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SOME CORRESPONDENCES FOR GALOIS SKEW GROUP RINGS

G. Szeto and L. Xue

Abstract: Let S be a ring with 1, G a finite automorphism group of S, and $S * G$ a skew group ring of G over S which is Azumaya. Two correspondence theorems are shown between a class of subgroups of G and a class of the Azumaya algebras contained in $S * G$.

1 – Introduction

The fundamental theorem for Galois extensions of fields was generalized to commutative rings with no idempotents but 0 and 1 ([5], Theorem 1.1, p. 80). For Galois extensions of noncommutative rings, there were some kind of correspondence theorems for special types of Galois extensions between different classes of separable extensions contained in the Galois extension ([2], [4]). Recently, we studied the following two types of Galois extensions:

(**i**) Galois extensions of Azumaya algebras called the Azumaya Galois extensions ([1], [2]), and

(**ii**) Galois skew group rings which are Azumaya ([1], [2], [3], [7]).

Let $S * G$ be an G'-Galois extension of $(S * G)^{G'}$ with an inner group G' induced by the elements in G. If $S * G$ is Azumaya, in the present paper, we shall show two correspondence theorems between a class of subgroups of G and a class of the Azumaya algebras contained in $S * G$. The results are given in section 3. Moreover, let Z be the center of $S * G$, H a normal subgroup of G such that the

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center of ZH is ZI_H where I_H is the center of H. Then in section 4, we shall show that the G'-Galois extension $S * G$ is a composition of the following Galois extensions:

(i) $S * G$ is an I'_H-Galois extension of $(S * G)^{I'_H}$,

(ii) $(S * G)^{I'_H}$ is an Azumaya H'/I'_H-Galois extension of $(S * G)^{H'}$, and

(iii) $(S * G)^{H'}$ is an G'/H'-Galois extension of $(S * G)^{G'}$.

2 – Preliminaries

Throughout, we assume that S is a ring with 1, G $(= \{g_1, g_2, ..., g_n\}$ with g_1 is identity) a finite automorphism group of S of order n for some integer n invertible in S, S^G the subring of the elements fixed under each element in G, and $S * G$ a skew group ring of G over S. We denote the inner automorphism group of $S * G$ induced by the elements in G by G', $H < G$ means that H is a subgroup of G, $V_A(B)$ is the commutator subring of the subring B in a ring A. Following [1], [2], and [5], we call S an G-Galois extension of S^G if there exist elements $\{c_i, d_i$ in S, $i = 1, 2, ..., k$ for some integer $k\}$ such that $\sum c_i\, g_j(d_i) = \delta_{1,j}$ for each $g_j \in G$. Let B be a subring of a ring A with 1. A is called a separable extension of B if there exist $\{a_i, b_i$ in A, $i = 1, 2, ..., m$ for some integer $m\}$ such that $\sum a_i\, b_i = 1$, and $\sum s\, a_i \otimes b_i = \sum a_i \otimes b_i\, s$ for all s in A where $\otimes$ is over B. An Azumaya algebra is a separable extension of its center. A ring A is called an H-separable extension of B if $A \otimes_B A$ is isomorphic to a direct summand of a finite direct sum of A as an A-bimodule. It is known that an Azumaya algebra is an H-separable extension and an H-separable extension is a separable extension. A ring S is call an Azumaya G-Galois extension of S^G if it is an G-Galois extension of S^G which is C^G-Azumaya algebra where C is the center of S ([1], [2]).

3 – Correspondence theorems

Let Z be the center of $S * G$. Assume that $S * G$ is an G'-Galois extension which is an Azumaya Z-algebra. We shall show two correspondence theorems. At first, we define two classes of subgroups of G.

Definition 1. For H, $K < G$, $H \sim K$ if ZH and ZK have the same center, that is, $D_H = D_K$ where D_H and D_K are the center of ZH and ZK respectively. □

We note that $\sim$ is an equivalence relation on the class of subgroups of G, the equivalence class of H is denoted by $[H \sim]$, and $\mathcal{C} = \{[H \sim] \mid H < G\}$.

