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EXISTENCE FOR ELLIPTIC EQUATIONS IN L^1 HAVING LOWER ORDER TERMS WITH NATURAL GROWTH

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Abstract: We deal with the following type of nonlinear elliptic equations in a bounded subset $\Omega \subset \mathbb{R}^N$:

$$\text{(P)} \qquad \begin{cases} -\mathrm{div}\Big(a(x,u,\nabla u)\Big) + g(x,u,\nabla u) = \chi & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \,, \end{cases}$$

where both $a(x,s,\xi)$ and $g(x,s,\xi)$ are Carathéodory functions such that $a(x,s,\cdot)$ is coercive, monotone and has a linear growth, while $g(x,s,\xi)$ has a quadratic growth with respect to ξ and satisfies a sign condition on s, that is $g(x,s,\xi)\,s \geq 0$ for every s in $\mathbb{R}$. The datum χ is assumed in $L^1(\Omega) + H^{-1}(\Omega)$. We prove the existence of a weak solution u of (P) which belongs to the Sobolev space $W_0^{1,q}(\Omega)$ for every $q < \frac{N}{N-1}$, by adapting to the framework of L^1 data a technique used in [6], which simply relies on Fatou lemma combined with the sign assumption on g.

1 – Introduction and statement of the result

An extensive literature has dealt with the Dirichlet problem in a bounded subset $\Omega \subset \mathbb{R}^N$, $N \geq 2$,

$$\text{(1.1)} \qquad \begin{cases} A(u) + g(x,u,\nabla u) = \chi & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \,, \end{cases}$$

where A is a pseudomonotone operator in $H_0^1(\Omega)$ of the type introduced by J. Leray and J.L. Lions (see [9]) and $g(x,s,\xi)$ is a Carathéodory function having at most quadratic growth with respect to the gradient:

$$(g_1) \qquad |g(x,s,\xi)| \leq b(|s|)\,\Big(h(x) + |\xi|^2\Big)\,, \quad \forall\, s \in \mathbb{R}, \;\; \forall\, \xi \in \mathbb{R}^N, \;\; \text{a.e. } x \in \Omega\,,$$

with $h(x)$ in $L^1(\Omega)$ and $b : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing continuous function.

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Starting with the paper [5], where χ is taken in $L^\infty(\Omega)$, existence results for problem (1.1) have been proved under a sign assumption on g:

$$(g_2) \qquad g(x,s,\xi)\, s \geq 0\,, \quad \forall\, s \in \mathbb{R}, \quad \forall\, \xi \in \mathbb{R}^N, \quad \text{a.e. } x \in \Omega\,,$$

and in [6] it is found a solution of (1.1) if χ only belongs to $H^{-1}(\Omega)$.

Here we consider the case in which

$$\chi \in L^1(\Omega) + H^{-1}(\Omega)\,.$$

In this setting a solution can not in general be expected to belong to $H_0^1(\Omega)$, and this is the main difficulty when trying to extend the previous results. Nevertheless, a solution of (1.1) belonging to $H_0^1(\Omega)$ has been obtained in [3] and in [4] if it is assumed in addition that $g(x,s,\xi)\,\mathrm{sign}(s) \geq \gamma|\xi|^2$ for every $|s| \geq L$, where L, $\gamma > 0$ (hence, for example all functions going to zero at infinity are not included). A more general result has been finally proved in [10] under the only assumptions (g_1) and (g_2); by approximating (1.1) with more regular problems a distributional solution is obtained in the Sobolev space $W_0^{1,q}(\Omega)$ for every $q < \frac{N}{N-1}$. This latter result, which applies to the extended framework in which χ is a positive Radon measure, however essentially relies on the proof that the truncations of the approximating solutions are compact in the strong topology of $H_0^1(\Omega)$, which is a fundamental result in its own but rather technical in its proof, indeed in the paper quoted above an assumption of positiveness on the datum is made for simplicity.

The aim of this note is to provide a simpler proof of the existence of a solution of (1.1) when χ belongs to $L^1(\Omega) + H^{-1}(\Omega)$, by applying the same method used in [6] for variational data, and recently adapted in [11] for unilateral problems in L^1, which only relies on a tricky use of Fatou lemma combined with the sign condition (g_2). In this sense we point out that the existence of a solution of (1.1) with L^1 data can be obtained without proving the strong convergence in $H_0^1(\Omega)$ of the truncations of the approximating solutions and this technique also allows to handle more easily the case of changing sign data and solutions.

