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UNIFORM CONVERGENCE RESULTS FOR CERTAIN TWO-DIMENSIONAL CAUCHY PRINCIPAL VALUE INTEGRALS *

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Abstract: A general uniform convergence theorem for numerical integration of certain two-dimensional Cauchy principal value integrals is proved. A special instance of this theorem is given as corollary.

1 – Introduction

In this paper we study the uniform convergence with respect to the parameters ϑ and ζ, of numerical methods for evaluating the Cauchy principal value (CPV) integral

$$J(f;\vartheta,\zeta) \;:=\; \fint_{-1}^{1} \fint_{-1}^{1} w_1(x)\, w_2(x)\, \frac{f(x,\widetilde{x})}{(x-\vartheta)\,(\widetilde{x}-\zeta)}\; dx\, d\widetilde{x}\;, \qquad \vartheta\in(-1,\,1)\,,\quad \zeta\in(-1,\,1)\;, \tag{1.1}$$

where w_1, w_2 are the Jacobi weight functions

$$w_1(x):=(1-x)^{\alpha_1}\,(1+x)^{\beta_1}\,,\qquad w_2(\widetilde{x}):=(1-\widetilde{x})^{\alpha_2}\,(1+\widetilde{x})^{\beta_2}\;, \qquad \alpha_i,\,\beta_i>-1,\;\; i=1,\,2\;. \tag{1.2}$$

In some recent papers [2, 3, 8], integrals of type (1.1), has been approximated by cubature rules based on quasi-uniform tensor product spline spaces.

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Supposing that $f \in H_p(\mu, \mu)$, $0 < \mu \leq 1$, $0 \leq p < m-1$, where m denotes the order of the splines, in [2, 3] the authors proved a convergence theorem for cubature rules obtained by substituting the integrand function f with quasi-interpolating or nodal spline operators; they gave besides, an upper bound for the remainder term. By subtracting out the singularities, in [8], it has been proved the uniform convergence of cubature rules based on quasi-interpolating spline-projectors for (ϑ, ζ) belonging to any closed region contained in the integration domain.

Since in many applications it is necessary to have at one's disposal rules uniformly converging for all $(\vartheta, \zeta) \in (-1,1)\times(-1,1)$, the main object of this paper is to give general conditions for obtaining such rules. We shall do that in Section 2, proving the general convergence theorem 1.

In Section 3 we shall apply the obtained results to a particular approximation operator and we shall give the relative convergence results as corollary of the above general convergence theorem.

2 – A general uniform convergence theorem

In this section we shall state and prove a general uniform convergence theorem for two-dimensional CPV integrals (1.1), that can be written in the form

$$\begin{aligned} J(f;\vartheta,\zeta) = & \fint_{-1}^{1} \fint_{-1}^{1} w_1(x)\, w_2(\widetilde{x})\, \frac{f(x,\widetilde{x}) - f(\vartheta,\zeta)}{(x-\vartheta)\,(\widetilde{x}-\zeta)}\, dx\, d\widetilde{x} \\ & + f(\vartheta,\zeta)\, W_1(\vartheta)\, W_2(\zeta)\ . \end{aligned} \tag{2.1}$$

where

$$W_1(\vartheta) = \fint_{-1}^{1} \frac{w_1(x)}{x-\vartheta}\, dx\,, \qquad W_2(\zeta) = \fint_{-1}^{1} \frac{w_2(\widetilde{x})}{\widetilde{x}-\zeta}\, d\widetilde{x}\ . \tag{2.2}$$

We denote $\Omega = I\times\widetilde{I}$, with $I = [-1,1]$, $\widetilde{I} = [-1,1]$, and we define some partitions $X_N = \{x_i\}_{i=1}^{N}$, $\widetilde{X}_{\widetilde{N}} = \{\widetilde{x}_{\widetilde{i}}\}_{\widetilde{i}=1}^{\widetilde{N}}$ of I and $\widetilde{I}$ respectively. Connected with the sequence of tensor product partitions $\{X_N\times\widetilde{X}_{\widetilde{N}}\colon\ N = N_1, N_2, ...;\ \widetilde{N} = \widetilde{N}_1, \widetilde{N}_2, ...\}$, we consider a sequence of real positive numbers $\{\Delta_{N\widetilde{N}}\}$ such that

$$\lim_{\substack{N\to\infty \\ \widetilde{N}\to\infty}} \Delta_{N\widetilde{N}} = 0\ . \tag{2.3}$$

Let $\Phi = \{\varphi_{i\widetilde{i}}(x,\widetilde{x})\}_{i=1,2,\cdots N}^{\widetilde{i}=1,2,\cdots\widetilde{N}}$ be a set of basis functions real continuous in Ω and let $F_{N\widetilde{N}}$ be a linear operator defined by Φ and approximating to f.