Definition 2. For $H,\ K < G$ with center I_H and I_K respectively, $H \approx K$ if ZH and ZK are central Galois algebras over the same center with inner Galois groups induced by the elements in H and K respectively. □

We note that $\approx$ is an equivalence relation on a class of subgroups of G, the equivalence class of H is denoted by $[H \approx]$, and $\mathcal{D} = \{[H \approx] \mid \text{for some } H < G\}$.

We begin with the well known commutator theorem for Azumaya algebras ([5], Theorem 4.3) and some consequences of the theorem.

Proposition ([5], Theorem 4.3). *Let A be a central separable R-algebra. Suppose B is any separable subalgebra of A containing R. Set $C = V_A(B)$, then C is a separable subalgebra of A and $V_A(C) = B$. If B is also central, so is C and the R-algebra map $B \otimes C \to A$ by $b \otimes c \to bc$ is an isomorphism.* ■

Lemma 3.1. *Let A be an Azumaya algebra over its center C. If B is a separable subalgebra of A, then*

(**1**) *B and $V_A(B)$ are Azumaya algebras over the same center, and*

(**2**) *let D be the center of B, then $V_A(D)$ is an Azumaya D-algebra such that $V_A(D) = BV_A(B)$.*

Proof: (**1**) By the Proposition, $V_A(V_A(B)) = B$, so the center of $V_A(B) \subset B \cap V_A(B)$. Clearly, $B \cap V_A(B) \subset$ the center of $V_A(B)$. Hence $B \cap V_A(B) =$ the center of $V_A(B)$. Similarly, $B \cap V_A(B) =$ the center of B. Thus B and $V_A(B)$ are Azumaya algebras over the same center.

(**2**) By the Proposition again, $V_A(D)$ is an Azumaya D-algebra such that $V_A(D) = B \otimes_D V_E(B)$ where $E = V_A(D)$. But $V_E(B) = E \cap V_A(B) = V_A(B)$, hence $V_A(D) \cong B \otimes_D V_A(B) \cong BV_A(B)$. ■

Lemma 3.2. *Let $H < G$ and D_H the center of ZH. Then*

(**1**) *$V_{S*G}(D_H) = (S*G)^{H'}(ZH)$ as an Azumaya D_H-algebra, and*

(**2**) *$V_{S*G}(D_H)$ is the maximum Azumaya D_H-algebra contained in $S*G$.*

Proof: Since n is a unit in S, the order of H is a unit in Z. Hence ZH is a separable subalgebra of $S*G$; and so $V_{S*G}(ZH)\,(= (S*G)^{H'})$ is a separable subalgebra of $S*G$ such that we have $V_{S*G}((S*G)^{H'}) = ZH$ by the commutator theorem for the Azumaya algebra $S*G$ ([5], Theorem 4.3). Thus $(S*G)^{H'}$ and ZH have the same center D_H by Lemma 3.1. Since D_H is a commutative separable subalgebra of $S*G$, $V_{S*G}(D_H)$ is an Azumaya D_H-algebra by Lemma 3.1 again. Moreover, since $(S*G)^{H'}$ and ZH are contained in $V_{S*G}(D_H)$ as Azumaya D_H-subalgebras, denoting $V_{S*G}(D_H)$ by A_H, we have that $V_{A_H}((S*G)^{H'}) = A_H \cap V_{S*G}((S*G)^{H'}) = A_H \cap (ZH) = ZH$. But then, $A_H = (S*G)^{H'} \otimes_{D_H} ZH \cong (S*G)^{H'} ZH$ (Lemma 3.1). This proves (1).

Again, since D_H is a commutative separable subalgebra of $S*G$, $V_{S*G}(D_H)$ is a separable subalgebra of $S*G$. But $V_{S*G}(V_{S*G}(D_H)) = D_H$, so $V_{S*G}(D_H)$ is a maximum Azumaya D_H-algebra contained in $S*G$. This completes the proof of (2). ∎

We denote the class of the Azumaya algebras contained in $S*G$ by $\mathcal{A}$. Then we show the first correspondence theorem from $\mathcal{C}$ to $\mathcal{A}$.