We assume that Ω is a bounded open subset of $\mathbb{R}^N$, $N \geq 2$, and we set

$$A(u) \equiv -\mathrm{div}\Big(a(x,u,\nabla u)\Big)\,,$$

where $a(x,s,\xi)$ is a Carathéodory function such that, for all s in $\mathbb{R}$, all ξ, η in $\mathbb{R}^N$ and almost every x in Ω, it satisfies:

$$(a_1) \qquad a(x,s,\xi)\cdot\xi \geq \alpha\,|\xi|^2\,, \quad \alpha > 0\,,$$

$$(a_2) \qquad |a(x,s,\xi)| \leq \beta\Big(d(x) + |s| + |\xi|\Big)\,, \quad \beta > 0\,,$$

$$(a_3) \qquad \Big[a(x,s,\xi) - a(x,s,\eta)\Big]\cdot[\xi - \eta] > 0\,, \quad \forall\, \xi \neq \eta\,,$$

with $d(x) \in L^2(\Omega)$. We will prove the following theorem.

Theorem 1.1. *Let assumptions (a_1)–(a_3) hold true and let $g(x,s,\xi)$ satisfy (g_1)–(g_2). Then for every χ in $L^1(\Omega)+H^{-1}(\Omega)$ there exists a function u in $W_0^{1,q}(\Omega)$ for every $q<\frac{N}{N-1}$ which is a solution of (1.1) in the sense of distributions.*

We finally remark that the problem of existence of a solution of (1.1) with g having a quadratic or a subquadratic growth with respect to ξ has also been investigated in [7], [8].

2 – Proof of the result

Before giving the proof of our result, let us recall the definition of truncation, that is, for every $k>0$, $T_k(s)=\min\{k,\max\{u,-k\}\}$; moreover we want to point out that the technique we adopt, based on the use of Fatou lemma, was first introduced in [1], then used in [6] and in [11].

Proof of Theorem 1.1: First of all we write $\chi=f-\operatorname{div}(F)$, with f in $L^1(\Omega)$ and F in $L^2(\Omega)^N$, and we take two sequences $\{f_n\}\subset L^\infty(\Omega)$ and $\{F_n\}\subset L^\infty(\Omega)^N$ such that

$$\begin{aligned} &f_n\to f \quad \text{strongly in } L^1(\Omega)\,,\\ &F_n\to F \quad \text{strongly in } L^2(\Omega)^N\,. \end{aligned} \tag{2.1}$$

In [6] it is proved that there exists u_n in $H_0^1(\Omega)\cap L^\infty(\Omega)$ solution of

$$\begin{cases} -\operatorname{div}\Big(a(x,u_n,\nabla u_n)\Big)+g(x,u_n,\nabla u_n)=f_n-\operatorname{div}(F_n) & \text{in } \Omega\,,\\ u_n=0 & \text{on } \partial\Omega\,. \end{cases} \tag{2.2}$$

If we take $T_k(u_n)$ as test function in (2.2) we obtain, applying Young's inequality,

$$\int_\Omega a(x,u_n,\nabla u_n)\,\nabla T_k(u_n)\,dx+\int_\Omega g(x,u_n,\nabla u_n)\,T_k(u_n)\,dx\ \le$$
$$\le\int_\Omega f_n\,T_k(u_n)\,dx+\frac{\alpha}{2}\int_\Omega|\nabla T_k(u_n)|^2\,dx+c_0\int_\Omega|F_n|^2\,dx\ ,$$

where c_0 (like all the following c_i's) denotes a positive constant not depending on n and k. Using assumption (a_1) and the sign condition on g, we get:

$$\frac{\alpha}{2}\int_\Omega|\nabla T_k(u_n)|^2\,dx\,+\,k\int_{\{|u_n|\ge k\}}|g(x,u_n,\nabla u_n)|\,dx\ \le \tag{2.3}$$
$$\le\int_\Omega f_n\,T_k(u_n)\,dx\,+\,c_0\int_\Omega|F_n|^2\,dx\ .$$

First of all (2.3) implies that for every fixed $k > 0$ the sequence $\{T_k(u_n)\}$ is bounded in $H_0^1(\Omega)$ (though not uniformly in k), and for $k = 1$ we have

$$\int_{\{|u_n|\geq 1\}} |g(x,u_n,\nabla u_n)|\,dx \;\leq\; c_1 \;,$$

which yields

$$\begin{aligned}\int_\Omega |g(x,u_n,\nabla u_n)|\,dx \;&\leq\; b(1)\int_\Omega \Big(h(x)+|\nabla T_1(u_n)|^2\Big)\,dx \;+\; \int_{\{|u_n|\geq 1\}} |g(x,u_n,\nabla u_n)|\,dx \\ &\leq\; c_2 \;.\end{aligned}$$