We assume

$$r_{N\widetilde{N}}(x,\widetilde{x}) = f(x,\widetilde{x}) - F_{N\widetilde{N}}(x,\widetilde{x}) \ . \tag{2.4}$$

Denoting by

$$S_{N\widetilde{N}}(x) = \int_{-1}^{1} w_2(\widetilde{x}) \, \frac{r_{N\widetilde{N}}(x,\widetilde{x}) - r_{N\widetilde{N}}(x,\zeta)}{\widetilde{x}-\zeta} \, d\widetilde{x} \ , \tag{2.5}$$

$$T_{N\widetilde{N}}(\widetilde{x}) = \int_{-1}^{1} w_1(x) \, \frac{r_{N\widetilde{N}}(x,\widetilde{x}) - r_{N\widetilde{N}}(\vartheta,\widetilde{x})}{x-\vartheta} \, dx \ , \tag{2.6}$$

and considering the cubature rule

$$\begin{aligned} J_{N\widetilde{N}}(f;\vartheta,\zeta) &= \fint_{-1}^{1} \fint_{-1}^{1} w_1(x)\, w_2(\widetilde{x}) \, \frac{F_{N\widetilde{N}}(x,\widetilde{x}) - F_{N\widetilde{N}}(\vartheta,\zeta)}{(x-\vartheta)\,(\widetilde{x}-\zeta)} \, dx\, d\widetilde{x} \\ &\quad + f(\vartheta,\zeta)\, W_1(\vartheta)\, W_2(\zeta) \ , \end{aligned} \tag{2.7}$$

the error term $E_{N\widetilde{N}}(f;\vartheta,\zeta) = J(f;\vartheta,\zeta) - J_{N\widetilde{N}}(f;\vartheta,\zeta)$, given by

$$E_{N\widetilde{N}}(f;\vartheta,\zeta) = \fint_{-1}^{1} \fint_{-1}^{1} w_1(x)\, w_2(\widetilde{x}) \, \frac{r_{N\widetilde{N}}(x,\widetilde{x}) - r_{N\widetilde{N}}(\vartheta,\zeta)}{(x-\vartheta)\,(\widetilde{x}-\zeta)} \, dx\, d\widetilde{x} \ , \tag{2.8}$$

can be written in the form [8],

$$\begin{aligned} E_{N\widetilde{N}}(f;\vartheta,\zeta) &= \int_{-1}^{1} w_1(x) \, \frac{S_{N\widetilde{N}}(x) - S_{N\widetilde{N}}(\vartheta)}{x-\vartheta} \, dx \\ &\quad + W_1(\vartheta)\, S_{N\widetilde{N}}(\vartheta) + W_2(\zeta)\, T_{N\widetilde{N}}(\zeta) \ . \end{aligned} \tag{2.9}$$

We say that $f \in H(\mu,\mu)$, $0 < \mu \le 1$, if f is a continuous function in Ω such that for all $(x_1,\widetilde{x}_1), (x_2,\widetilde{x}_2) \in \Omega$ there results $|f(x_1,\widetilde{x}_1) - f(x_2,\widetilde{x}_2)| \le C[|x_1 - x_2|^\mu + |\widetilde{x}_1 - \widetilde{x}_2|^\mu]$, with C real constant.

For proving theorem 1 below, we need some lemmas [6].

