PORTUGALIAE MATHEMATICA
Vol. 57 Fasc. 2 – 2000

EXISTENCE THEOREM OF PERIODIC POSITIVE SOLUTIONS FOR THE RAYLEIGH EQUATION OF RETARDED TYPE

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Abstract: In this paper, by using the coincidence degree theory, we give four sufficient conditions on the existence of periodic positive solutions of the following non-autonomous Rayleigh equation of retarded type

$$x'' + f(t, x'(t-\sigma)) + g(t, x(t-\tau)) \;=\; p(t) \;.$$

1 – Introduction

In the papers [1–5] the authors studied the existence of periodic solutions of the Rayleigh equation

$$x''(t) + f(x') + g(x) = p(t) \;. \tag{1}$$

So, for the existence of periodic positive solutions of (1), we can not see any results for it. Since theory of the existence of periodic positive solutions of the differential equation with retarded argument will play an important role in mathematical ecology, in this paper, we discuss the existence of periodic positive solutions of the non-autonomous Rayleigh equation of retarded type

$$x''(t) + f(t, x'(t-\sigma)) + g(t, x(t-\tau)) \;=\; p(t) \;, \tag{2}$$

where $\sigma, \tau \geq 0$ are constants, f and $g \in C(R^2, R)$, $f(t,x)$ and $g(t,x)$ are functions with period 2π for t, $p \in C(R,R)$, $p(t) = p(t+2\pi)$ for $t \in R$ and $\int_0^{2\pi} p(t) = 0$. Using coincidence degree theory developed by Mawhin [6], we find four sufficient conditions for the existence of periodic positive solutions of (2).

Received: March 16, 1998; *Revised*: June 19, 1998.
AMS Subject Classification: 34K15.
Keywords and Phrases: Coincidence degree; Rayleigh equation; Periodic positive solution.

2 – Main results

The following results provide sufficient conditions for the existence of periodic positive solutions of (2).

Theorem 1. *Suppose that there exist constants $H > 0$, $M > 0$, $\alpha > 0$, $\beta > 0$ and $\gamma > 0$, such that*

(**i**) $|f(t,x)| \le H$ *and* $|g(t,x)| \le \alpha |x|^\gamma + \beta$, *for* $(t,x) \in R^2$;

(**ii**) $g(t,x) > H$, *for* $t \in R$ *and* $x \ge M$;

(**iii**) $f(t,0) = 0$ *and* $g(t,0) < 0$, *for* $t \in R$.

Then there exists a periodic positive solution with period 2π *of (2).*

Theorem 2. *Suppose that there exist constants $H > 0$, $M > 0$, $\alpha > 0$, $\beta > 0$ and $\gamma > 0$, such that*

(**i**) $|f(t,x)| \le H$ *and* $|g(t,x)| \le \alpha |x|^\gamma + \beta$, *for* $(t,x) \in R^2$;

(**ii**) $g(t,x) < -H$, *for* $t \in R$ *and* $x \ge M$;

(**iii**) $f(t,0) = 0$ *and* $g(t,0) > 0$, *for* $t \in R$.

Then there exists a periodic positive solution with period 2π *of (2).* ∎

Theorem 3. *Suppose that there exist constants $M > 0$, $\alpha > 0$, $\beta > 0$ and $\gamma > 0$, such that*

(**i**) $f(t,x) \ge 0$ *and* $|g(t,x)| \le \alpha |x|^\gamma + \beta$, *for* $(t,x) \in R^2$;

(**ii**) $g(t,x) > 0$, *for* $t \in R$ *and* $x \ge M$;

(**iii**) $f(t,0) = 0$ *and* $g(t,0) < 0$, *for* $t \in R$.