Since $g(x,u_n,\nabla u_n)$ is bounded in $L^1(\Omega)$, we can apply all the compactness results for equations with $L^1(\Omega)+H^{-1}(\Omega)$ data (see [12], [2], [4] and the references cited therein), that is there exist a function u in $W_0^{1,q}(\Omega)$ for every $q < \frac{N}{N-1}$ and a subsequence of u_n, not relabeled, such that

$$\begin{aligned} u_n \to u \qquad & \text{strongly in } \; W_0^{1,q}(\Omega) \;\; \text{for every } q < \tfrac{N}{N-1} \;, \\ \nabla u_n \to \nabla u \qquad & \text{a.e. in } \; \Omega \;, \\ T_k(u_n) \to T_k(u) \qquad & \text{weakly in } \; H_0^1(\Omega) \;\; \text{for every } k > 0 \;. \end{aligned}$$

As a consequence of Fatou lemma, we also have that $g(x,u,\nabla u)$ is in $L^1(\Omega)$; moreover from (2.3) we get, for every $M > 0$,

$$\frac{\alpha}{2}\int_\Omega |\nabla T_k(u_n)|^2\,dx \;\leq\; k\int_{\{|u_n|>M\}} |f_n|\,dx \;+\; M\int_{\{|u_n|\leq M\}} |f_n|\,dx \;+\; c_0\int_\Omega |F_n|^2\,dx \;,$$

hence we deduce:

$$\frac{\alpha}{2}\int_\Omega \frac{|\nabla T_k(u_n)|^2}{k}\,dx \;\leq\; \int_{\{|u_n|>M\}} |f_n|\,dx \;+\; c_3\,\frac{M+1}{k} \;.$$

If we let first k tend to infinity, then M go to infinity, we conclude, thanks to the equi–integrability of the f_n's,

$$\lim_{k\to+\infty}\int_\Omega \frac{|\nabla T_k(u_n)|^2}{k}\,dx \;=\; 0 \qquad \text{uniformly on } \; n \;. \tag{2.4}$$

This is the basic estimate we will use afterwards: now we define

$$B(s) \;\equiv\; \int_0^s b(|t|)\,dt\,, \qquad \forall\, s\in\mathbb{R} \;,$$

and we take a function $H \in C^1(\mathbb{R})$ such that

$$\begin{aligned} H(s) &\equiv 0 \quad \text{if } |s| \geq 1\,, \\ H(s) &\equiv 1 \quad \text{if } |s| \leq \frac{1}{2}\,, \end{aligned} \qquad 0 \leq H(s) \leq 1, \quad \forall\, s \in \mathbb{R}\,.$$

Next we take, as in [6], $v = \psi\, e^{-\frac{B(u_n^-)}{\alpha}} H(\frac{u_n}{k})$ as test function in (2.2) with ψ in $H_0^1(\Omega) \cap L^\infty(\Omega)$, $\psi \geq 0$. It is essential to note that, by the properties of H, v is identically zero on the subset $\{x \in \Omega : |u_n| \geq k\}$; then we have:

$$\begin{aligned} &\int_\Omega a(x,u_n,\nabla u_n)\,\nabla\psi\, e^{-\frac{B(u_n^-)}{\alpha}}\, H\Big(\frac{u_n}{k}\Big)\,dx \;+ \\ &\quad + \frac{1}{\alpha} \int_{\{u_n \leq 0\}} a(x,u_n,\nabla u_n)\,\nabla T_k(u_n)\, b(u_n^-)\, e^{-\frac{B(u_n^-)}{\alpha}} \psi\, H\Big(\frac{u_n}{k}\Big)\,dx \;+ \\ &\qquad\qquad + \int_\Omega g(x,u_n,\nabla u_n)\, e^{-\frac{B(u_n^-)}{\alpha}}\, \psi\, H\Big(\frac{u_n}{k}\Big)\,dx \;= \\ &= \int_\Omega f_n\, e^{-\frac{B(u_n^-)}{\alpha}}\, \psi\, H\Big(\frac{u_n}{k}\Big)\,dx \;- \\ &\quad - \frac{1}{k}\int_\Omega a(x,u_n,\nabla u_n)\,\nabla T_k(u_n)\, H'\Big(\frac{u_n}{k}\Big)\, e^{-\frac{B(u_n^-)}{\alpha}}\, \psi\,dx \;+ \\ &\qquad\qquad + \int_\Omega F_n\, \nabla\Big[e^{-\frac{B(u_n^-)}{\alpha}}\, \psi\, H\Big(\frac{u_n}{k}\Big)\Big]\,dx\,. \end{aligned}$$