Lemma 1. *Let $g \in H(\sigma,\sigma)$ in Ω, $0 \le \sigma < 1$. The function*

$$\varphi(x,\widetilde{x}) = \frac{g(x,\widetilde{x}) - g(x,\widetilde{x}_0)}{|x-\widetilde{x}_0|^{\bar{\epsilon}}} \ , \qquad \forall\, \bar{\epsilon}, \quad 0 < \bar{\epsilon} < \sigma \ ,$$

satisfies the $H(\sigma-\bar{\epsilon})$ condition for the variable x uniformly with respect to $\widetilde{x}$, and $H(\sigma-\bar{\epsilon})$ condition for the variable $\widetilde{x}$ uniformly with respect to x. ∎

If we define the functions

$$s(x) \;=\; \int_{-1}^{1} w_2(\widetilde{x})\, \frac{g(x,\widetilde{x})-g(x,\widetilde{x}_0)}{\widetilde{x}-\widetilde{x}_0}\; d\widetilde{x} \tag{2.10}$$

and

$$t(\widetilde{x}) \;=\; \int_{-1}^{1} w_1(x)\, \frac{g(x,\widetilde{x})-g(x_0,\widetilde{x})}{x-x_0}\; dx \;, \tag{2.11}$$

we have:

Lemma 2. *Suppose $g \in H(\sigma,\sigma)$ in Ω, $0 \le \sigma < 1$. The functions s and t, defined in (2.10), (2.11), satisfy a Hölder condition of order $\sigma-\bar{\epsilon}$, where $\bar{\epsilon}$ is an arbitrary real number such that $0<\bar{\epsilon}<\sigma$, i.e.*

$$|s(x)-s(x_0)| \;\le\; K_0\, |x-x_0|^{\sigma-\bar{\epsilon}} \;, \tag{2.12}$$

$$|t(\widetilde{x})-t(\widetilde{x}_0)| \;\le\; K_1\, |\widetilde{x}-\widetilde{x}_0|^{\sigma-\bar{\epsilon}} \;. \;\blacksquare \tag{2.13}$$

Theorem 1. *Let $f \in H(\mu,\mu)$ in Ω, and assume that the approximation $F_{N\widetilde{N}}$ to f is such that*

(i) $r_{N\widetilde{N}}(x,\pm1)=0,\ \forall\, x\in I,\ \ r_{N\widetilde{N}}(\pm1,\widetilde{x})=0,\ \forall\,\widetilde{x}\in\widetilde{I}$,

(ii) $\|r_{N\widetilde{N}}\|_\infty = O(\Delta^{\nu}_{N\widetilde{N}}),\ \ 0<\nu\le\mu$,

(iii) $r_{N\widetilde{N}} \in H(\sigma,\sigma),\ \ 0<\sigma\le\mu$.

If

$$\rho+\gamma-\bar{\epsilon} \;>\; 0 \tag{2.14}$$

where $\rho:=\min(\sigma,\nu)$, $\gamma:=\min(\alpha_1,\alpha_2,\beta_1,\beta_2,0)$ and $\bar{\epsilon}$ is a positive real number as small as we like, then

$$\begin{aligned} E_{N\widetilde{N}}(f;\vartheta,\zeta) \to 0 \quad & \text{as}\ \ N\to\infty,\ \widetilde{N}\to\infty\ , \\ & \text{uniformly for all}\ \ (\vartheta,\zeta)\in(-1,1)\times(-1,1)\ . \end{aligned} \tag{2.15}$$

Proof: We can write:

$$\begin{aligned} E_{N\widetilde{N}}(f;\vartheta,\zeta) \;&=\; \int_{-1}^{1} w_1(x)\, \frac{S_{N\widetilde{N}}(x)-S_{N\widetilde{N}}(\vartheta)}{x-\vartheta}\; dx \;+\; W_1(\vartheta)\, S_{N\widetilde{N}}(\vartheta) \;+\; W_2(\zeta)\, T_{N\widetilde{N}}(\zeta) \\ &=\; T_1+T_2+T_3\ . \end{aligned}$$

Taking into account the condition (iii), and assuming in lemma 2 $g(x,\widetilde{x}) = r_{N\widetilde{N}}(x,\widetilde{x})$, $x_0=\vartheta$, $\widetilde{x}_0=\zeta$, the functions $S_{N\widetilde{N}}(x)$, $T_{N\widetilde{N}}(\widetilde{x})$ defined in (2.5), (2.6), satisfy (2.12) and (2.13) respectively. Furthermore, for $0 < \epsilon^* < \sigma$, by (ii) and lemma 5 in [6], there results:

$$|S_{N\widetilde{N}}(x)| \le \bar{C}\, \Delta_{N\widetilde{N}}^{\nu(1-\frac{\epsilon^*}{\sigma})} \qquad \forall \zeta \in (-1,1) \ , \tag{2.16}$$

$$|T_{N\widetilde{N}}(\widetilde{x})| \le \bar{C}_1\, \Delta_{N\widetilde{N}}^{\nu(1-\frac{\epsilon^*}{\sigma})} \qquad \forall \vartheta \in (-1,1) \ . \tag{2.17}$$