Theorem 3.3. *Let $f\colon \mathcal{C} \to \mathcal{A}$ by $f([H \sim]) = (S*G)^{H'}(ZH)$ for $[H \sim] \in \mathcal{C}$. Then f is an injection.*

Proof: Let $H \sim K$ be subgroups of G. Then H and K have the same center $D_H = D_K$. By Lemma 3.2, $V_{S*G}(D_H) = (S*G)^{H'}(ZH) = V_{S*G}(D_K) = (S*G)^{K'}(ZK)$, so f is well defined. Next, suppose that $f([H \sim]) = f([L \sim])$ for some subgroups H and L. Then $(S*G)^{H'}(ZH) = (S*G)^{L'}(ZL)$ as Azumaya algebras. Hence ZH and ZL have the same center (for $(S*G)^{H'}$, ZH, $(S*G)^{L'}$, and ZL have the same center by the commutator theorem for Azumaya algebras ([5], Theorem 4.3)). Thus, $[H \sim] = [L \sim]$; and so f is an injection. ∎

Next, we show that the above $f([H \sim])$ becomes an Azumaya Galois extension when D_H is ZI_H generated by Z and the center I_H of H. Consequently, we obtain the second correspondence theorem between $\mathcal{D}$ and the class of the Azumaya H'/I'_H-Galois extensions contained in $S*G$ as studied in [1] and [2]. We begin with several lemmas.

Lemma 3.4. *Let H be a subgroup of G. Then, the following statements are equivalent.*

(1) *ZH is a central Galois algebra with Galois group H'/I'_H, where I_H is the center of H,*

(2) *$(S*G)^{I'_H} = (S*G)^{H'}H$ as an Azumaya ZI_H-algebra, and*

(3) *The center of ZH is ZI_H.*

Proof: We first claim that $\{g \in H \mid g'(d) = d \text{ for all } d \in ZH\} = I_H$. In fact, $g'(d) = d$ for all $d \in ZH$ implies that $g'(g_i) = g_i$ for all $g_i \in H$. Hence $g \in I_H$. The converse is clear.

(1) $\longrightarrow$ **(2)** By Lemma 3.2, $(S*G)^{H'}$ and ZH have the same center. But $(S*G)^{I'_H}$ is an Azumaya ZI_H-algebra, hence $(S*G)^{H'}ZH$ is an Azumaya algebra. Clearly $Z \subset (S*G)^{H'}$, so $(S*G)^{H'}ZH = (S*G)^{H'}H$. Since $ZH\,(= V_{S*G}(V_{S*G}(ZH)) = V_{S*G}((S*G)^{H'}))$ is a central Galois H'/I'_H-algebra, $(S*G)^{I'_H} = (S*G)^{H'}ZH$ ([6], Theorem 6-(3)).

(2) $\longrightarrow$ **(1)** $(S*G)^{I'_H} = (S*G)^{H'}H = (S*G)^{H'}ZH = (S*G)^{H'}V_{S*G}((S*G)^{H'})$, so ZH is a central H'/I'_H-Galois algebra ([6], Theorem 6-(3)).

(1) $\longrightarrow$ **(3)** is given in the proof of (1) $\longrightarrow$ (2).

(3) $\longrightarrow$ **(1)** Since the center of ZH is ZI_H, the center of $V_{S*G}(ZH)$ $(= (S*G)^{H'})$ is ZI_H. Clearly, ZH and $(S*G)^{H'} \subset V_{S*G}(ZI_H)$. Denote $V_{S*G}(ZI_H)\,(= (S*G)^{I'_H})$ by A. Then A is an Azumaya ZI_H-algebra and $ZH = V_{S*G}(V_{S*G}(ZH)) = V_A(V_{S*G}(ZH))$. Since A is an Azumaya ZI_H-algebra, $A = (S*G)^{I'_H} = (S*G)^{H'}ZH$ as Azumaya ZI_H-algebras (Lemma 3.1). Thus ZH is a central H'/I'_H-Galois algebra ([6], Theorem 6-(3)). ∎

Corollary 3.5. *If ZH is a central Galois algebra with Galois group H'/I'_H, then $(S*G)^{H'}H$ is an Azumaya Galois extension with Galois group H'/I'_H.*

Proof: Since $(S*G)^{H'}H = (S*G)^{H'}ZH \cong (S*G)^{H'} \otimes ZH$ as an Azumaya ZI_H-algebra, where $\otimes$ is over ZI_H such that ZH is a central H'/I'_H-Galois algebra over ZI_H by Lemma 3.1, $(S*G)^{H'}H$ is an H'/I'_H-Galois extension of $(S*G)^{H'}$ which is an Azumaya ZI_H-algebra, that is, $(S*G)^{H'}H$ is an Azumaya Galois extension of $(S*G)^{H'}$. ∎

When two subgroups H and K of G such that ZH and ZK are central Galois algebras as given in Lemma 3.4, we shall show a sufficient and necessary condition under which $(S*G)^{H'}H = (S*G)^{K'}K$.