Using assumption (a_2) we obtain:

$$\begin{aligned} &\int_\Omega a(x,u_n,\nabla u_n)\,\nabla\psi\, e^{-\frac{B(u_n^-)}{\alpha}}\, H\Big(\frac{u_n}{k}\Big)\,dx \;+ \\ &\quad + \frac{1}{\alpha} \int_{\{u_n \leq 0\}} a(x,u_n,\nabla u_n)\,\nabla T_k(u_n)\, b(u_n^-)\, e^{-\frac{B(u_n^-)}{\alpha}}\, \psi\, H\Big(\frac{u_n}{k}\Big)\,dx \;+ \\ &\qquad\qquad + \int_\Omega g(x,u_n,\nabla u_n)\, e^{-\frac{B(u_n^-)}{\alpha}}\, \psi\, H\Big(\frac{u_n}{k}\Big)\,dx \;= \\ &= \int_\Omega f_n\, e^{-\frac{B(u_n^-)}{\alpha}}\, \psi\, H\Big(\frac{u_n}{k}\Big)\,dx \;+ \\ &\quad + \int_\Omega F_n\, \nabla\Big[e^{-\frac{B(u_n^-)}{\alpha}}\, \psi\, H\Big(\frac{u_n}{k}\Big)\Big]\,dx \;+ \\ &\qquad + c_4\, \|\psi\|_{L^\infty(\Omega)}\, \frac{1}{k} \int_\Omega \Big[d(x)^2 + |T_k(u_n)|^2 + |\nabla T_k(u_n)|^2\Big]\,dx\,. \end{aligned} \tag{2.5}$$

Setting

$$\delta_k \equiv \sup_n \left(\frac{1}{k} \int_\Omega \Big[d(x)^2 + |T_k(u_n)|^2 + |\nabla T_k(u_n)|^2 \Big] \right) dx \ ,$$

we have by (2.4) that δ_k goes to zero as k tends to infinity: then from (2.5) we get

$$\begin{aligned} &\int_\Omega a(x,u_n,\nabla u_n)\,\nabla\psi\, e^{-\frac{B(u_n^-)}{\alpha}}\, H\Big(\frac{u_n}{k}\Big)\,dx \ - \\ &\quad - \frac{1}{\alpha}\int_\Omega a(x,u_n,\nabla u_n)\,\nabla T_k(u_n^-)\, b(u_n^-)\, e^{-\frac{B(u_n^-)}{\alpha}}\,\psi\, H\Big(\frac{u_n}{k}\Big)\,dx \ + \\ &\qquad\qquad + \int_\Omega g(x,u_n,\nabla u_n)\, e^{-\frac{B(u_n^-)}{\alpha}}\,\psi\, H\Big(\frac{u_n}{k}\Big)\,dx \ = \\ &\quad = \int_\Omega f_n\, e^{-\frac{B(u_n^-)}{\alpha}}\,\psi\, H\Big(\frac{u_n}{k}\Big)\,dx \ + \\ &\qquad\qquad + \int_\Omega F_n\,\nabla\Big[e^{-\frac{B(u_n^-)}{\alpha}}\,\psi\, H\Big(\frac{u_n}{k}\Big)\Big]\,dx \ + \ c_4\,\|\psi\|_{L^\infty(\Omega)}\,\delta_k \ . \end{aligned} \tag{2.6}$$

In order to pass to the limit as n tends to infinity, first of all we observe that by definition of $H(s)$ we have

$$\begin{aligned} &\int_\Omega a(x,u_n,\nabla u_n)\,\nabla\psi\, e^{-\frac{B(u_n^-)}{\alpha}}\, H\Big(\frac{u_n}{k}\Big)\,dx \ = \\ &\qquad = \int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\,\nabla\psi\, e^{-\frac{B(u_n^-)}{\alpha}}\, H\Big(\frac{u_n}{k}\Big)\,dx \ . \end{aligned}$$

Since $\nabla T_k(u_n)$ almost everywhere converges to $\nabla T_k(u)$ then $a(x,T_k(u_n),\nabla T_k(u_n))$ weakly converges to $a(x,T_k(u),\nabla T_k(u))$ in $L^2(\Omega)^N$, while $\nabla\psi\, e^{-\frac{B(u_n^-)}{\alpha}}\, H(\frac{u_n}{k})$ strongly converges in $L^2(\Omega)^N$, hence we deduce that

$$\begin{aligned} &\lim_{n\to+\infty}\int_\Omega a(x,u_n,\nabla u_n)\,\nabla\psi\, e^{-\frac{B(u_n^-)}{\alpha}}\, H\Big(\frac{u_n}{k}\Big)\,dx \ = \\ &\qquad = \int_\Omega a(x,u,\nabla u)\,\nabla\psi\, e^{-\frac{B(u^-)}{\alpha}}\, H\Big(\frac{u}{k}\Big)\,dx \ . \end{aligned} \tag{2.7}$$

Moreover using that $T_k(u_n)$ converges to $T_k(u)$ almost everywhere in Ω and weakly in $H_0^1(\Omega)$, which implies that $\nabla[e^{-\frac{B(u_n^-)}{\alpha}}\,\psi\, H(\frac{u_n}{k})]$ weakly converges to