Consider first T_2. Because (i), in a neighbourhood of $x = 1$, we have

$$S_{N\widetilde{N}}(\vartheta) = \int_{-1}^{1} w_2(\widetilde{x})\, \frac{\Big[r_{N\widetilde{N}}(\vartheta,\widetilde{x}) - r_{N\widetilde{N}}(\vartheta,\zeta)\Big] - \Big[r_{N\widetilde{N}}(1,\widetilde{x}) - r_{N\widetilde{N}}(1,\zeta)\Big]}{\widetilde{x}-\zeta}\, d\widetilde{x}$$

then, by condition (iii) and lemma 1, there results

$$|S_{N\widetilde{N}}(\vartheta)| \le C\, |\vartheta - 1|^{\sigma-\bar{\epsilon}} \int_{-1}^{1} \frac{w_2(\widetilde{x})}{|\widetilde{x}-\zeta|^{1-\bar{\epsilon}}}\, d\widetilde{x} = O(1-\vartheta)^{\sigma-\bar{\epsilon}} \ . \tag{2.18}$$

Besides, in a neighbourhood of $x = 1$, by §4.62 of [10],

$$W_1(\vartheta) = \begin{cases} O\Big((1-\vartheta)^{\alpha_1}\Big) + c & \text{if } \alpha_1 \text{ is not an integer,} \\ O\Big(|\log(1-\vartheta)|\Big) & \text{if } \alpha_1 \text{ is an integer .} \end{cases} \tag{2.19}$$

Hence, we can find $\delta > 0$ sufficiently small so that for all $\vartheta \in [1-\delta, 1]$ and $\forall\, \epsilon > 0$, $T_2 = O((1-\vartheta)^{\sigma-\bar{\epsilon}+\alpha_1}\, |\log(1-\vartheta)|) < \epsilon$ uniformly in ϑ if (2.14) holds.
Similarly, we can find $\bar{\delta} > 0$ such that for all $\vartheta \in [-1, -1+\bar{\delta}]$

$$T_2 = O\Big((1+\vartheta)^{\sigma-\bar{\epsilon}+\beta_1}\, |\log(1+\vartheta)|\Big) < \epsilon \quad \text{uniformly in } \vartheta \ .$$

Finally, since $W_1(\vartheta) = O(1)$ in $[-1+\bar{\delta}, 1-\delta]$ and $\|S_{N\widetilde{N}}\|_\infty = o(1)$ as $N \to \infty$, $\widetilde{N} \to \infty$, we conclude that

$$T_2 = o(1) \quad \text{as } N \to \infty,\ \widetilde{N} \to \infty \ , \quad \text{uniformly for all } (\vartheta, \zeta) \in (-1,1)\times(-1,1) \ .$$

In the same way we can prove that

$$T_3 = o(1) \quad \text{as } N \to \infty,\ \widetilde{N} \to \infty \ , \quad \text{uniformly for all } (\vartheta, \zeta) \in (-1,1)\times(-1,1) \ .$$

Consider now

$$\begin{aligned} |T_1| &= \left| \int_{-1}^{1} w_1(x)\, \frac{S_{N\widetilde{N}}(x) - S_{N\widetilde{N}}(\vartheta)}{x-\vartheta}\, dx \right| \\ &\le \left| \int_{U} w_1(x)\, \frac{S_{N\widetilde{N}}(x) - S_{N\widetilde{N}}(\vartheta)}{x-\vartheta}\, dx \right| \\ &\quad + \left| \int\limits_{\substack{|x-\vartheta|\ge \Delta_{N\widetilde{N}} \\ x\notin U}} w_1(x)\, \frac{S_{N\widetilde{N}}(x) - S_{N\widetilde{N}}(\vartheta)}{x-\vartheta}\, dx \right| \\ &\quad + \left| \int\limits_{\substack{|x-\vartheta|\le \Delta_{N\widetilde{N}} \\ x\notin U}} w_1(x)\, \frac{S_{N\widetilde{N}}(x) - S_{N\widetilde{N}}(\vartheta)}{x-\vartheta}\, dx \right| \\ &= T_{11} + T_{12} + T_{13} \end{aligned}$$