Lemma 3.6. *Let H and K be subgroups of G such that ZH and ZK are central Galois algebras as given in Lemma 3.4. Then $(S*G)^{H'}H = (S*G)^{K'}K$ if and only if $ZI_H = ZI_K$.*

Proof: By Lemma 3.4, $(S*G)^{H'}H = (S*G)^{I'_H}$ as ZI_H-algebras and $(S*G)^{K'}K = (S*G)^{I'_K}$ as ZI_K-algebras, so that $(S*G)^{H'}H = (S*G)^{K'}K$ implies that $ZI_H = ZI_K$. Conversely, $ZI_H = ZI_K$ implies that $V_{S*G}(ZI_H) = V_{S*G}(ZI_K)$, so $(S*G)^{I'_H} = (S*G)^{I'_K}$; and so $(S*G)^{H'}H = (S*G)^{K'}K$. ∎

Next we show that any Azumaya H'/I'_H-Galois extension A of A^{H'/I'_H} with center ZI_H is of the form $(S*G)^{I'_H}$.

Lemma 3.7. *If A is an Azumaya H'/I'_H-Galois extension of $(S*G)^{H'}$ with center ZI_H for some subgroup H of G, then $A = (S*G)^{H'}(ZH) = (S*G)^{I'_H}$.*

Proof: Since $V_{S*G}((S*G)^{H'}) = ZH$ which is a separable subalgebra of $S*G$, the center of ZH is ZI_H by Lemma 3.1 and $A \subset V_{S*G}(ZI_H) = (S*G)^{I'_H}$. Hence $(S*G)^{I'_H} = (S*G)^{H'}(ZH) = (S*G)^{H'}H$ as an Azumaya algebra by Lemma 3.4. Thus ZH is a central Galois H'/I'_H-algebra ([6], Theorem 6-(3)); and so $(S*G)^{I'_H}$ is an Azumaya H'/I'_H-Galois extension of $(S*G)^{H'}$ by Corollary 3.5. Noting that A is an H'/I'_H-Galois extension of $(S*G)^{H'}$ by hypothesis and that $A \subset (S*G)^{I'_H}$, we conclude that $A = (S*G)^{I'_H}$. ∎

We denote the class of Azumaya H'/I'_H-Galois extension of $(S*G)^{H'}$ with center ZI_H for some $H < G$ in $S*G$ by $\mathcal{B}$.

Theorem 3.8. *There exists a one-to-one correspondence between $\mathcal{D}$ and $\mathcal{B}$.*

Proof: Let f: $[H \approx] \to (S*G)^{H'}(ZH)$ be a map from $\mathcal{D}$ to $\mathcal{B}$. Then f is well defined by Theorem 3.3 and Lemma 3.6. Also, by Lemma 3.7, for any $A \in \mathcal{B}$, $A = (S*G)^{H'}(ZH) = (S*G)^{I'_H}$. Moreover, since $ZH = V_{S*G}((S*G)^{H'})$, ZH is a central Galois algebra with Galois group H'/I'_H ([6], Theorem 6-(3)) with center ZI_H. Thus $f[H \approx]) = A$. This implies that f is an surjection. By theorem 3.3, f is also an injection. ∎

4 – Structure of $S*G$

In this section, assume that $S*G$ is an G'-Galois extension and an Azumaya Z-algebra. Let H be a normal subgroup of G such that $[H \approx] \in \mathcal{D}$. We shall show that $S*G$ is a composition of three Galois extensions, and give expressions of $(S*G)^{G'}$ and Z when G is Abelian.