$\nabla[e^{-\frac{B(u^-)}{\alpha}}\,\psi\,H(\frac{u}{k})]$ in $L^2(\Omega)^N$, we obtain, by (2.1),

$$
\begin{aligned}
\lim_{n\to+\infty}\int_\Omega f_n\, e^{-\frac{B(u_n^-)}{\alpha}}\,\psi\,H\Big(\frac{u_n}{k}\Big)\,dx + \int_\Omega F_n\,\nabla\Big[e^{-\frac{B(u_n^-)}{\alpha}}\,\psi\,H\Big(\frac{u_n}{k}\Big)\Big]\,dx &= \\
= \int_\Omega f\, e^{-\frac{B(u^-)}{\alpha}}\,\psi\,H\Big(\frac{u}{k}\Big)\,dx + \int_\Omega F\,\nabla\Big[e^{-\frac{B(u^-)}{\alpha}}\,\psi\,H\Big(\frac{u}{k}\Big)\Big]\,dx\,. &
\end{aligned}
\tag{2.8}
$$

It remains to deal with the second and third integrals in (2.6); but note that the sequence $\{e^{-\frac{B(u_n^-)}{\alpha}}\,\psi\,H(\frac{u_n}{k})\,[-\frac{1}{\alpha}\,a(x,u_n,\nabla u_n)\,\nabla T_k(u_n^-)\,b(u_n^-) \;+\; g(x,u_n,\nabla u_n)]\}$ converges almost everywhere in Ω and thanks to (g_1), (g_2) and (a_1) it satisfies

$$
\begin{aligned}
e^{-\frac{B(u_n^-)}{\alpha}}\,\psi\,H\Big(\frac{u_n}{k}\Big)\Big[-\frac{1}{\alpha}\,a(x,u_n,\nabla u_n)\,\nabla T_k(u_n^-)\,b(u_n^-) + g(x,u_n,\nabla u_n)\Big] &\geq \\
\geq e^{-\frac{B(u_n^-)}{\alpha}}\,\psi\,H\Big(\frac{u_n}{k}\Big)\Big[g(x,u_n,\nabla u_n)\,\chi_{\{u_n\geq 0\}} & \\
+ \Big(|\nabla T_k(u_n)|^2\,b(|u_n|) - |g(x,u_n,\nabla u_n)|\Big)\,\chi_{\{u_n\leq 0\}}\Big] & \\
\geq -C_k\,h(x) \in L^1(\Omega)\,, &
\end{aligned}
$$

where C_k is a positive constant depending on k. Therefore we can apply Fatou lemma and conclude that

$$
\begin{aligned}
\liminf_{n\to+\infty}\int_\Omega e^{-\frac{B(u_n^-)}{\alpha}}\,\psi\,H\Big(\frac{u_n}{k}\Big)\Big[-\frac{1}{\alpha}\,a(x,u_n,\nabla u_n)\,\nabla T_k(u_n^-)\,b(u_n^-)+g(x,u_n,\nabla u_n)\Big]dx &\geq \\
\geq \int_\Omega e^{-\frac{B(u^-)}{\alpha}}\,\psi\,H\Big(\frac{u}{k}\Big)\Big[-\frac{1}{\alpha}\,a(x,u,\nabla u)\,\nabla T_k(u^-)\,b(u^-) + g(x,u,\nabla u)\Big]\,dx\,. &
\end{aligned}
\tag{2.9}
$$

By means of (2.7), (2.8) and (2.9) we obtain passing to the limit on n in (2.6):

$$
\begin{aligned}
&\int_\Omega a(x,u,\nabla u)\,\nabla\psi\, e^{-\frac{B(u^-)}{\alpha}}\,H\Big(\frac{u}{k}\Big)\,dx \;- \\
&\quad - \frac{1}{\alpha}\int_\Omega a(x,u,\nabla u)\,\nabla T_k(u^-)\,b(u^-)\,e^{-\frac{B(u^-)}{\alpha}}\,\psi\,H\Big(\frac{u}{k}\Big)\,dx \;+ \\
&\qquad + \int_\Omega g(x,u,\nabla u)\,e^{-\frac{B(u^-)}{\alpha}}\,\psi\,H\Big(\frac{u}{k}\Big)\,dx \;\leq \\
&\leq \int_\Omega f\, e^{-\frac{B(u^-)}{\alpha}}\,\psi\,H\Big(\frac{u}{k}\Big)\,dx + \int_\Omega F\,\nabla\Big[e^{-\frac{B(u^-)}{\alpha}}\,\psi\,H\Big(\frac{u}{k}\Big)\Big]\,dx \;+\; c_4\,\|\psi\|_{L^\infty(\Omega)}\,\delta_k\,.
\end{aligned}
\tag{2.10}
$$