where $U := [-1, -1+\widetilde{s}] \cup [1-\bar{s}, 1]$, for some $\widetilde{s}$, $\bar{s}$ to be determined below. There results

$$\begin{aligned} \left| \int_{-1}^{-1+\widetilde{s}} w_1(x)\, \tfrac{S_{N\widetilde{N}}(x) - S_{N\widetilde{N}}(\vartheta)}{x-\vartheta}\, dx \right| &\le K_0 \int_{-1}^{-1+\widetilde{s}} w_1(x)\, |x-\vartheta|^{\sigma-\bar{\epsilon}-1}\, dx \\ &= O\left(\int_{-1}^{-1+\widetilde{s}} (1+x)^{\gamma+\sigma+\bar{\epsilon}-1}\, dx \right) < \epsilon \end{aligned}$$

for $\widetilde{s}$ sufficiently small. Similarly we have

$$\left| \int_{1-\bar{s}}^{1} w_1(x)\, \frac{S_{N\widetilde{N}}(x) - S_{N\widetilde{N}}(\vartheta)}{x-\vartheta}\, dx \right| < \epsilon$$

for $\bar{s}$ sufficiently small, and then

$$T_{11} < 2\,\epsilon\,, \quad \forall\, \epsilon > 0\,. \tag{2.20}$$

Considering the term T_{12}, we have

$$
\begin{aligned}
T_{12} &= \left| \int\limits_{\substack{|x-\vartheta|\geq\Delta_{N\widetilde{N}} \\ x\notin U}} w_1(x)\, \frac{S_{N\widetilde{N}}(x) - S_{N\widetilde{N}}(\vartheta)}{x-\vartheta}\, dx \right| \\
&= O\left(\Delta_{N\widetilde{N}}^{\nu\left(1-\frac{\epsilon^*}{\sigma}\right)} \ln(\Delta_{N\widetilde{N}})\right) = o(1) \qquad \text{as } N\to\infty,\ \widetilde{N}\to\infty\ .
\end{aligned}
\tag{2.21}
$$

Finally, by (2.12), (2.13),

$$
\begin{aligned}
T_{13} &= \left| \int\limits_{\substack{|x-\vartheta|\leq\Delta_{N\widetilde{N}} \\ x\notin U}} w_1(x)\, \frac{S_{N\widetilde{N}}(x) - S_{N\widetilde{N}}(\vartheta)}{x-\vartheta}\, dx \right| \\
&\leq C_2 \int\limits_{\substack{|x-\vartheta|\leq\Delta_{N\widetilde{N}} \\ x\notin U}} |x-\vartheta|^{\sigma-\bar{\epsilon}-1}\, dx = o(1)
\end{aligned}
\tag{2.22}
$$

as $N\to\infty$, $\widetilde{N}\to\infty$ uniformly $\forall\,(\vartheta,\zeta)\in(-1,1)\times(-1,1)$.
The thesis follows considering that T_1, T_2, T_3 can be made arbitrarily small, uniformly in $(\vartheta,\zeta)\in(-1,1)\times(-1,1)$, as $N\to\infty$, $\widetilde{N}\to\infty$. ∎

We remark that the theorem 1 extends to the two-dimensional case the results of theorem 1 in [7].

3 – Particular example of theorem 1

In this section we derive uniform convergence results for the below defined approximation $\bar{W}_{N\widetilde{N}}$ to f, which we state as corollary.

