j+k}\left\{2\,f\left(\frac{k+1/2}{2^j}\right) - f\left(\frac{k+1}{2^j}\right) - f\left(\frac{k}{2^j}\right)\right\} 2^{(j+1)\alpha} \\
&= \int_{[0,1]} f(t)\,\mu(dt) + \int_{[0,1]^2} \frac{2\,f(t) - f(t+u) - f(t-u)}{u^\alpha}\,\nu(dt, du)
\end{aligned}$$

thus φ has the representation (5). ∎

Remark. It is clear that the decomposition (5) is not unique. The dual $(\mathrm{H}^0_\alpha)'$ is in fact isomorphic to a quotient of the Banach space $\mathcal{M}[0,1] \oplus \mathcal{M}[0,1]^2$. The interest of the decomposition (5) is to clarify the structure of a linear continuous functional on H^0_α. Its first component μ *charges* the values of f, like a linear continuous functional on $C[0,1]$. The second component ν *charges* the second differences of f with weight $u^{-\alpha}$. Roughly speaking, ν charges the Hölderian increments of f. □

2.2.2. Examples of functionals

We give now some examples of continuous functionals on $\mathrm{H}^0_\alpha[0,1]$.

Example 1: This example borrowed to Ciesielski, requires the introduction of a particular class of continuous linear functionals on $\mathcal{C}[0,1]$. Let $f \in \mathcal{C}[0,1]$ and g a function with bounded variation $V(g)$ on $[0,1]$. We consider the integral $\varphi(f) = \int_0^1 g\,df$ whose existence is not obvious since f is not supposed to be of bounded variation, so we can not consider $\varphi(f)$ as a Stieltjes integral. In fact we construct it as follows

$$\varphi(f) = \int_0^1 g(t)\,df(t) = \lim_{N\to\infty} \int_0^1 g_N(t)\,df(t) = \lim_{N\to\infty} \sum_{n=0}^{N} a_n\,b_n\ , \tag{6}$$

where $g_N = \sum_{n=0}^N a_n \chi_n$ is the partial sum of the Haar series of g and

$$b_n = \int_0^1 \chi_n(t)\,df(t) := \lambda_n(f)$$

is the n-th Schauder coefficient of f. Remark that if f is C^1, the integral so defined coincides with the classical definition (i.e. in Riemann's sense) since $\int_0^1 \chi_n(t)\,f'(t)\,dt = \lambda_n(f)$. The existence of the limit in (6) follows from the following elementary lemma.

Lemma 2. *If g is a function with bounded variation $V(g)$ on $[0,1]$ and $(a_n(g))_{n\geq 0}$ is the sequence of its Haar coefficients then $\sum_{n=0}^\infty |a_n(g)| < \infty$.*

Proof: We have $\chi_0(t) = \mathbf{1}_{[0,1]}(t)$ and for $j \geq 0$, $0 \leq k < 2^j$,

$$\chi_{2^j+k} = 2^{j/2}\left(\mathbf{1}_{[\frac{k}{2^j},\frac{k+1/2}{2^j}[} - \mathbf{1}_{[\frac{k+1/2}{2^j},\frac{k+1}{2^j}[}\right).$$

Clearly, $|a_0(g)| = |\int_0^1 g(t)\,dt| < \infty$ since g is bounded. Next

$$\begin{aligned} a_{2^j+k}(g) &= \int_0^1 \chi_{2^j+k}(t)\, g(t)\, dt \\ &= \frac{1}{2\cdot 2^{j/2}} \int_0^1 \left[g\Big(\frac{t+2k}{2^{j+1}}\Big) - g\Big(\frac{t+2k+1}{2^{j+1}}\Big)\right] dt \ . \end{aligned}$$

which can easily be bounded by

$$\sum_{k=0}^{2^j-1} |a_{2^j+k}(g)| \ \le\ \frac{1}{2\cdot 2^{j/2}}\, V(g) \ .$$

Hence

$$\sum_{j\ge 0}\sum_{k=0}^{2^j-1} |a_{2^j+k}(g)| \ \le\ \frac{V(g)}{2} \sum_{j\ge 0}\Big(\frac{1}{\sqrt{2}}\Big)^j \ <\ \infty\ ,$$

whence the conclusion follows. ∎

Using these continuous linear functionals φ on $\mathcal{C}[0,1]$,with other regularity conditions on g, Ciesielski has obtained continuous functionals on $\mathrm{H}^0_\alpha[0,1]$.

Theorem 7 (Ciesielski [6]). *Let* $0<\alpha<1$, $\alpha+\beta = 1$ *and* $g\in H^\beta[0,1]$. *We suppose moreover that* g *has a bounded variation* $V(g)$ *on* $[0,1]$. *The linear functional*

$$\varphi(f) \ = \int_0^1 g(t)\, df(t)$$

is continuous on $\mathrm{H}^0_\alpha[0,1]$ *and its norm satisfies the inequality*

$$\|\varphi\|_{(\mathrm{H}^0_\alpha)'} \ \le\ \Big|\int_0^1 g(t)\,dt\Big| + A(\alpha)\,\Big[V(g)\,\|g\|_\beta\Big]^{1/2}$$

where $A(\alpha) = \frac{1}{\sqrt{2}\,(2^{\alpha/2}-1)}$. ∎

Example 2:

$$\varphi(f) \ = \int_0^1 \frac{f(t)}{t^{1+\beta}}\, dt\,, \qquad \beta<\alpha\ .$$

It is clear that φ is a linear functional and

$$|\varphi(f)| \ \le\ \|f\|_\alpha \int_0^1 \frac{dt}{t^{1+\beta-\alpha}} \ <\ \infty \qquad \text{for}\ \ \beta<\alpha\ .$$

Example 3: This one is more general than the former. For $0 < t_0 < 1$ and $0 < \beta < \alpha$, we consider

$$\begin{aligned}\varphi(f) &= \text{v.p.} \int_0^1 \frac{f(t)\,\mathrm{sgn}(t-t_0)}{|t-t_0|^{1+\beta}}\,dt \\ &= \lim_{\varepsilon\to 0}\left\{\int_0^{t_0-\varepsilon} \frac{-f(t)}{(t_0-t)^{1+\beta}}\,dt + \int_{t_0+\varepsilon}^1 \frac{f(t)}{(t-t_0)^{1+\beta}}\,dt\right\} .\end{aligned}$$

The continuity of φ can be checked by elementary computations.

Example 4:

$$\varphi(f) = \int_0^1 \frac{1}{u} \int_{\{|t-s|\le u\}} \frac{f(t)-f(s)}{|t-s|^\alpha}\,ds\,dt\,\mu(du)$$

where μ is a signed measure on $[0,1]$. φ is a linear functional and

$$\begin{aligned}|\varphi(f)| &\le \|f\|_\alpha \int_0^1 \frac{1}{u}\int_{\{|t-s|\le u\}} ds\,dt\,|\mu|(du) \\ &\le 2\,|\mu|\,([0,1])\,\|f\|_\alpha ,\end{aligned}$$

thus φ is continuous.