Theorem 4.1. *If H is a normal subgroup of G such that the center of ZH is ZI_H where I_H is the center of H, then*

(1) *H' and I'_H are normal group of G', and*

(2) *$S*G$ is a composition of the following Galois extensions:*

- **(i)** *$S*G$ is an I'_H-Galois extension of $(S*G)^{I'_H}$,*
- **(ii)** *$(S*G)^{I'_H}$ is an Azumaya H'/I'_H-Galois extension of $(S*G)^{H'}$, and*
- **(iii)** *$(S*G)^{H'}$ is an G'/H'-Galois extension of $(S*G)^{G'}$.*

Proof: (**1**) Since $(g\,h\,g^{-1})' = g'\,h'(g^{-1})'$ for any $g \in G$, and $h \in H$, H' is normal in G' whenever H is normal in G. Moreover, since H is normal in G, g' is an automorphism of H for each $g \in G$. Therefore $g'(I_H) = I_H$ since I_H is the center of H. Thus I_H is normal in G, and so I'_H is normal in G'.

(**2**) By hypothesis, $S * G$ is an G'-Galois extension of $(S * G)^{G'}$, so (i) $S * G$ is an I'_H-Galois extension of $(S * G)^{I'_H}$. For (ii) since ZI_H is the center of ZH, $(S * G)^{I'_H} = V_{S*G}(ZI_H)$ is an Azumaya ZI_H-algebra such that $(S * G)^{H'}$ and ZH are contained in $(S * G)^{I'_H}$. Hence by the argument given in the proof of Lemma 3.7, $(S * G)^{I'_H} = (S * G)^{H'}(ZH))$. Noting that $ZH = V_{S*G}((S * G)^{H'}) = V_{S*G}(V_{S*G}(ZH))$ ([5], Theorem 4.3), we conclude that ZH is an H'/I'_H-Galois extension of ZI_H. Thus $(S * G)^{I'_H}$ is an H'/I'_H-Galois extension of $(S * G)^{H'}$. By (1), H' is a normal subgroup of G', so $(S * G)^{H'}$ is an G'/H'-Galois extension of $(S * G)^{G'}$. This proves (iii). ∎

Next, we give expressions of $(S * G)^{G'}$ and Z when G is Abelian.

Theorem 4.2. *If G is Abelian, then $(S * G)^{G'} = S^G G$, a group ring of G over S^G and $Z = \sum_{i=1}^{n}(D \cap J_i)\, g_i$ where D is the center of S^G and $J_i = \{s \in S \mid ts = s\, g_i(t)$ for all $t \in S\}$.*

Proof: $(S * G)^{G'} = S^G G$ is clear. Now let $x \in Z$, $x = \sum_{i=1}^{n} s_i\, g_i$, then for each $g_i \in G$, $g_i\, x = x\, g_i$, so $s_i \in S^G$. Also for each $t \in S$, $tx = xt$ implies that $s_i \in J_i$. In particular, for each $t \in S^G$, $s_i \in D$ (= the center of S^G). Thus $s_i \in D \cap J_i$ for each i. Conversely, $\sum_{i=1}^{n} s_i\, g_i$, where $s_i \in D \cap J_i$ for each i, is clearly in Z. ∎

We conclude the paper with an G'-Galois skew group ring $S * G$ which is Azumaya such that G is Abelian as constructed at the end of [7].

Let S be the quaternion algebra $Q[i, j, k]$ over the rational field Q and G $(= \{g_1, g_i, g_j, g_k \mid\ g_1 =$ the identity, $g_i =$ the inner automorphism group of S induced by i, $g_j =$ the inner automorphism group of S induced by j, $g_k =$ the inner automorphism group of S induced by $k\})$. Then

(**1**) $S * G$ is an G'-Galois because S is G-Galois with a Galois system $\{2^{-1},\, 2^{-1}i,\, 2^{-1}j,\, 2^{-1}k;\ 2^{-1},\, -2^{-1}i,\, -2^{-1}j,\, -2^{-1}k\}$,

(**2**) $S * G$ is Azumaya because $S * G$ is separable over S and S is Azumaya over Q,

(**3**) G is an Abelian, so G' is an Abelian,

(**4**) $(S * G)^{G'} = S^G G$, where $S^G = Q$,

(**5**) $J_1 = Q1$, $J_i = Qi$, $J_j = Qj$ and $J_k = Qk$.

(**6**) $Z = Q$ by Theorem 4.2.

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George Szeto and Lianyong Xue,
Department of Mathematics, Bradley University,
Peoria, Illinois 61625 – U.S.A.