Consider the sequence of two-dimensional nodal spline operators studied in [3]:

$$
W^*_{N\widetilde{N}}(f;x,\widetilde{x}) = \sum_{i=p_j}^{q_j} \sum_{\widetilde{i}=\widetilde{p}_{\widetilde{j}}}^{\widetilde{q}_{\widetilde{j}}} \omega_{m\widetilde{m}i\widetilde{i}}(x,\widetilde{x})\ f(\tau_j,\widetilde{\tau}_{\widetilde{j}})\ ,
\tag{3.1}
$$

$$
(x,\widetilde{x})\in[\tau_j,\tau_{j+1}]\times[\widetilde{\tau}_{\widetilde{j}},\widetilde{\tau}_{\widetilde{j}+1}]\,,\qquad j=0,1,\ldots N-1\,,\qquad \widetilde{j}=0,1,\ldots,\widetilde{N}-1\ .
$$

For constructing (3.1), we firstly need to consider the partitions

$$X_{mN} := \Big\{-1 \equiv x_0 < x_1 < \cdots < x_{(m-1)N} \equiv 1\Big\} ,$$

$$\widetilde{X}_{\widetilde{m}\widetilde{N}} := \Big\{-1 \equiv \widetilde{x}_0 < \widetilde{x}_1 < \cdots < \widetilde{x}_{(\widetilde{m}-1)\widetilde{N}} \equiv 1\Big\}$$

of I, $\widetilde{I}$ respectively. The nodes τ_i, $\widetilde{\tau}_{\widetilde{i}}$ are defined by

$$\tau_i := x_{(m-1)i} , \quad 0 \le i \le N, \ \text{ with } N \ge m-1 ,$$

$$\widetilde{\tau}_{\widetilde{i}} := \widetilde{x}_{(\widetilde{m}-1)\widetilde{i}} , \quad 0 \le \widetilde{i} \le \widetilde{N}, \ \text{ with } \widetilde{N} \ge \widetilde{m}-1 ,$$

and the functions $\omega_{m\widetilde{m}i\widetilde{i}}$ are such that $\omega_{m\widetilde{m}i\widetilde{i}} = \omega_{mi}(x) \times \omega_{\widetilde{m}\widetilde{i}}(\widetilde{x})$ where

$$\omega_{mi}(x) := \begin{cases} \displaystyle\prod_{\substack{k=0\\k\neq i}}^{m} \frac{x-\tau_k}{\tau_i-\tau_k} & x \in [-1, \tau_{i_1-1}], \ (i \le m-1) , \\ s_i(x) & x \in [\tau_{i_1-1}, \tau_{N-i_0+1}], \ (N \ge m) , \\ \displaystyle\prod_{\substack{k=0\\k\neq N-i}}^{m} \frac{x-\tau_{N-k}}{\tau_i-\tau_{N-k}} & x \in [\tau_{N-i_0+1}, 1], \ (i \ge N-(m-1)) , \end{cases}$$

with $i_0 := [\frac{m}{2}]+1$, $i_1 := m - [\frac{m}{2}]$. For the functions s_i we have

$$s_i(x) := \sum_{r=0}^{m} \sum_{t=-i_0}^{-i_0+m-1} \alpha_{irt} \, B^m_{(m-1)(i+t)+r}(x) , \qquad i = 0, 1, \dots, N ,$$

where $\{B^m_i(x)\}_{i=1-m}^{(m-1)N-1}$ are the normalized B-splines of order m.

We recall that

$$0 < B^m_i(x) \le 1 \ \text{ if } x \in (x_i, x_{i+m}) , \qquad B^m_i(x) = 0 \ \text{ otherwise} ,$$

except that $B^m_{1-m}(-1) = 1$, $B_{(m-1)N-1}(1) = 1$.

We besides assume

$$p_j := \begin{cases} 0 & \text{if } j = 0, 1, \dots, i_1-2 , \\ j - i_1 + 1 & \text{if } j = i_1-1, \dots, N-i_0 , \\ N - m + 1 & \text{if } j = N-i_0+1, \dots, N-1 , \end{cases}$$

$$q_j := \begin{cases} m-1 & \text{if } j = 0, 1, \dots, i_1-2 , \\ j + i_0 & \text{if } j = i_1-1, \dots, N-i_0 , \\ N & \text{if } j = N-i_0+1, \dots, N-1 . \end{cases}$$

Likewise we define $\widetilde{\omega}_{\widetilde{m}\widetilde{i}}(\widetilde{x})$, $\widetilde{p}_{\widetilde{j}}$, $\widetilde{q}_{\widetilde{j}}$. Without loss of generality, we can assume $m=\widetilde{m}$, i.e. we use splines of the same order on both axes.