Let us now define $p(k)$ such that $B(p(k)) = \alpha \log \frac{1}{\sqrt{\delta_k}}$; this is possible since $B'(s) = b(|s|)$, hence B is one–to–one, and from the fact that δ_k goes to zero as k tends to infinity it follows that

$$\lim_{k\to+\infty} p(k) = +\infty \ . \tag{2.11}$$

We choose, again following [6], $\psi = e^{\frac{B(u^-)}{\alpha}} H(\frac{u}{p(k)})\,\varphi^+$ in (2.10), with φ in $C_c^\infty(\Omega)$; since $H(\frac{u}{p(k)}) \equiv 0$ if $|s| \geq p(k)$, we have in fact that ψ belongs to $H_0^1(\Omega) \cap L^\infty(\Omega)$, it is positive and

$$\|\psi\|_{L^\infty(\Omega)} \leq \|\varphi\|_{L^\infty(\Omega)}\, e^{\frac{B(p(k))}{\alpha}} \leq \|\varphi\|_{L^\infty(\Omega)} \frac{1}{\sqrt{\delta_k}} \ .$$

Then we have from (2.10):

$$\begin{aligned}
&\int_\Omega a(x,u,\nabla u)\,\nabla\varphi^+ H\left(\frac{u}{k}\right) H\left(\frac{u}{p(k)}\right) dx + \int_\Omega g(x,u,\nabla u)\,\varphi^+ H\left(\frac{u}{k}\right) H\left(\frac{u}{p(k)}\right) dx \leq \\
&\qquad \leq \int_\Omega f\,\varphi^+ H\left(\frac{u}{k}\right) H\left(\frac{u}{p(k)}\right) dx + \int_\Omega F\,\nabla\left[\varphi^+ H\left(\frac{u}{p(k)}\right) H\left(\frac{u}{k}\right)\right] dx + \\
&\quad + c_4\, \|\varphi\|_{L^\infty(\Omega)} \sqrt{\delta_k} - \frac{1}{p(k)} \int_\Omega a(x,u,\nabla u)\,\nabla T_{p(k)}(u)\, H'\left(\frac{u}{p(k)}\right) H\left(\frac{u}{k}\right) \varphi^+\, dx \ .
\end{aligned} \tag{2.12}$$

Last term in (2.12) can be dealt with using (a_2), so that

$$\begin{aligned}
-\frac{1}{p(k)} &\int_\Omega a(x,u,\nabla u)\,\nabla T_{p(k)}(u)\, H'\left(\frac{u}{p(k)}\right) H\left(\frac{u}{k}\right) \varphi^+\, dx \leq \\
&\leq c_5\, \|\varphi\|_{L^\infty(\Omega)} \frac{1}{p(k)} \int_\Omega \left[d(x)^2 + |T_{p(k)}(u)|^2 + |\nabla T_{p(k)}(u)|^2\right] dx \ ,
\end{aligned}$$

and since

$$\begin{aligned}
\int_\Omega &\left[d(x^2) + |T_{p(k)}(u)|^2 + |\nabla T_{p(k)}(u)|^2\right] dx \leq \\
&\leq \liminf_{n\to+\infty} \int_\Omega \left[d(x)^2 + |T_{p(k)}(u_n)|^2 + |\nabla T_{p(k)}(u_n)|^2\right] dx \ ,
\end{aligned}$$

recalling the definition of δ_k, we get from (2.12):

$$
\begin{aligned}
&\int_\Omega a(x,u,\nabla u)\,\nabla\varphi^+ H\Big(\frac{u}{k}\Big)\,H\Big(\frac{u}{p(k)}\Big)\,dx + \int_\Omega g(x,u,\nabla u)\,\varphi^+ H\Big(\frac{u}{k}\Big)\,H\Big(\frac{u}{p(k)}\Big)\,dx \le \\
&\le \int_\Omega f\,\varphi^+ H\Big(\frac{u}{k}\Big)\,H\Big(\frac{u}{p(k)}\Big)\,dx + \int_\Omega F\,\nabla\Big[\varphi^+ H\Big(\frac{u}{p(k)}\Big)\,H\Big(\frac{u}{k}\Big)\Big]\,dx \\
&\quad + c_4\,\|\varphi\|_{L^\infty(\Omega)}\sqrt{\delta_k} + c_5\,\|\varphi\|_{L^\infty(\Omega)}\,\delta_{p(k)}\,.
\end{aligned}
\tag{2.13}
$$