Example 5: We list here some non linear functionals on H_α^0 whose continuity is obvious

$$\varphi_1(f) = \|f\|_\alpha , \qquad \varphi_2(f) = w_\alpha(f,\delta) , \qquad \varphi_3(f) = \sup_{t\in[0,1]} \frac{|f(t)-f(t_0)|}{|t-t_0|^\alpha} .$$

Example 6: The p-variation in the sense of Wiener for $p \ge 1/\alpha$.

Let f be a function $[0,1] \to \mathbb{R}$. We suppose that there exists $c = c(f)$ such that for any family $(I_k,\, k \ge 1)$ of disjointed intervals $(I_k =]a_k, b_k[)$ of $[0,1]$

$$\left(\sum_k \Big|f(b_k) - f(a_k)\Big|^p\right)^{1/p} \le c < \infty . \tag{7}$$

Then we say that f has a bounded p-variation and we define the p-variation of f the infimum $V_p(f)$ of constants c satisfying (7). If $f \in \mathrm{H}_\alpha$, $V_p(f)$ is finite for any $p \ge 1/\alpha$, in fact

$$\begin{aligned}\left(\sum_k \Big|f(b_k) - f(a_k)\Big|^p\right)^{1/p} &= \left(\sum_k (b_k-a_k)^{p\alpha} \frac{\Big|f(b_k)-f(a_k)\Big|^p}{(b_k-a_k)^{p\alpha}}\right)^{1/p} \\ &\le \left(\|f\|_\alpha^p\right)^{1/p} \left(\sum_k (b_k-a_k)^{p\alpha}\right)^{1/p} \\ &\le \|f\|_\alpha \left(\sum_k (b_k-a_k)\right)^{1/p} ,\end{aligned}$$

since $p\,\alpha \geq 1$ and $b_k - a_k \leq 1$. As $\sum_k (b_k - a_k) \leq 1$, we obtain

$$\Big(\sum_k \Big|f(b_k) - f(a_k)\Big|^p\Big)^{1/p} \leq \|f\|_\alpha < \infty\ .$$

Hence $V_p(f) \leq \|f\|_\alpha$ and since V_p satisfies the triangular inequality, $V_p(f)$ is continuous.

2.2.3. Examples of operators

Example 1: *Fractional integral*

We consider the fractional integration operator with order β, of Riemann–Liouville (cf. for example M. Riesz [23])

$$I^\beta f(x) \;=\; \frac{1}{\Gamma(\beta)} \int_0^x (x-t)^{\beta-1}\, f(t)\, dt\,, \qquad x \in [0,1]\ . \tag{8}$$

The integral converges at least for $\beta > 0$ and f continuous. The operator I^β satisfies the following properties

$$I^\beta(I^\gamma) = I^{\beta+\gamma}\,, \qquad \frac{d}{dx}(I^{\beta+1}) = I^\beta\ .$$

Proposition 1. *We suppose $\alpha,\beta > 0$ and $\alpha+\beta < 1$. Then I^β is a continuous linear operator from $\mathrm{H}_\alpha[0,1]$ in $H_{\alpha+\beta}[0,1]$.*

Proof: The result follows from an easy adaptation of the proof of theorem 14 in Hardy Littlewood [16] by expliciting in terms of $\|f\|_\alpha$ the constants involved in the $O(h^{\alpha+\beta})'s$. ∎

Example 2: *Fractional derivation*

The operator of fractional derivation of order β of a function f is formally defined by

$$D^\beta f(x) \;=\; \frac{d}{dx}(I^{1-\beta} f)(x) \tag{9}$$

where $I^{1-\beta}$ is the operator of fractional integration of order $1-\beta$ of Riemann–Liouville.

We begin by a result of Hardy–Littlewood on the existence and definition of the operator D^β on $\mathrm{H}_\alpha[0,1]$.

Theorem 8 (Hardy–Littlewood [16]). *If $0 < \beta < \alpha \le 1$ and $f \in \mathrm{H}_\alpha[0,1]$, then $D^\beta f$ exists and $f = I^\beta D^\beta f$. Moreover*

$$D^\beta(\mathrm{H}_\alpha[0,1]) = \mathrm{H}_{\alpha-\beta}[0,1] \quad \textit{and} \quad D^\beta(\mathrm{H}^0_\alpha[0,1]) = \mathrm{H}^0_{\alpha-\beta}[0,1]\ . \ \blacksquare$$

Proposition 2. *D^β is a continuous operator from $\mathrm{H}_\alpha[0,1]$ in $\mathrm{H}_{\alpha-\beta}[0,1]$.*

Proof: By theorem 8, $I^\beta D^\beta = \mathrm{Id}_{\mathrm{H}_\alpha}$ for $0 < \beta < \alpha$. We verify that $D^\beta I^\beta = \mathrm{Id}_{\mathrm{H}_{\alpha-\beta}}$. Let $g \in \mathrm{H}_{\alpha-\beta}$,

$$\begin{aligned} D^\beta I^\beta g(x) &= \frac{d}{dx}\Big\{I^{1-\beta}(I^\beta g)\Big\}(x) = \frac{d}{dx}(I^1 g) = \\ &= \frac{d}{dx}\left(\int_0^x g(t)\,dt\right) = g(x)\ . \end{aligned}$$

We deduce that I^β is a bijection from $\mathrm{H}_{\alpha-\beta}$ on H_α. I^β is a continuous linear operator and is bijective from $\mathrm{H}_{\alpha-\beta}$ on H_α. Then its inverse D^β is also continuous from H_α in $\mathrm{H}_{\alpha-\beta}$, by a classical corollary of the theorem of the open map (cf. for example Brézis [5], corollary II.6 p. 19). ∎

3 – Random elements in H_α

3.1. Weak convergence in H_α

We consider in this section processes with Hölderian paths as random elements of the functional space $\mathrm{H}_\alpha[0,1]$. We observe directly the whole path, which corresponds to select at random a function ξ with distribution P_ξ. This situation is frequent for example in studying invariance principles where we can observe directly all the path $\xi_n(t)$ (polygon line). The study of weak convergence of random elements of H^0_α is based on the following result.

Proposition 3. *The weak convergence in H^0_α of a sequence of processes $(\xi_n,\ n \ge 1)$ is equivalent to the tightness in H^0_α of the sequence of distributions $P_n = P\xi_n^{-1}$ of random elements ξ_n and the convergence of the finite-dimensional distributions of ξ_n.*

Proof: Clearly tightness and convergence of finite dimensional distributions are necessary conditions for the weak-Hölder convergence of a sequence of processes $(\xi_n,\ n \ge 1)$. On the other hand, if a sequence of distributions $(P_n)_{n\ge1}$

is tight, there exists at least a subsequence of $(P_n)_{n\geq 1}$ which converges to the distribution P_ξ, of some random element ξ of H^0_α. It suffices then to prove that the limit distribution is unique. For that, recall that if $\mathcal{X}$ is a separable Banach space, its Borelian σ-field $\mathcal{B}_\mathcal{X}$ coincides with its cylindrical σ-field $\mathcal{C}_\mathcal{X}$, spanned by the functionals φ of the topological dual $\mathcal{X}'$.