Then there exists a periodic positive solution with period 2π *of (2).*

Theorem 4. *Suppose that there exist constants $M > 0$, $\alpha > 0$, $\beta > 0$ and $\gamma > 0$, such that*

(**i**) $f(t,x) \ge 0$ *and* $|g(t,x)| \le \alpha |x|^\gamma + \beta$, *for* $(t,x) \in R^2$;

(**ii**) $g(t,x) < 0$, *for* $t \in R$ *and* $x \ge M$;

(**iii**) $f(t,0) = 0$ *and* $g(t,0) > 0$, *for* $t \in R$.

Then there exists a periodic positive solution with period 2π *of (2).* ∎

To prove above results we need preliminaries. Set

$$X := \Big\{ x \in C^1(R,R) \mid \ x(t+2\pi) = x(t) \Big\}$$

and define the norm on X as $\|x\|_1 = \max\limits_{t\in[0,2\pi]} \{|x(t)|,\, |x'(t)|\}$. Similarly, set

$$Z := \Big\{ z \in C(R,R) \mid \ z(t+2\pi) = z(t) \Big\}$$

and define the norm on Z as $\|z\|_0 = \max\limits_{t\in[0,2\pi]} |z(t)|$. Then both $(X, \|\cdot\|_1)$ and $(Z, \|\cdot\|_0)$ are Banach space. Define respectively the operators L and N as

$$L\colon\ X \cap C^2(R,R) \to Z\,, \qquad x(t) \mapsto x''(t) \ ,$$

and

$$N\colon\ X \to Z\,, \qquad x(t) \ \mapsto\ -f(t, x'(t-\sigma)) - g(t, x(t-\tau)) + p(t) \ .$$

We know that $\operatorname{Ker} L = R$. Define respectively the projective operators P and Q as

$$P\colon\ X \to \operatorname{Ker} L\,, \qquad x \mapsto Px = \frac{1}{2\pi} \int_0^{2\pi} x(t)\, dt \ ,$$

and

$$Q\colon\ Z \to Z/\operatorname{Im}\,, \qquad z \mapsto Qz = \frac{1}{2\pi} \int_0^{2\pi} z(t)\, dt \ .$$

Then we have $\operatorname{Im} P = \operatorname{Ker} Q = \operatorname{Im} L$. For some positive number D, set

$$\Omega := \Big\{ x \in X \mid \ 0 < x(t) < D,\ |x'(t)| < D \Big\} \ .$$

The following two lemmas which will be used in the proofs of our main results are extracted from [6].

Lemma 1. *L is the Fredholm operator with index null.* ∎

Lemma 2. *N is L-compact on $\bar\Omega$.* ∎

Proof of Theorem 1: Consider the equation

$$x''(t) + \lambda\, f(t, x'(t-\sigma)) + \lambda\, g(t, x(t-\tau)) \ = \ \lambda\, p(t) \tag{3}$$

where $\lambda \in (0,1)$. Suppose that $x(t)$ is a periodic positive solution with period 2π of (3). By integrating (3) from 0 to 2π we find

$$\int_0^{2\pi} \Big[f(t, x'(t-\sigma)) + g(t, x(t-\tau)) \Big]\, dt \ = \ 0 \ . \tag{4}$$

Using condition (i) we have

$$\int_0^{2\pi} \Big| f(t, x'(t-\sigma)) \Big| \, dt \;\le\; 2\pi H \; . \tag{5}$$

It follows from (i) and (4) that

$$\begin{aligned} \int_0^{2\pi} \Big[g(t, x(t-\tau)) - H \Big] \, dt &\le \int_0^{2\pi} \Big[g(t, x(t-\tau)) - \Big| f(t, x'(t-\sigma)) \Big| \Big] \, dt \\ &\le \int_0^{2\pi} \Big[g(t, x(t-\tau)) + f(t, x'(t-\sigma)) \Big] \, dt \;=\; 0 \; . \end{aligned} \tag{6}$$