Now we pass to the limit as k tends to infinity; we have

$$
\begin{aligned}
&\int_\Omega F\,\nabla\Big[\varphi^+ H\Big(\frac{u}{p(k)}\Big)\,H\Big(\frac{u}{k}\Big)\Big]\,dx = \\
&= \int_\Omega F\,\nabla\varphi^+ H\Big(\frac{u}{k}\Big)\,H\Big(\frac{u}{p(k)}\Big)\,dx + \frac{1}{k}\int_\Omega F\,\nabla T_k(u)\,H'\Big(\frac{u}{k}\Big)\,H\Big(\frac{u}{p(k)}\Big)\,\varphi^+\,dx \\
&\quad + \frac{1}{p(k)}\int_\Omega F\,\nabla T_{p(k)}(u)\,H'\Big(\frac{u}{p(k)}\Big)\,H\Big(\frac{u}{k}\Big)\,\varphi^+\,dx\,,
\end{aligned}
$$

and since assumption (a_2) implies

$$
\begin{aligned}
&\Bigg|\frac{1}{k}\int_\Omega F\,\nabla T_k(u)\,H'\Big(\frac{u}{k}\Big)\,H\Big(\frac{u}{p(k)}\Big)\,\varphi^+\,dx \; + \\
&\qquad\qquad + \frac{1}{p(k)}\int_\Omega F\,\nabla T_{p(k)}(u)\,H'\Big(\frac{u}{p(k)}\Big)\,H\Big(\frac{u}{k}\Big)\,\varphi^+\,dx\Bigg| \le \\
&\le c_6\left(\frac{1}{k}\int_\Omega \Big(|F|^2 + |\nabla T_k(u)|^2\Big)\,dx + \frac{1}{p(k)}\int_\Omega \Big(|F|^2 + |\nabla T_{p(k)}(u)|^2\Big)\,dx\right),
\end{aligned}
$$

we get, in virtue of (2.11), (2.4) and the fact that $H(\frac{u}{k})\,H(\frac{u}{p(k)})$ converges to 1 almost everywhere in Ω,

$$
\lim_{k\to+\infty}\int_\Omega F\,\nabla\Big[\varphi^+ H\Big(\frac{u}{p(k)}\Big)\,H\Big(\frac{u}{k}\Big)\Big]\,dx = \int_\Omega F\,\nabla\varphi^+\,dx\,.
$$

As far as the other terms in (2.13) are concerned, it is enough to use the Lebesgue theorem, so that we finally obtain, recalling that δ_k and $\delta_{p(k)}$ go to zero,

$$
\int_\Omega a(x,u,\nabla u)\,\nabla\varphi^+\,dx + \int_\Omega g(x,u,\nabla u)\,\varphi^+\,dx \le \int_\Omega f\,\varphi^+\,dx + \int_\Omega F\,\nabla\varphi^+\,dx\,,
\tag{2.14}
$$

for every φ in $C_c^\infty(\Omega)$.

To obtain the second half of the desired inequality, we will take $v = \psi\, e^{-\frac{B(u_n^+)}{\alpha}}\, H(\frac{u_n}{k})$ as test function in (2.2) with ψ in $H_0^1(\Omega) \cap L^\infty(\Omega)$, $\psi \leq 0$; as before, we will subsequently choose $\psi = -\varphi^-\, e^{\frac{B(u^+)}{\alpha}} H(\frac{u}{p(k)})$, with $p(k)$ defined above. The same arguments used before then allow to conclude that

$$
\begin{aligned}
-\int_\Omega a(x,u,\nabla u)\,\nabla\varphi^-\,dx \;-\; \int_\Omega g(x,u,\nabla u)\,\varphi^-\,dx \;\leq \\
\leq\; -\int_\Omega f\,\varphi^-\,dx \;-\; \int_\Omega F\,\nabla\varphi^-\,dx\;,
\end{aligned}
\tag{2.15}
$$

for every φ in $C_c^\infty(\Omega)$, and adding (2.14) and (2.15) we get

$$
\int_\Omega a(x,u,\nabla u)\,\nabla\varphi\,dx + \int_\Omega g(x,u,\nabla u)\,\varphi\,dx \;\leq\; \int_\Omega f\,\varphi\,dx + \int_\Omega F\,\nabla\varphi\,dx\,, \quad \forall\,\varphi \in C_c^\infty(\Omega)\,,
$$

hence taking $-\varphi$ it is proved that u is a distributional solution of (1.1). ∎

Remark 2.1. The same method provides a proof of the existence of a solution of

$$
\begin{cases} A(u) + g(x,u,\nabla u) = \chi & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases}
\tag{1.1}
$$

if A is an operator in the Sobolev space $W_0^{1,p}(\Omega)$ and $g(x,s,\cdot)$ has a growth of order p; to be more precise, let $p > 1$, and let g satisfy

$$
|g(x,s,\xi)| \leq b(|s|)\left(h(x) + |\xi|^p\right), \quad \forall\, s \in \mathbb{R},\ \ \forall\, \xi \in \mathbb{R}^N,\ \text{ a.e. } x \in \Omega\,,
\tag{2.16}
$$

$$
g(x,s,\xi)\,s \geq 0\,, \quad \forall\, s \in \mathbb{R},\ \ \forall\, \xi \in \mathbb{R}^N,\ \text{ a.e. } x \in \Omega\,,
\tag{2.17}
$$

with $h(x)$ in $L^1(\Omega)$, and set $A(u) \equiv -\mathrm{div}(a(x,u,\nabla u))$, where a is a Carathéodory function such that