We remark that $W^*_{N\widetilde{N}}$ is a spline operator with the following properties:

(**a**) $W^*_{N\widetilde{N}}$ is local, in the sense that $W^*_{N\widetilde{N}}(f;x,\widetilde{x})$ depends only on the values of f in a small neighborhood of $(x,\widetilde{x})$;

(**b**) $W^*_{N\widetilde{N}}$ satisfies the relations: $W^*_{N\widetilde{N}}(f;\tau_i,\widetilde{\tau}_{\widetilde{i}})=f(\tau_i,\widetilde{\tau}_{\widetilde{i}})$;

(**c**) $W^*_{N\widetilde{N}}$ has the optimal order polynomial reproduction property, that means $W^*_{N\widetilde{N}}p=p$ for all $p\in\mathcal{P}^2_m$, where $\mathcal{P}^2_m$ is the set of bivariate polynomials of total order m.

We say that the collection of product partitions

$$\left\{X_{mN}\times\widetilde{X}_{\widetilde{m}\widetilde{N}},\quad N=N_1,N_2,...,\ \widetilde{N}=\widetilde{N}_1,\widetilde{N}_2,...\right\}$$

of $\Omega=I\times\widetilde{I}$, is quasi-uniform (q.u.) if there exists a positive constant A such that $\Delta_i/\delta_j\le A$ for i and j equal to N or $\widetilde{N}$, where

$$\begin{aligned}\Delta_N &= \max\nolimits_{0\le i\le(m-1)N-1}(x_{i+1}-x_i)\,, & \Delta_{\widetilde{N}} &= \max\nolimits_{0\le\widetilde{i}\le(m-1)\widetilde{N}-1}(\widetilde{x}_{\widetilde{i}+1}-\widetilde{x}_{\widetilde{i}})\,,\\ \delta_N &= \min\nolimits_{0\le i\le(m-1)N-1}(x_{i+1}-x_i)\,, & \delta_{\widetilde{N}} &= \min\nolimits_{0\le\widetilde{i}\le(m-1)\widetilde{N}-1}(\widetilde{x}_{\widetilde{i}+1}-\widetilde{x}_{\widetilde{i}})\,,\end{aligned}$$

and we shall call a sequence of spline spaces quasi-uniform if they are based on a sequence of q.u. partitions.

We define $\Delta_{N\widetilde{N}}=\Delta_N+\Delta_{\widetilde{N}}$, and suppose that

$$\Delta_N\to 0\ \text{ as }\ N\to\infty\,,\qquad \Delta_{\widetilde{N}}\to 0\ \text{ as }\ \widetilde{N}\to\infty\,. \tag{3.2}$$

We shall use the results in [3] for deducing the following

Proposition 1. *Suppose $f\in C(\Omega)$, for any sequence of q.u. nodal spline spaces $\{W^*_{N\widetilde{N}}\}$, there results:*

$$\|f-W^*_{N\widetilde{N}}\|_\infty\ \le\ K\,\omega(f;\Delta_{N\widetilde{N}};\Omega) \tag{3.3}$$

where K is a constant depending only on m and A, and

$$\omega(W^*_{N\widetilde{N}};\Delta_{N\widetilde{N}};\Omega)\ =\ O\Big(\omega(f;\Delta_{N\widetilde{N}};\Omega)\Big) \tag{3.4}$$

with

$$\omega(\phi;\Delta;\Theta)\quad=\max_{\substack{|h|,|\widetilde{h}|\le\Delta\\ (x,\widetilde{x}),(x+h,\widetilde{x}+\widetilde{h})\in\Theta}}\left|\phi(x+h,\,\widetilde{x}+\widetilde{h})-\phi(x,\widetilde{x})\right|\,.\ \blacksquare$$