By applying (i) we have

$$|g(t,x)| \le M_1 \quad \text{for} \;\; t \in R, \;\; x \in (0, M) \; , \tag{7}$$

where $M_1 = \alpha \, M^\gamma + \beta$. Set

$$G_+(t) \;=\; \max\Big\{ g(t, x(t-\tau)) - H, \, 0 \Big\} \tag{8}$$

and

$$G_-(t) \;=\; \max\Big\{ H - g(t, x(t-\tau)), \, 0 \Big\} \; . \tag{9}$$

Both $G_+(t)$ and $G_-(t)$ are nonnegative continuous functions, and

$$g(t, x(t-\tau)) - H \;=\; G_+(t) - G_-(t) \; , \tag{10}$$

$$\Big| g(t, x(t-\tau)) - H \Big| \;=\; G_+(t) + G_-(t) \; . \tag{11}$$

Note that $x(t)$ is a periodic positive solution with period 2π of (3). By (ii), when $g(t, x(t-\tau)) < H$, we find that $t \in R$ and $0 < x(t-\tau) < M$. Hence by (7), we have $|g(t, x(t-\tau))| \le M_1$ and for any $t \in [0, 2\pi]$,

$$G_-(t) \;=\; |G_-(t)| \;\le\; H + M_1 \; . \tag{12}$$

From (6), (10) and (12) we see

$$\int_0^{2\pi} G_+(t) \, dt \;\le\; \int_0^{2\pi} G_-(t) \, dt \;\le\; 2\pi (H + M_1) \; . \tag{13}$$

Using (11) and (13) we have

$$\int_0^{2\pi} \Big| g(t, x(t-\tau)) - H \Big| \, dt \;\le\; 4\pi (H + M_1) \; . \tag{14}$$

Thus

$$\int_0^{2\pi} \Big|g(t,x(t-\tau))\Big|\,dt \;\le\; M_2 \;, \tag{15}$$

where $M_2 = 6\pi H + 4\pi M_1$. Since $x(2\pi) = x(0)$, there exists $t_0 \in [0, 2\pi]$, such that $x'(t_0) = 0$. Then by (3), (5) and (15), we conclude for any $t \in [0, 2\pi]$ that

$$\begin{aligned}
|x'(t)| \;&=\; \left|\int_{t_0}^{t} x''(s)\,ds\right| \;\le\; \int_0^{2\pi} |x''(t)|\,dt \\
&\le\; \lambda \int_0^{2\pi} \Big|f(t,x'(t-\sigma))\Big|\,dt \;+\; \lambda \int_0^{2\pi} \Big|g(t,x(t-\tau))\Big|\,dt \;+\; \lambda \int_0^{2\pi} |p(t)|\,dt \\
&\le\; 2\pi H + M_2 + 2\pi m \;,
\end{aligned}$$

where $m = \max\limits_{t\in[0,2\pi]} |p(t)|$. Taking $M_3 = 2\pi H + M_2 + 2\pi m$, for any $t \in [0, 2\pi]$, we have

$$|x'(t)| \;\le\; M_3 \;. \tag{16}$$

By (ii) and (6), there exists $t_1 \in [0, 2\pi]$ such that $0 < x(t_1 - \tau) < M$. Taking $t_1 - \tau = 2n\pi + t_2$ (n is one integer), and $t_2 \in [0, 2\pi]$, we have $0 < x(t_2) < M$, and for any $t \in [0, 2\pi]$

$$0 \;<\; x(t) \;=\; x(t_2) + \int_{t_2}^{t} x'(s)\,ds \;\le\; M + \int_0^{2\pi} |x'(t)|\,dt \;\le\; M + 2\pi M_3 \;. \tag{17}$$