Writing

$$L_\xi(\varphi) \;=\; \mathbb{E}\exp(i\langle\xi,\varphi\rangle) \;=\; \mathbb{E}\exp(i\,\varphi(\xi))\,, \qquad \varphi\in\mathcal{X}'\ ,$$

for the characteristic functional of ξ, we have

$$L_\xi(\varphi)=L_\zeta(\varphi) \quad\Longleftrightarrow\quad \xi \text{ and } \zeta \text{ have the same distribution}\ .$$

By Lebesgue's Dominated convergence theorem, we see easily that the characteristic functional is continuous on $\mathcal{X}'$. It suffices then to prove the equality in distribution of ξ and ζ.

We return to H^0_α, let φ be an element of the dual $(\mathrm{H}^0_\alpha)'$. By theorem 5 there exists a sequence $a=(a_n)\in\ell^1(\mathbb{N})$ such that

$$\varphi(f) \;=\; a_0\,f(1)+\sum_{n=1}^{\infty} a_n\,2^{(j+1)\alpha}\,\lambda_n(f)\,, \qquad f\in\mathrm{H}^0_\alpha[0,1]\ . \tag{10}$$

Moreover $\|\varphi\|_{(\mathrm{H}^0_\alpha)'}\leq\|S_\alpha\|\,\|a\|_{\ell^1}$. By this inequality and Cauchy's criterion we verify that the series (10) converges for the topology of the norm of $(\mathrm{H}^0_\alpha)'$. So the set of functionals $\{\lambda_n,\,n\geq 0\}$ defined by (3) is total in $(\mathrm{H}^0_\alpha)'$. It follows immediately that the family of evaluation to dyadic points $(f\mapsto f(k\,2^{-j}))$ is total in $(\mathrm{H}^0_\alpha)'$. To conclude, suppose that $(P_n)_{n\geq 1}$ has two subsequences with distributions which converge respectively to P_ξ and P_ζ. By the convergence of finite-dimensional distributions of $(P_n)_{n\geq 1}$, we deduce the equality of P_ξ and P_ζ. ∎

3.2. Tightness in $\mathrm{H}_\alpha[0,1]$

In the sequel it is more convenient to work with H^0_α instead of H_α. As the canonical injection of $\mathrm{H}^0_\alpha[0,1]$ in $\mathrm{H}_\alpha[0,1]$ is continuous, weak convergence in the former implies weak convergence in the latter. A first sufficient condition for the tightness in H^0_α is given by

Theorem 9 (Kerkyacharian, Roynette [18]). *Let $(\xi_n)_{n\geq 1}$ be a sequence of processes vanishing at 0 and suppose there are $\gamma>0$, $\delta>0$ and $c>0$ such that*

$$\forall\,\lambda>0\,, \qquad P\Big(|\xi_n(t)-\xi_n(s)|>\lambda\Big)\;\leq\;\frac{c}{\lambda^\gamma}\,|t-s|^{1+\delta}\ . \tag{11}$$

Then $(\xi_n)_{n\geq 1}$ is tight in $\mathrm{H}^0_\alpha[0,1]$ for $0<\alpha<\delta/\gamma$. ∎

In the applications, this condition is essentially used in its moments version, obtained via Markov's inequality from (11).

Corollary 1 (Lamperti [19]). *Let $(\xi_n)_{n\geq 1}$ be a sequence of processes vanishing at 0. Suppose there are $\gamma>0$, $\delta>0$ and $c>0$ such that*

$$\mathbb{E}\,|\xi_n(t)-\xi_n(s)|^\gamma \;\le\; c\,|t-s|^{1+\delta}\;. \tag{12}$$

Then the sequence $(\xi_n)_{n\geq 1}$ is tight in $\mathrm{H}^0_\alpha[0,1]$ for $0<\alpha<\delta/\gamma$. ∎

On the other hand the Hölder version of Ascoli's theorem gives the following sufficient and necessary condition which can be useful to test the optimality of certain results.

Theorem 10 (Račkauskas, Suquet [24]). *Let $(\xi_n)_{n\geq 1}$ be a sequence of random elements of $\mathrm{H}^0_\alpha[0,1]$. $(\xi_n)_{n\geq 1}$ is tight if and only if*

$$\forall\,\varepsilon>0\,,\qquad \lim_{\delta\to 0}\sup_{n\geq 1} P\Big(w_\alpha(\xi_n,\delta)\geq\varepsilon\Big)=0\;. \;∎$$

Last, we have obtained the following result, inspired from a Davydov's theorem in the $D[0,1]$ setting (Skorokhod space) [10], which allows more flexibility in the handling of moment inequalities.

Theorem 11 (Hamadouche [14]). *Let $(\xi_n(t))_{n\geq 1}$ be a sequence of random elements of $\mathrm{H}^0_\alpha[0,1]$, satisfying the following conditions*

a) *There exists constants $a>1$, $b>1$, $c>0$ and a sequence of positive numbers $(a_n)\downarrow 0$ such that*

$$\mathbb{E}\,|\xi_n(t)-\xi_n(s)|^a \;\le\; c\,|t-s|^b\;, \tag{13}$$

for all $|t-s|\geq a_n$, $0\leq s,t\leq 1$ and $n\geq 1$.

b) *For any $\varepsilon>0$, $\lim\limits_{n\to\infty} P\{w_\alpha(\xi_n,a_n)>\varepsilon\}=0$.*

Then for all $\alpha<a^{-1}(\min(a,b)-1)$, $(\xi_n)_{n\geq 1}$ is tight in $\mathrm{H}^0_\alpha[0,1]$.

Sketched Proof: For a complete proof, we refer to [14]. In fact, we introduce a new process ζ_n defined by linear interpolation at the points $t_k=k\,a_n$ $(0\leq k\leq k_n)$ with $k_n=[\frac{1}{a_n}]$ and $t_{k_n+1}=1$. The paths of ζ_n are polygon lines and therefore are in $\mathrm{H}^0_\alpha[0,1]$ for all $\alpha\leq 1$. We use a) to show the tightness of $\{\zeta_n, n\geq 1\}$ and b) to prove the convergence in probability to 0 of $\|\xi_n-\zeta_n\|_\alpha$. The tightness of $\{\xi_n,\, n \geq 1\}$ will follow by the sequential characterization of tightness in the Polish space H^0_α.