Using the method introduced in [5], we modify the operator (3.1) as

$$
\begin{aligned}
\bar{W}_{N\widetilde{N}}(f;x,\widetilde{x}) \;=\; & W^*_{N\widetilde{N}}(f;x,\widetilde{x}) \\
& + \Big[f(-1,\widetilde{x}) - W^*_{N\widetilde{N}}(f;-1,\widetilde{x})\Big]\, B^m_{1-m}(x) \\
& + \Big[f(1,\widetilde{x}) - W^*_{N\widetilde{N}}(f;1,\widetilde{x})\Big]\, B^m_{(m-1)N-1}(x) \\
& + \Big[f(x,-1) - W^*_{N\widetilde{N}}(f;x,-1)\Big]\, \widetilde{B}^m_{1-m}(\widetilde{x}) \\
& + \Big[f(x,1) - W^*_{N\widetilde{N}}(f;x,1)\Big]\, \widetilde{B}^m_{(m-1)\widetilde{N}-1}(\widetilde{x}) \;,
\end{aligned} \tag{3.5}
$$

and we prove the following

Lemma 3. *Let $f \in H(\mu,\mu)$ in Ω, and let $\{\bar{W}_{N\widetilde{N}}\}$ be a sequence of q.u. spline operators defined in (3.5). There results:*

(1) $r_{N\widetilde{N}}(\mp 1,\widetilde{x}) = 0,\ \forall\, \widetilde{x} \in \widetilde{I},\ \ r_{N\widetilde{N}}(x,\mp 1) = 0,\ \forall\, x \in I;$

(2) $\|r_{N\widetilde{N}}\|_\infty = O(\Delta^\mu_{N\widetilde{N}});$

(3) $r_{N\widetilde{N}} \in H(\mu,\mu)$ *in* Ω

with $r_{N\widetilde{N}}(x,\widetilde{x}) = f(x,\widetilde{x}) - \bar{W}_{N\widetilde{N}}(x,\widetilde{x})$.

Proof: Property (1) derives by the definition (3.5) and by the interpolating property of $W^*_{N\widetilde{N}}$. From the definition (3.5) we have $\|r_{N\widetilde{N}}\|_\infty \le 3\,\|f - W^*_{N\widetilde{N}}\|$ and by (3.3) property (2) holds. Property (3) follows considering that for $(x_1,\widetilde{x}_1),(x_2,\widetilde{x}_2) \in \Omega$ we can write

$$
\begin{aligned}
& \Big|r_{N\widetilde{N}}(x_1,\widetilde{x}_1) - r_{N\widetilde{N}}(x_2,\widetilde{x}_2)\Big| \le \\
& \le \Big|f_{N\widetilde{N}}(x_1,\widetilde{x}_1) - f_{N\widetilde{N}}(x_2,\widetilde{x}_2)\Big| + \Big|\bar{W}_{N\widetilde{N}}(x_1,\widetilde{x}_1) - \bar{W}_{N\widetilde{N}}(x_2,\widetilde{x}_2)\Big| \;.
\end{aligned} \tag{3.6}
$$

Then, by the definition of $\bar{W}_{N\widetilde{N}}$, using (3.3) and (3.4), it is possible to prove that in each of the different cases, according to the positions of the points $(x_1,\widetilde{x}_1)$, $(x_2,\widetilde{x}_2)$ in Ω, there results [9]:

$$
\Big|\bar{W}_{N\widetilde{N}}(f;x_1,\widetilde{x}_1) - \bar{W}_{N\widetilde{N}}(f;x_2,\widetilde{x}_2)\Big| \le \bar{C}\,\omega(f;\Delta_{N\widetilde{N}};\Omega) \;.
$$

Therefore the claim follows by (3.6) and the hypothesis on f. ∎

Corollary 1. *Let $f \in H(\mu,\mu)$ in Ω and let $\{F_{N\widetilde{N}}\}$ be a sequence of q.u. spline spaces $\{\bar{W}_{N\widetilde{N}}\}$ defined in (3.5). Assume that (3.2) holds, then*

$$
E_{N\widetilde{N}}(f;\vartheta,\zeta) \to 0 \ \text{ as } N,\widetilde{N} \to \infty \quad \text{uniformly in } (\vartheta,\zeta) \in (-1,1)\times(-1,1) \;.
$$

Proof: Condition (i) of theorem 1 follows by (1) of lemma 3. By conditions (2) and (3) of the same lemma, condition (ii) (with $\Delta_{N\widetilde{N}} = \Delta_N + \widetilde{\Delta}_{\widetilde{N}}$, $\nu = \mu$) and condition (iii) (with $\sigma = \mu$) hold. ∎

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