Taking $M_4 = M + 2\pi M_3$, by (16) and (17), for any $t \in [0, 2\pi]$, we have $0 < x(t) \le M_4$ and $|x'(t)| \le M_4$. Let $M_5 > 0$, $M_5 > M_4$ and

$$\Omega_1 := \Big\{ x \in X \mid\; 0 < x(t) < M_5,\; |x'(t)| < M_5 \Big\} \;.$$

By Lemma 1 and Lemma 2, we know that L is the Fredholm operator with index null and N is L-compact on $\bar\Omega_1$ (see [6]). In terms of valuation of bound of periodic positive solution as above, we know that for any $x \in \partial\Omega_1 \cap \operatorname{dom} L$ and $\lambda \in (0, 1)$, $Lx \ne \lambda Nx$. Since for any $x \in \partial\Omega_1 \cap \operatorname{Ker} L$, $x = M_5\ (> M)$ or $x = 0$, then in view of (ii), (iii) and $\int_0^{2\pi} p(t) = 0$, we have

$$\begin{aligned}
QNx \;&=\; \frac{1}{2\pi}\int_0^{2\pi} \Big[-f(t,x'(t-\sigma)) - g(t,x(t-\tau)) + p(t)\Big]\,dt \\
&=\; \frac{1}{2\pi}\int_0^{2\pi} \Big[-f(t,0) - g(t,x(t-\tau)) + p(t)\Big]\,dt \\
&=\; \frac{1}{2\pi}\int_0^{2\pi} \Big[-g(t,x(t-\tau))\Big]\,dt \;=\; -\frac{1}{2\pi}\int_0^{2\pi} g(t,x)\,dt \;\ne\; 0 \;.
\end{aligned}$$

Set $\delta = M_5/2$ and define the transformation F as

$$F(x,\mu) \;=\; \mu(x-\delta) + (1-\mu)\,\frac{1}{2\pi}\int_0^{2\pi} g(t,x)\,dt, \qquad \text{for} \;\; \mu \in [0,1] \;.$$

Then we have

$$F(0,\mu) \;=\; -\mu\,\delta + (1-\mu)\,\frac{1}{2\pi}\int_0^{2\pi} g(t,0)\,dt \;<\; 0, \qquad \text{for} \;\; \mu \in [0,1]$$

and

$$F(M_5,\mu) \;=\; M_5\,\mu/2 + (1-\mu)\,\frac{1}{2\pi}\int_0^{2\pi} g(t,M_5)\,dt \;>\; 0, \qquad \text{for} \;\; \mu \in [0,1] \;.$$

Hence for any $x \in \partial\Omega_1 \cap \operatorname{Ker} L$ and $\mu \in [0,1]$, $F(x,\mu) \neq 0$. Thus $F(x,\mu)$ is a homotopic transformation and

$$\begin{aligned} \deg\Big\{QNx,\, \Omega_1 \cap \operatorname{Ker} L,\, 0\Big\} \;&=\; \deg\Big\{-\frac{1}{2\pi}\int_0^{2\pi} g(t,x)\,dt,\;\Omega_1 \cap \operatorname{Ker} L,\;0\Big\} \\ &=\; \deg\Big\{-x+\delta,\, \Omega_1 \cap \operatorname{Ker} L,\, 0\Big\} \;\neq\; 0 \;. \end{aligned}$$

In view of Mawhin continuation theorem in [6], there exists a periodic positive solution with period 2π of (2). This completes the proof. ∎

By the same way, we can prove Theorem 2.

Proof of Theorem 3: Suppose that $x(t)$ is a periodic positive solution with period 2π of (3), then (4) holds. By (i) and (4) we have

$$\int_0^{2\pi} g(t, x(t-\tau))\,dt \;\leq\; 0 \;. \tag{18}$$

Taking $M_6 = \alpha\,M^\gamma + \beta$, by (i) for $t \in R$ and $0 < x < M$ we find

$$|g(t,x)| \;\leq\; M_6 \;. \tag{19}$$

Define the functions as

$$\bar{G}_+(t) = \max\Big\{g(t,x(t-\tau)),\,0\Big\} \qquad \text{and} \qquad \bar{G}_-(t) = \max\Big\{-g(t,x(t-\tau)),\,0\Big\} \;.$$