Tightness of $\{\zeta_n,\, n \geq 1\}$ is obtained by the sufficient condition of Lamperti (corollary 1) for $\gamma = a$, $\delta = \min(a, b) - 1$, by discussing the location of s and t with respect to the grid $(k\, a_n,\, 0 \leq k \leq n)$. This discussion gives us too the following estimation

$$\|\zeta_n - \xi_n\|_\alpha \;\leq\; 4\, w_\alpha(\xi_n, a_n) \ ,$$

from which the convergence of the finite-dimensional distributions follows. ∎

4 – Some dependence tools

Recall that the strong mixing coefficient between two σ-fields $\mathcal{A}$ and $\mathcal{B}$ is defined by

$$\alpha(\mathcal{A}, \mathcal{B}) \;=\; \sup_{(A,B)\,\in\,\mathcal{A}\times\mathcal{B}} \Big|P(A \cap B) - P(A)\, P(B)\Big| \ . \tag{14}$$

Let $(X_n)_{n\geq 1}$ be a sequence of random variables defined on the same probability space. We define the strong mixing coefficient α_n by

$$\alpha_n \;=\; \sup\Big\{\alpha(\mathcal{F}_1^k, \mathcal{F}_{n+k}^{+\infty}),\ k \in \mathbb{N}^*\Big\} \tag{15}$$

where $\mathcal{F}_j^l$ is the σ-field spanned by the variables $(X_i,\ j \leq i \leq l)$. The sequence $(X_n)_{n\geq 1}$ is said α-mixing or strong mixing if α_n goes to zero as n goes to infinity. For a recent review about mixing, we refer to [11].

We say that $X_1, X_2, \cdots, X_m$ is a finite sequence of associated random variables if

$$\mathrm{Cov}\Big(f(X_1, \ldots, X_m),\, g(X_1, \ldots, X_m)\Big) \geq 0 \ , \tag{16}$$

for any pair f, g of functions $\mathbb{R}^m \to \mathbb{R}$ coordinatewise non decreasing such that this covariance exists. A sequence $(X_n)_{n\geq 1}$ is said associated if any finite subsequence is associated. Known results about association show that the dependence structure of a sequence of associated random variables is strongly determined by its covariance structure that is by the coefficient

$$u(n) \;=\; \sup_{k\in\mathbb{N}^*} \sum_{j:\,|j-k|\geq n} \mathrm{Cov}(X_j, X_k) \ . \tag{17}$$

Remark. If $(X_n)_{n\geq 1}$ is a sequence of stationary variables

$$u(n) \;=\; 2 \sum_{j\geq n+1} \mathrm{Cov}(X_1, X_j) \ . \ \square$$

The invariance principles under dependence rely on some central limit theorems and classical moment inequalities, we begin with.

Theorem 12 (Davydov [9]). *Let X, Y be real random variables with $\mathbb{E}X = \mathbb{E}Y = 0$ and finite variances. For $p, q, r \geq 1$ and $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$,*

$$|\operatorname{Cov}(X,Y)| \;\leq\; 8\;\alpha(X,Y)^{1/p}\;\mathbb{E}^{1/q}|X|^q\;\mathbb{E}^{1/r}|Y|^r\;. \blacksquare \tag{18}$$

Theorem 13 (Yokoyama [25]). *Let $(X_j)_{j\geq 1}$ be a strictly stationary sequence of α-mixing random variables such that $\mathbb{E}X_1 = 0$, $\mathbb{E}|X_1|^{\gamma+\varepsilon} < \infty$ for $\gamma > 2$, $\varepsilon > 0$ and*

$$\sum_{n=0}^{\infty}(n+1)^{\gamma/2-1}\,\alpha_n^{\varepsilon/(\gamma+\varepsilon)} \;<\; \infty\;.$$

Then there exists $C > 0$ such that

$$\mathbb{E}|X_1 + X_2 + \cdots + X_n|^{\tau} \;\leq\; C\,n^{\tau/2}\;. \blacksquare \tag{19}$$

Theorem 14 (Odaïra, Yoshihara [22]). *Let $(X_j)_{j\geq 1}$ be a sequence of α-mixing random variables satisfying for some constants $\varepsilon > 0$, $\gamma > 2$ the following conditions*

$$\sum_{n=1}^{\infty}\alpha_n^{\varepsilon/(\gamma+\varepsilon)} \;<\; \infty\;,$$

$$\sup_{j\geq 1}\mathbb{E}|X_j|^{\gamma+\varepsilon} \;<\; \infty\;.$$

Then $(X_j)_{j\geq 1}$ satisfies the functional central limit theorem in $D[0,1]$. $\blacksquare$

This last result has been improved by Doukhan, Massart and Rio [12]. Their method to prove the tightness does not seem transposable to the Hölderian functional framework.

We return now to theorems about association.

Theorem 15 (Birkel[3]). *Let $(X_j)_{j\geq 1}$ be a sequence of associated and centered random variables such that $\sup_{j\geq 1}\mathbb{E}|X_j|^{\gamma+\varepsilon} < \infty$ for $\gamma > 2$ and $\varepsilon > 0$. We suppose that the coefficient $u(n)$ defined by (17) satisfies*

$$u(n) \;=\; O\Big(n^{-(\gamma-2)(\gamma+\varepsilon)/(2\varepsilon)}\Big)\;. \tag{20}$$

Then there exists some constant b such that for all $n \geq 1$

$$\sup_m E|S_{n+m} - S_m|^{\gamma} \leq b\,n^{\gamma/2}, \quad \text{where} \;\; S_k = \sum_{j=1}^{k} X_j\;. \blacksquare \tag{21}$$

Theorem 16 (Newman, Wright [20]). *Let $(X_j)_{j\geq 1}$ be a strictly stationary sequence of centered and associated random variables with finite variance such that*

$$\sigma^2 \;=\; \mathbb{E}(X_1^2) + \,2\sum_{j\geq 2} \operatorname{Cov}(X_1, X_j) \;<\; \infty \;.$$

For all $n \geq 1$, we define the process

$$W_n(t) \;=\; \frac{1}{\sigma\sqrt{n}} \left(\sum_{k=1}^{j} X_k + (n\,t-j)\,X_{j+1}\right), \qquad \frac{j}{n} \leq t < \frac{j+1}{n}, \quad 0 \leq j < n \;.$$

Then W_n converges weakly in $C[0,1]$ to the Brownian motion W.
Hence the finite-dimensional distributions of W_n converge to those of W. ∎

5 – Invariance principles in H_α^0

5.1. Polygonal smoothing of partial sums process

We present here two extensions of Lamperti's invariance principle to dependent random variables.

Theorem 17. *Let $(X_j)_{j\geq 1}$ be a strictly stationary sequence of α-mixing and centered random variables. We suppose that there are $\gamma > 2$ and $\varepsilon > 0$ such that $\mathbb{E}|X_1|^{\gamma+\varepsilon} < \infty$ and*

$$\sum_{n=1}^{\infty} (n+1)^{\gamma/2-1}\,[\alpha_n]^{\varepsilon/(\gamma+\varepsilon)} \;<\; \infty \;. \tag{22}$$

Define for all $n \in \mathbb{N}^*$ and $0 \leq j < n$

$$\xi_n(t) \;=\; \frac{1}{\sigma\sqrt{n}} \left[\sum_{k=1}^{j} X_k + (n\,t-j)\,X_{j+1}\right], \qquad \frac{j}{n} \leq t < \frac{j+1}{n} \;, \tag{23}$$

where

$$\sigma^2 \;=\; \mathbb{E}X_1^2 + 2\sum_{j=2}^{\infty} \operatorname{Cov}(X_1, X_j) \;<\; \infty \;. \tag{24}$$

Then ξ_n converges weakly to the Brownian motion in H_α^0 for all $\alpha < 1/2 - 1/\gamma$.