$$
a(x,s,\xi)\cdot\xi \geq \alpha\,|\xi|^p\,, \qquad \alpha > 0\,,
\tag{2.18}
$$

$$
|a(x,s,\xi)| \leq \beta\left(d(x) + |s|^{p-1} + |\xi|^{p-1}\right), \qquad \beta > 0\,,
\tag{2.19}
$$

$$
\Big[a(x,s,\xi) - a(x,s,\eta)\Big]\cdot[\xi - \eta] > 0\,, \qquad \forall\,\xi \neq \eta\,,
\tag{2.20}
$$

for all s in $\mathbb{R}$, all ξ, η in $\mathbb{R}^N$ and almost every x in Ω, with $d(x) \in L^{p'}(\Omega)$ ($\frac{1}{p} + \frac{1}{p'} = 1$). Then in the same way as above if χ is in $L^1(\Omega) + W^{-1,p'}(\Omega)$ we obtain a distributional solution u of (1.1). This solution belongs to $W_0^{1,q}(\Omega)$ for every $q < \frac{N(p-1)}{N-1}$ if $p > 2-\frac{1}{N}$; since if $p \le 2-\frac{1}{N}$ we have $\frac{N(p-1)}{N-1} \le 1$, in this case we should say that $|\nabla u|$ is in the Marcinkiewicz space $M^{\frac{N(p-1)}{N-1}}(\Omega)$, nevertheless it is always true that $a(x,u,\nabla u)$ belongs to $L^q(\Omega)^N$ for every $q < \frac{N}{N-1}$, hence the weak formulation makes sense and u is a solution in the sense of distributions. □

Remark 2.2. It should be noted that the proof of Theorem 1.1 essentially relies on the estimate (2.4) for the approximating solutions:

$$\lim_{k\to+\infty} \int_\Omega \frac{|\nabla T_k(u_n)|^2}{k}\,dx = 0 \qquad \text{uniformly on } n ,$$

which is not true if the sequence f_n only weakly converges to a Radon measure μ. In this sense this method, differently from the one used in [10] and based on the strong convergence in $H_0^1(\Omega)$ of the truncations of the approximating solutions, better points out the difference between a datum in $L^1(\Omega)+H^{-1}(\Omega)$ or in the space of bounded Radon measures. □

REFERENCES

[**1**] Bensoussan, A. and Frehse, J. – Jeux differentiels stochastiques et systémes d'équations aux derivées partielles non lineaires, *C.R. Acad. Sc. Paris,* 293, Serie I, (1981), 187–190.

[**2**] Boccardo, L. and Gallouët, T. – Nonlinear elliptic equations with right hand side measures, *Comm. Partial Differential Equations,* 17(3&4) (1992), 641–655.

[**3**] Boccardo, L. and Gallouët, T. – Strongly nonlinear elliptic equations having natural growth terms and L^1 data, *Nonlinear Anal. T.M.A.,* 19 (1992), 573–579.

[**4**] Boccardo, L.; Gallouët, T. and Orsina, L. – Existence and nonexistence of solutions for some nonlinear elliptic equations, *Journal D'Analyse Mathematique,* 73 (1997), 203–223.

[**5**] Boccardo, L.; Murat, F. and Puel, J.P. – Existence de solutions faibles des équations elliptiques quasi-lineaires à croissance quadratique, *Nonlinear P.D.E. and their applications, Collège de France Seminar,* Vol. IV, (H. Brezis & J.L. Lions, Eds.), Research Notes in Mathematics 84, Pitman, London, (1983), 19–73.

[**6**] Boccardo, L.; Murat, F. and Puel, J.P. – Existence de solutions non bornées pour certaines equations quasi-linéaires, *Portugaliae Math.,* 41 (1982), 507–534.

[**7**] Del Vecchio, T. – Strongly nonlinear problems with Hamiltonian having natural growth, *Houst. Journ. of Math.,* 16 (1990), 7–24.

[**8**] Grenon-Isselkou, N. and Isselkou, O.-A. – Existence results for some quasilinear elliptic equations with right hand side in L^1, *Portugaliae Math.,* 56 (1999), 39–57.

[9] Leray, J. and Lions, J.-L. – Quelques résultats de Višik sur les problèmes elliptiques non linéaires par les méthodes de Minty-Browder, *Bull. Soc. Math. France,* 93 (1965), 97–107.

[10] Porretta, A. – Some remarks on the regularity of solutions for a class of elliptic equations with measure data, *Houston Journ. of Math.,* to appear.

[11] Saad, M. – *A unilateral problem for elliptic equations with at most quadratic growth with respect to gradient and for L^1-data,* preprint.

[12] Stampacchia, G. – Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus, *Ann. Inst. Fourier (Grenoble),* 15(1) (1965), 189–258.

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