Both $\bar{G}_+(t)$ and $\bar{G}_-(t)$ are nonnegative continuously functions and

$$g(t,x(t-\tau)) \;=\; \bar{G}_+(t) - \bar{G}_-(t) \;; \tag{20}$$

$$\Big|g(t,x(t-\tau))\Big| \;=\; \bar{G}_+(t) + \bar{G}_-(t) \;. \tag{21}$$

By (ii), when $g(t,x(t-\tau)) \le 0$, we have $t \in R$ and $0 < x(t-\tau) < M$; and then by (19), $|g(t,x(t-\tau))| \le M_6$. Hence for any $t \in [0,2\pi]$,

$$\bar{G}_-(t) \;=\; |\bar{G}_-(t)| \;\le\; M_6 \;. \tag{22}$$

Using (18), (20) and (22) it follows that

$$\int_0^{2\pi} \bar{G}_+(t)\,dt \;\le\; \int_0^{2\pi} \bar{G}_-(t)\,dt \;\le\; 2\pi M_6 \;. \tag{23}$$

By (21) and (23) we have

$$\int_0^{2\pi} \Big|g(t,x(t-\tau))\Big|\,dt \;\le\; M_7 \;, \tag{24}$$

where $M_7 = 4\pi M_6$. Then from (i), (4) and (24) we find that

$$\begin{aligned}\int_0^{2\pi} \Big|f(t,x'(t-\sigma))\Big|\,dt \;&=\; \int_0^{2\pi} f(t,x'(t-\sigma))\,dt \\ &= -\int_0^{2\pi} g(t,x(t-\tau))\,dt \;\le\; \int_0^{2\pi} \Big|g(t,x(t-\sigma))\Big|\,dt \;\le\; M_7 \;.\end{aligned}$$

It follows that

$$\int_0^{2\pi} \Big|f(t,x'(t-\sigma))\Big|\,dt \;\le\; M_7 \;. \tag{25}$$

Since $x(2\pi) = x(0)$, there exists $t_3 \in [0,2\pi]$ such that $x'(t_3) = 0$. Hence by (3), (24) and (25) for any $t \in [0,2\pi]$ we have

$$\begin{aligned}|x'(t)| \;&=\; \left|\int_{t_3}^t x''(s)\,ds\right| \;\le\; \int_0^{2\pi} |x''(t)|\,dt \\ &\le\; \lambda \int_0^{2\pi} \Big|f(t,x'(t-\sigma))\Big|\,dt \;+\; \lambda \int_0^{2\pi} \Big|g(t,x(t-\tau))\Big|\,dt \;+\; \lambda \int_0^{2\pi} |p(t)|\,dt \\ &\le\; 2M_7 + 2\pi m \;,\end{aligned}$$

where $m = \max\limits_{t\in[0,2\pi]} |p(t)|$. Taking $M_8 = 2M_7 + 2\pi m$, for any $t \in [0,2\pi]$ we see

$$|x'(t)| \;\le\; M_8 \;. \tag{26}$$

By (ii) and (18), there exists $t_4 \in [0,2\pi]$ such that $0 < x(t_4-\tau) < M$. Taking $t_4 - \tau = 2k\pi + t_5$ (k is an integer), $t_5 \in [0,2\pi]$, we have $0 < x(t_5) < M$ and for any $t \in [0,2\pi]$

$$0 \;<\; x(t) \;=\; x(t_5) + \int_{t_5}^t x'(s)\,ds \;\le\; M + \int_0^{2\pi} |x'(t)|\,dt \;\le\; M + 2\pi M_8 \;. \tag{27}$$

Taking $M_9 = M + 2\pi M_8$ and using (26) and (27), for any $t \in [0, 2\pi]$, we have $0 < x(t) \le M_9$ and $|x'(t)| \le M_9$. Let $M_{10} > 0$, $M_{10} > M_9$ and

$$\Omega_2 := \left\{ x \in X \mid \ 0 < x(t) < M_{10}, \ |x'(t)| < M_{10} \right\} .$$

We can prove the remainder parts by the same way of Theorem 1. The proof is complete. ∎

By the same way, we can prove Theorem 4.

ACKNOWLEDGEMENT – The authors are grateful to the referee for his/her valuable suggestions for the improvement of this paper.

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