Proof: We prove that under assumptions of theorem 17

$$\mathbb{E}|\xi_n(t) - \xi_n(s)|^\gamma \;\leq\; K\,|t-s|^{1+\delta} \qquad \text{with } \;1+\delta = \frac{\gamma}{2} > 1 \;.$$

First, if $j/n \le s \le t \le j+1/n$, we have $|\xi_n(t)-\xi_n(s)| = |t-s|\,|X_{j+1}|\sqrt{n}$ (we suppose that $\sigma = 1$). Next

$$\mathbb{E}|\xi_n(t)-\xi_n(s)|^\gamma \;\le\; |t-s|^\gamma\,(\sqrt{n})^\gamma\,\mathbb{E}|X_1|^\gamma \;\le\; |t-s|^{\gamma/2}\,(\mathbb{E}|X_1|^{\gamma+\varepsilon})^{\gamma/(\gamma+\varepsilon)}\;,$$

since $n|t-s| \le 1$.

Now if for some j and k, $(j-1)/n \le s \le j/n \le (j+k)/n \le t \le (j+k+1)/n$, we have by convexity

$$\mathbb{E}|\xi_n(t)-\xi_n(s)|^\gamma \;\le$$

$$\le\; 3^{\gamma-1}\left(\mathbb{E}\left|\xi_n(s)-\xi_n\left(\frac{j}{n}\right)\right|^\gamma + \mathbb{E}\left|\xi_n\left(\frac{j}{n}\right)-\xi_n\left(\frac{j+k}{n}\right)\right|^\gamma + \mathbb{E}\left|\xi_n\left(\frac{j+k}{n}\right)-\xi_n(t)\right|^\gamma\right).$$

We shall just estimate the middle term, the two others terms can be treated as in the precedent case.

$$\text{(25)}\qquad \mathbb{E}\left|\xi_n\left(\frac{j}{n}\right)-\xi_n\left(\frac{j+k}{n}\right)\right|^\gamma \;=\; \mathbb{E}\left|\frac{1}{\sqrt{n}}\Big(X_{j+1}+X_{j+2}+\cdots+X_{j+k}\Big)\right|^\gamma .$$

By theorem 13,

$$\text{(26)}\qquad \mathbb{E}\left|\xi_n\left(\frac{j}{n}\right)-\xi_n\left(\frac{j+k}{n}\right)\right|^\gamma \le K'\left(\frac{k}{n}\right)^{\gamma/2} \le K'|t-s|^{\gamma/2}\quad \text{since}\;\; |t-s| \ge \frac{k}{n}\,.$$

Finally, we obtain

$$\text{(27)}\qquad \mathbb{E}|\xi_n(t)-\xi_n(s)|^\gamma \;\le\; K|t-s|^{1+\gamma}\quad \text{with}\;\; 1+\delta = \frac{\gamma}{2} > 1\,.$$

Thus by theorem 9 and Markov's inequality, the sequence of distributions $(P_n)_{n\ge1}$ of processes ξ_n is tight in H^0_α, for any $\alpha < \delta/\gamma = 1/2 - 1/\gamma$.

To conclude, the finite-dimensional distributions of ξ_n converges to those of W using the theorem 14 whose assumptions are more general that those of theorem 17 since for $\gamma > 2$,

$$\forall\, n \ge 1\,,\qquad (\alpha_n)^{\varepsilon/(\gamma+\varepsilon)} < (n+1)^{\gamma/2-1}(\alpha_n)^{\varepsilon/(\gamma+\varepsilon)}. \;\blacksquare$$

Theorem 18. *Let $(X_j)_{j\ge1}$ be a strictly stationary sequence of centered and associated random variables such that $\mathbb{E}|X_1|^{\gamma+\varepsilon} < \infty$ for $\gamma > 2$ and $\varepsilon > 0$. Suppose that*

$$\text{(28)}\qquad u(n) \;=\; 2\sum_{j\ge n+1}\mathrm{Cov}(X_1,X_j) \;=\; O\Big(n^{-(\gamma-2)(\gamma+\varepsilon)/(2\varepsilon)}\Big)$$

and

$$0 < \sigma^2 = \mathbb{E}|X_1|^2 + u(1) < \infty .$$

Then $(\xi_n)_{n\geq 1}$ *converges weakly to the Brownian motion* W *in* H_α^0 *for all* $\alpha < 1/2 - 1/\gamma$.

Proof: The tightness is proved like in the precedent case, using Birkel's moment inequality (theorem 15) instead of Yokoyama's one. The convergence of finite-dimensional distributions follows from theorem 16. ∎

5.2. Convolution smoothing of partial sums process

Let $(X_j)_{j\geq 1}$ be a sequence of independent random variables, identically distributed such that $\mathbb{E}X_1 = 0$ and $\mathbb{E}|X_1|^\gamma < \infty$ for some $\gamma > 2$. We denote again $\sigma^2 = \mathbb{E}X_1^2$, $S_i = \sum_{k=1}^{i} X_k$, $S_0 = 0$ and we consider the Donsker–Prokhorov's normalized partial sums process:

$$\xi_n(t) = \frac{1}{\sigma\sqrt{n}} S_{[nt]}\,, \qquad t \in [0,1] \,, \tag{29}$$

where $[nt]$ is the integer part of nt. For the sake of convenience, we shall use in the one of the following expressions of ξ_n:

$$\xi_n(t) \; = \; \frac{1}{\sigma\sqrt{n}} \sum_{i=1}^{n} S_i \,\mathbf{1}_{[\frac{i}{n},\frac{i+1}{n}[}(t) \,,$$

$$\xi_n(t) \; = \; \frac{1}{\sigma\sqrt{n}} \sum_{k=1}^{n} X_k \,\mathbf{1}_{[\frac{k}{n},1]}(t) \,.$$

Let K be a probability density on the real line such that

$$\int_{\mathbb{R}} |u|\, K(u)\, du \; < \; \infty \tag{30}$$

and $(b_n)_{n\geq 1}$ a sequence of positive numbers such that $\lim_{n\to\infty} b_n = O$ and

$$\frac{1}{b_n} = O(n^{\tau/2})\,, \qquad 0 < \tau < \frac{1}{2} \,. \tag{31}$$

We define the sequence $(K_n)_{n\geq 1}$ of convolution kernels by

$$K_n(t) = \frac{1}{b_n} K\!\left(\frac{t}{b_n}\right), \qquad t \in \mathbb{R} \,. \tag{32}$$

We consider the smoothed partial sums process defined by:

$$\zeta_n(t) = (\xi_n * K_n)(t) - (\xi_n * K_n)(0), \qquad t \in [0,1] . \tag{33}$$

The term $(\xi_n * K_n)(0)$ is subtracted in order to have a process with paths vanishing at zero. We will impose some conditions on K_n to ensure that any path of ζ_n belongs to $\mathrm{H}_{1/2}$ and so to H^0_α for $\alpha < 1/2$. These conditions are provided by the following lemma.

Lemma 3. *Let f be a bounded measurable function with support in $[0,1]$ and K a convolution kernel satisfying*

$$K \in L^1([-1,1]) \cap L^{1/2}([-1,1]) , \tag{34}$$

$$|K(x) - K(y)| \le a(K)\,|x-y| , \qquad x,y \in [-1,1] , \tag{35}$$

for some constant $a(K)$. Then the restriction to $[0,1]$ of $f*K - f*K(0)$ is in $\mathrm{H}_{1/2}[0,1]$.

Proof: Clearly $f*K$ is bounded. On the other hand

$$\begin{aligned}
\Big|f*K(x) - f*K(y)\Big| &\le \int_{[0,1]} |f(u)|\,\Big|K(x-u) - K(y-u)\Big|\,du \\
&\le \|f\|_\infty\, a(K)^{1/2}\,|x-y|^{1/2} \int_{[0,1]} \Big|K(x-u) - K(y-u)\Big|^{1/2} du \\
&\le 2\,\|f\|_\infty\, a(K)^{1/2}\,|x-y|^{1/2} \int_{[-1,1]} |K(v)|^{1/2}\,dv \\
&\le c(K)\,|x-y|^{1/2} .
\end{aligned}$$

Hence

$$\|f*K\|_{1/2} = w_{1/2}(f*K,\,1) < \infty . \blacksquare$$

Theorem 19. *Let $(X_j)_{j\ge1}$ be a sequence of independent random variables, identically distributed such that $\mathbb{E}X_1 = 0$ and $\mathbb{E}|X_1|^\gamma < \infty$ for some $\gamma > 2$. Suppose that the convolution kernels K_n satisfy (32), (30), (34) and (35). Then the sequence of smoothed partial sums processes ζ_n defined by (33) converges weakly to the Brownian motion W in $\mathrm{H}^0_\alpha[0,1]$ for all $\alpha < 1/2 - \max(\tau, 1/\gamma)$.*

Proof: By lemma 3, ζ_n is in $\mathrm{H}^0_\alpha[0,1]$, for all $\alpha < 1/2$.

Tightness

We apply theorem 11 with $a_n = 1/n$. This leads us to study separately the cases $t-s \geq 1/n$ and $0 < t-s < 1/n$. without loss of generality we can assume that $s < t$.

First case: $|t-s| \geq 1/n$.

$$\begin{aligned}\mathbb{E}|\zeta_n(t)-\zeta_n(s)|^\gamma &= \mathbb{E}\Big|\xi_n * K_n(t) - \xi_n * K_n(s)\Big|^\gamma \\ &= \mathbb{E}\left|\frac{1}{\sigma\sqrt{n}}\int_{\mathbb{R}}\Big(S_{[n(t-u)]} - S_{[n(s-u)]}\Big)\, K_n(u)\, du\right|^\gamma \\ &= \mathbb{E}\left|\frac{1}{\sigma\sqrt{n}}\int_{\mathbb{R}}\sum_{i=[n(s-u)]+1}^{[n(t-u)]} X_i\, K_n(u)\, du\right|^\gamma .\end{aligned}$$

By Jensen's inequality with respect to $K_n(u)\,du$ and Fubini's theorem, we obtain

$$\mathbb{E}|\zeta_n(t)-\zeta_n(s)|^\gamma \leq \int_{\mathbb{R}}\mathbb{E}\left|\frac{1}{\sigma\sqrt{n}}\int_{\mathbb{R}}\sum_{i=[n(s-u)]+1}^{[n(t-u)]} X_i\right|^\gamma K_n(u)\, du .$$

Using Marcinkiewicz–Zygmund's inequality for the moments of sums of i.i.d. random variables, it follows

$$\begin{aligned}\mathbb{E}|\zeta_n(t)-\zeta_n(s)|^\gamma &\leq \int_{\mathbb{R}}\left(\frac{[n\,(t-u)]-[n\,(s-u)]}{n}\right)^{\gamma/2} c_\gamma\, K_n(u)\; du \\ &\leq \int_{\mathbb{R}}\left(|t-s| + \frac{2}{n}\right)^{\gamma/2} c_\gamma\, K_n(u)\; du ,\end{aligned} \tag{36}$$

since $[n(t-u)] - [n(s-u)] \leq n(t-s) + 2$. Hence, there is some constant c'_γ such that

$$\mathbb{E}|\zeta_n(t)-\zeta_n(s)|^\gamma \leq c'_\gamma\, |t-s|^{\gamma/2}, \quad \text{since } \ |t-s| \geq \frac{1}{n} .$$

Second case: $0 \leq t-s < 1/n$.

We proceed as follows

$$\begin{aligned}|\zeta_n(t)-\zeta_n(s)| &= \left|\int_{\mathbb{R}}\frac{1}{\sigma\sqrt{n}}\sum_{i=1}^{i} X_i\Big(K_n(t-u) - K_n(s-u)\Big)\, \mathbf{1}_{[\frac{i}{n},1]}(u)\; du\right| \\ &\leq \frac{a(K)}{b_n^2\,\sigma\sqrt{n}}\sum_{i=1}^{n}|X_i|\,|t-s|\left(1-\frac{i}{n}\right) .\end{aligned}$$

Since $1 - i/n \le 1$, it follows

$$|\zeta_n(t) - \zeta_n(s)| \;\le\; \frac{a(K)}{b_n^2\,\sigma\sqrt{n}} \sum_{i=1}^{n} |X_i|\,|t-s|$$

and hence

$$\frac{|\zeta_n(t) - \zeta_n(s)|}{|t-s|^\alpha} \;\le\; \frac{a(K)}{b_n^2\,\sigma\sqrt{n}} \sum_{i=1}^{n} |X_i|\,|t-s|^{1-\alpha}\ .$$

Thus

$$\begin{aligned} w_\alpha\Big(\zeta_n, \frac{1}{n}\Big) &= \sup_{|t-s|\le\frac{1}{n}} \frac{|\zeta_n(t) - \zeta_n(s)|}{|t-s|^\alpha} \\ &\le \frac{a(K)}{b_n^2\,\sigma\sqrt{n}} \sum_{i=1}^{n} |X_i| \Big(\frac{1}{n}\Big)^{1-\alpha} \qquad (37)\\ &\le \frac{a(K)}{\sigma\, b_n^2\, n^{1/2-\alpha}}\, \frac{1}{n} \sum_{i=1}^{n} |X_i|\ . \end{aligned}$$

By (31), $b_n^{-2}\, n^{\alpha-1/2}$ goes to 0 as soon as $1/2 - \alpha > \tau$. By the strong law of large numbers, $n^{-1}\sum_{i=1}^n |X_i|$ goes to $\mathbb{E}|X_1|$ almost surely. Hence $w_\alpha(\xi, 1/n)$ goes to 0 in probability as n goes to infinity, for all $\alpha < 1/2 - \tau$.

We conclude about tightness by the theorem 11, noticing that its hypothesis are satisfied for $a = \gamma$, $b = \gamma/2$, $(\gamma > 2)$, $c = c'_\gamma$ and $a_n = 1/n$.

We obtain then the tightness of (ζ_n) in $\mathrm{H}^0_\alpha[0,1]$ for all α satisfying $\alpha < 1/2 - \tau$ and $\alpha < 1/2 - 1/\gamma$, so for all $\alpha < 1/2 - \max(\tau, 1/\gamma)$.

Convergence of the finite-dimensional distributions of $\{\zeta_n,\, n \ge 1\}$

Since the finite-dimensional distributions of ξ_n converge to those of the Brownian motion by Lindeberg–Levy's central limit theorem, it will be the same for those of ζ_n if we prove for instance the convergence to 0 of $\mathbb{E}|\zeta_n(t) - \xi_n(t)|^2$ for all $t \in [0,1]$. We begin by noting that

$$\begin{aligned} \mathbb{E}\Big|\xi_n * K_n(t) - \xi_n(t)\Big|^2 &= \mathbb{E}\Big|\int_{\mathbb{R}} \Big(\xi_n(t-u) - \xi_n(t)\Big)\, K_n(u)\ du\Big|^2 \\ &= \mathbb{E}\Big|\int_{\mathbb{R}} \frac{1}{\sigma\sqrt{n}} \Big(S_{[n(t-u)]} - S_{[nt]}\Big)\, K_n(u)\ du\Big|^2 \\ &= \mathbb{E}\Big|\int_{\mathbb{R}} \frac{1}{\sigma\sqrt{n}} \sum_{i=[nt]+1}^{[n(t-u)]} X_i\, K_n(u)\ du\Big|^2\ . \end{aligned}$$

Applying the Jensen's inequality with respect to $K_n(u)du$ and Fubini's theorem, we obtain

$$\mathbb{E}\Big|\xi_n * K_n(t) - \xi_n(t)\Big|^2 \le \int_{\mathbb{R}} \mathbb{E}\Big|\frac{1}{\sigma\sqrt{n}} \sum_{i=[nt]+1}^{[n(t-u)]} X_i\Big|^2 K_n(u)\, du \,. \tag{38}$$

Using Marcinkiewicz–Zygmund's inequality, we obtain

$$\begin{aligned} \mathbb{E}\Big|\xi_n * K_n(t) - \xi_n(t)\Big|^2 &\le c\int_{\mathbb{R}} \frac{\big|[n\,(t-u)] - [n\,t]\big|}{n} K_n(u)\, du \\ &\le c\int_{\mathbb{R}} \Big(|u| + \frac{2}{n}\Big) K_n(u)\, du \,. \end{aligned} \tag{39}$$

Hence

$$\mathbb{E}\Big|\xi_n * K_n(t) - \xi_n(t)\Big|^2 \le c\left(b_n \int_{\mathbb{R}} |v|\, K(v)\, dv + \frac{2}{n}\right) .$$

Since K has the first order moment and b_n goes to 0 as n goes to infinity, we deduce that $\xi_n * K_n(t) - \xi_n(t)$ goes to 0 in $L^2(\Omega)$ for all $t \in [0,1]$. In particular for $t=0$, $\mathbb{E}|\xi_n * K_n(0)|^2$ goes to 0. Since for all $t \in [0,1]$,

$$\mathbb{E}|\zeta_n(t) - \xi_n(t)|^2 \le \frac{1}{2}\left(\mathbb{E}\Big|\xi_n * K_n(t) - \xi_n(t)\Big|^2 + \mathbb{E}\Big|\xi_n * K_n(0)\Big|^2\right) ,$$

this achieves the proof of the convergence of the finite-dimensional distributions and of Theorem 19. ∎

The arguments used in the proof above allow an extension of the result to dependent random variables case.

Theorem 20. *Let $(X_j)_{j\ge1}$ be a strictly stationary sequence of α-mixing centered random variables. Suppose that there exists $\gamma > 2$ and $\varepsilon > 0$ such that $\mathbb{E}|X_1|^{\gamma+\varepsilon} < \infty$,*

$$\sum_{n=1}^{\infty} (n+1)^{\gamma/2-1}\, [\alpha_n]^{\varepsilon/(\gamma+\varepsilon)} < \infty \,, \tag{40}$$

$$\sigma^2 = \mathbb{E}X_1^2 + 2\sum_{j\ge2} \mathrm{Cov}(X_1, X_j) < \infty \,. \tag{41}$$

We suppose moreover that the convolution kernels satisfy (32), (30), (34) and (35). Then the sequence of smoothed partial sums processes ζ_n defined by (33) converges weakly to the Brownian motion W in $\mathrm{H}^0_\alpha[0,1]$ for all $\alpha < 1/2 - \max(\tau, 1/\gamma)$.

Proof: The tightness is obtained as in the proof of theorem 19. In the case $|t-s| \geq 1/n$, we use the Yokoyama's inequality (theorem 13) instead of the Marcinkiewicz–Zygmund's one. To prove the convergence in probability to 0 of $w_\alpha(\zeta_n, 1/n)$, it suffices to observe that the estimates leading to (37) remain valid and to apply the Markov's inequality

$$P\left\{ \frac{a(K)}{\sigma\, b_n^2\, n^{1/2-\alpha}}\, \frac{1}{n} \sum_{i=1}^{n} |X_i| \geq \delta \right\} \leq \frac{a(K)}{\sigma\,\delta\, b_n^2\, n^{1/2-\alpha}}\, \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}|X_i| \ .$$

Since $\mathbb{E}|X_i| \leq M^{1/(\gamma+\varepsilon)}$, this estimate is well an $O(b_n^{-2}\, n^{\alpha-1/2})$ and goes to 0 for all $\alpha < 1/2 - \tau$.

Now the finite-dimensional distributions of ξ_n converge to those of the Brownian motion by the Odaïra–Yoshihara's theorem (theorem 14). The problem is then reduced to prove the convergence to 0 of $\mathbb{E}|\zeta_n(t) - \xi_n(t)|^2$ as in the independent case. The unique difference is the passage from (38) to (39), where instead of Marcinkiewicz–Zygmund's inequality, we use the following variance estimation, based on a Davydov's inequality (theorem 12)

$$\begin{aligned} \operatorname{Var}\Big(\sum_{i=1}^{m} X_i\Big) &= m \operatorname{Var} X_1 + 2 \sum_{j=2}^{m} \sum_{i=1}^{j-1} \operatorname{Cov}(X_1, X_{j-i+1}) \\ &\leq m \operatorname{Var} X_1 + 16\, m \sum_{k=1}^{\infty} \alpha_k^{1/p}\, \mathbb{E}^{1/q}|X_1|^q\, \mathbb{E}^{1/r}|X_1|^r \ . \end{aligned}$$

Taking $q = r = \gamma + \varepsilon$ in Davydov's inequality, it follows

$$\operatorname{Var}\Big(\sum_{i=1}^{m} X_i\Big) \leq m \operatorname{Var} X_1 + 16\, m \left(\mathbb{E}|X_1|^{\gamma+\varepsilon}\right)^{2/(\gamma+\varepsilon)} \sum_{k=1}^{\infty} \alpha_k^{(\gamma-2+\varepsilon)/(\gamma+\varepsilon)} \ .$$

Since $\gamma > 2$, the convergence of the series above follows from the assumptions (40) on mixing coefficients. We have then

$$\operatorname{Var}\Big(\sum_{i=1}^{m} X_i\Big) = O(m)$$

and applying this with $m = |[n\,(t-u)] - [n\,t]|$, we can conclude as in the independent case. ∎

Theorem 21. *Let $(X_j)_{j\geq 1}$ be a strictly stationary sequence of associated and centered random variables such that $\mathbb{E}|X_1|^{\gamma+\varepsilon} < \infty$ for some $\gamma > 2$ and $\varepsilon > 0$. Suppose that*

$$u(n) = 2 \sum_{j \geq n+1} \operatorname{Cov}(X_1, X_j) = O\Big(n^{-(\gamma-2)(\gamma+\varepsilon)/(2\varepsilon)}\Big) \tag{42}$$

and

$$0 < \sigma^2 = \mathbb{E}|X_1|^2 + u(1) < \infty .$$

Suppose moreover that the convolution kernels satisfy (32), (30), (34) and (35). Then the sequence of smoothed partial sums processes ζ_n defined by (33) converges weakly to the Brownian motion W in $\mathrm{H}^0_\alpha[0,1]$ for all $\alpha < 1/2 - \max(\tau, 1/\gamma)$.

Proof: It is similar to the α-mixing case, using Birkel's moment inequality (Theorem 15) instead of Yokoyama's one and the Newman–Wright's central limit theorem instead of Odaïra–Yoshihara's one. The variance inequality is now a direct consequence of hypothesis $u(1) < \infty$. ∎

REFERENCES

[1] Baldi, P. and Roynette, B. – Some exact equivalents for the Brownian motion in Hölder norm, *Probability Theory and Related Fields,* 93 (1993), 457–484.

[2] Billingsley, P. – *Convergence of Probability Measures*, J. Wiley, New-York, 1968.

[3] Birkel, T. – Moment bounds for associated sequences, *Ann. Probab.,* 16 (1988), 1184–1193.

[4] Birkel, T. – The invariance principle for associated processes, *Stochastic processes and their Applications,* 27 (1988), 57–71.

[5] Brézis, H. – *Analyse Fonctionnelle*, Masson, Paris, 1983.

[6] Ciesielski, Z. – On the isomorphisms of the spaces H_α and m, *Bull. Acad. Pol. Sci. Sér. Sci. Math. Astronom. Phys.,* 8 (1960), 217–222.

[7] Ciesielski, Z. – Hölder conditions for realizations of Gaussian processes, *Trans. Amer. Math. Soc.,* 99 (1961), 403–413.

[8] Ciesielski, Z.; Kerkyacharian, G. and Roynette, B. – Quelques espaces fonctionnels associés à des processes gaussiens, *Studia-Math.,* 107 (1993), 171–204.

[9] Davydov, Y. – The invariance principle for stationary processes, *Theory Probab. Appl.,* 15 (1970), 487–498.

[10] Davydov, Y. – Weak convergence of discontinuous processes to continuous ones, *Th. of Probab. and Math. Stat.,* Proc. of the seminar dedicated to the memory of Kolmogorov, March–May 1993, St. Petersbourg (I. Ibragimov, A. Zaitser, Eds.), Gordon and Breach, 1996, 15–16.

[11] Doukhan, P. – Mixing, *Lecture Notes in Statistics,* 85 (1994).

[12] Doukhan, P.; Massart, P. and Rio, E. – The functional central limit theorem for strongly mixing processes, *Ann. Inst. Henri Poincaré, Probab. Stat.,* 30 (1994), 63–82.

[13] Hamadouche, D. – Principle d'invariance dans les espaces hölderiens pour des variables α-mélangeantes ou associées,*Pub. IRMA Lille,* 37-IX (1995).

[14] Hamadouche, D. – Weak convergence of smoothed empirical process in Hölder spaces, *Stat. Probab. Letters,* 36 (1998), 393–400, with an erratum in *Stat. Probab. Letters,* 39 (1998), 385–386.

[15] Hamadouche, D. and Suquet, Ch. – Weak Hölder convergence of processes with application to the perturbed empirical process, *Applicationes Math.,* 26 (1999), 63–83.

[16] Hardy, G.H. and Littlewood, J.E. – Some properties of fractional integrals, *I. Math. Zeitschr.,* 27 (1927), 565–606.

[17] Ibragimov, I. – Sur la régularité des trajectoires des fonctions aléatoires, *C. R. Acad. Sci. Paris, Sér. A,* 289 (1979), 545–547.

[18] Kerkyacharian, G. and Roynette, B. – Une démonstration simple des théorèmes de Kolmogorov, Donsker et Ito-Nisio, *C. R. Acad. Sci. Paris, Sér. I,* 312 (1991), 877–882.

[19] Lamperti, J. – On convergence of stochastic processes, *Trans. Amer. Math. Soc.,* 104 (1962), 430–435.

[20] Newman, C.M. and Wright, A.L. – An invariance principle for certain dependent sequences, *Ann. Probab.,* 9 (1981), 671–675.

[21] Nobelis, Ph. – Fonctions aléatoires lipschitziennes, *Lecture Notes Math.,* 850 (1981), 38–43.

[22] Oodaïra, H. and Yoshihara, K.I. – Functional central limit theorems for strictly stationary processes satisfying the strong mixing condition, *Kodai Math. Sem. Rep.,* 24 (1972), 259–269.

[23] Riesz, M. – L'intégrale de Riemann–Liouville et le problème de Cauchy, *Acta Math.,* 81 (1949), 1–223.

[24] Račkauskas, A. and Suquet, Ch. – Central limit theorem in Hölder spaces, *Probab. and Math. Statist.,* 19 (1999), 133–152.

[25] Yokoyama, R. – Moment bounds for stationary mixing sequences, *Z. Wahrsch. Verw. Gebiete,* 52 (1980), 45–57.

Djamel Hamadouche,
Laboratoire de Statistique et Probabilités, Bât. M2,
Université des Sciences et Technologies de Lille,
F-59655 Villeneuve d'Ascq Cedex – FRANCE