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STABILIZATION OF A HYBRID SYSTEM: AN OVERHEAD CRANE WITH BEAM MODEL

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Abstract: The object of this paper is the study of a model of an overhead crane consisting of a beam carrying a load, fixed to a moving platform. Stabilization of the system is achieved by means of a monotone feedback law taking into account the position and the velocity of the platform. We then prove that strong stabilization occurs without global monotonicity, thanks to an energy estimate for solutions of a variational nonlinear nonautonomous evolution equation.

1 – Introduction

In this work, we shall study a model of an overhead crane with beam, which is a structure consisting of a platform moving along a straight rail and carrying a beam at the end of which a load is attached. With a feedback control law taking into account the position and the velocity of the platform, the objective is to drive it at a given point from a given configuration, so that the whole structure should be at rest. We propose a modelization which yields a hybrid system namely a partial differential equation coupled with dynamic equations at both ends. This study follows the overhead crane problem with a cable (*see* [6], [5] *or* [2]), where strong stabilization was achieved by means of the application of LaSalle's invariance principle, which usually requires some monotonicity. Two models will be studied, according to whether the beam is linked or clamped at the platform. In the sequel, we consider an orthonormal basis $(O;\vec{\imath},\vec{\jmath})$ for which $\vec{\imath}$ is parallel to the direction of the beam at vertical rest, $x=0$ being the platform's abscissa and $\vec{\jmath}$ is parallel to the rail. The following equations come from projections over $(O,\vec{\jmath})$ axis.

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According to the classical laws of linear elasticity, the vibrations of the beam, which is supposed homogeneous and of unit length, are governed by Euler–Bernoulli's equation. Thus, the transversal displacement $y(x,t)$ at time t of a point of the beam of curvilinear abscissa x satisfies the equation

$$y_{tt} + y_{xxxx} - (a\, y_x)_x = 0 \tag{1.1}$$

where a denotes the tension force due to the load and the lineic mass of the beam. We shall make first the following assumption on a:

$$a \in H^1(0,1) \quad \text{and} \quad a(x) \geq a_0 > 0 \ , \tag{1.2}$$

which will be sufficient for well-posedness of the problem. Other assumptions will be necessary for stabilization and will be mentioned later on.

The load M is supposed to be point mass and to have no moment of inertia, hence:

$$y_{xx}(1,t) = 0 \ . \tag{1.3}$$

The displacements of the ends of the beam are governed by Euler–Newton's equation. For the load, this leads to:

$$M y_{tt}(1,t) - y_{xxx}(1,t) = -a(1)\, y_x(1,t) \ . \tag{1.4}$$

Let us note $(0, F(t))$ the components of the force vector acting on the platform. If the beam is supposed to be linked to the platform, we have

$$y_{xx}(0,t) = 0 \tag{1.5}$$

and the displacement is governed by

$$m\, y_{tt}(0,t) + y_{xxx}(0,t) = F(t) + a(0)\, y_x(0,t) \ . \tag{1.6}$$

If the beam is clamped, we have

$$y_x(0,t) = 0 \ , \tag{1.7}$$

$$m\, y_{tt}(0,t) + y_{xxx}(0,t) = F(t) \ . \tag{1.8}$$

Thus, we have to consider two hybrid systems

$$\begin{cases} y_{tt} + y_{xxxx} - (a\, y_x)_x &= 0 \\ y_{xx}(1,t) &= 0 \\ M y_{tt}(1,t) - y_{xxx}(1,t) &= -a(1)\, y_x(1,t) \\ y_{xx}(0,t) &= 0 \\ m\, y_{tt}(0,t) + y_{xxx}(0,t) &= F(t) + a(0)\, y_x(0,t) \ , \end{cases} \tag{1.9}$$

$$
(1.10) \qquad \begin{cases} y_{tt} + y_{xxxx} - (a\, y_x)_x &= 0 \\ y_{xx}(1,t) &= 0 \\ My_{tt}(1,t) - y_{xxx}(1,t) &= -a(1)\, y_x(1,t) \\ y_x(0,t) &= 0 \\ m\, y_{tt}(0,t) + y_{xxx}(0,t) &= F(t)\ . \end{cases}
$$

For a solution of any of these two systems, we set

$$
(1.11) \qquad E(t) = \frac{1}{2}\Big(\int_0^1 \Big(y_{xx}^2(x,t) + y_t^2(x,t) + a(x)\, y_x^2(x,t) \Big)\, dx + \alpha\, y^2(0,t) + m\, y_t^2(0,t) + M y_t^2(1,t) \Big)
$$

where $\alpha > 0$ is constant.

In both cases, supposing y regular enough, integration by parts yields, using the boundary conditions at both ends:

$$
(1.12) \qquad E'(t) = y_t(0,t)\, \Big(F(t) + \alpha\, y(0,t) \Big)
$$

so that choosing a control law F of feedback type of the form

$$
(1.13) \qquad F(t) = f(y_t(0,t)) - \alpha\, y(0,t)
$$

where the function f satisfies

$$
(1.14) \qquad s\, f(s) \le 0 \quad \forall\, s \in \mathbb{R}\ ,
$$

systems (1.9) and (1.10) are dissipative since in both cases

$$
(1.15) \qquad E'(t) = y_t(0,t)\, f(y_t(0,t)) \le 0 \quad \forall\, t \ge 0\ .
$$

Let us briefly outline the content of this paper. The second part is devoted to the formulation of these systems into abstract evolution equations in Hilbert spaces and to their well-posedness. In the third part, which is more or less classical, strong stability is proved with global monotonicity for the feedback law by considering the ω-limit sets, La Salle's principle and a multiplier method. The major part of this work is section four, where strong stabilization is achieved with a hypothesis of *local* monotonicity, which leads to a possibly non contractive semigroup. Nonemptiness of ω-limit sets is obtained by showing that the orbits are bounded for the graph norm by means of a functional related to the energy which can be associated to weak solutions of a non autonomous variational equation of evolution.

2 – Well-posedness

We first study the system (1.9). According to a well known procedure, we transform this system into an equation of evolution with $(y,\, y_t,\, y_t(0,t),\, y_t(1,t))$ as unknown in an appropriate Hilbert space. Let us introduce

$$\mathbf{H} = H^2(0,1) \times L^2(0,1) \times \mathbb{R} \times \mathbb{R} \tag{2.1}$$

equipped with the following scalar product, for $U=(y,z,\eta,\xi)$ and $\tilde{U}=(\tilde{y},\tilde{z},\tilde{\eta},\tilde{\xi})$ in $\mathbf{H}$:

$$\langle U,\tilde{U}\rangle = \int_0^1 (y_{xx}\,\tilde{y}_{xx} + a\,y_x\,\tilde{y}_x + z\,\tilde{z}) + \alpha\,y(0)\,\tilde{y}(0) + m\,\eta\,\tilde{\eta} + M\,\xi\,\tilde{\xi} \tag{2.2}$$

the associated norm is denoted $\|\cdot\|$.

In this space, we consider the unbounded operator $\mathbf{A}$ defined by

$$D(\mathbf{A}) = \Big\{U=(y,z,\eta,\xi) \in \mathbf{H} \;/\; y \in H^4(0,1),\ z \in H^2(0,1) \\ y_{xx}(1)=0,\ y_{xx}(0)=0,\ \eta=z(0),\ \xi=z(1)\Big\} \tag{2.3}$$

$$\mathbf{A}(U) = \Big[-z,\ y_{xxxx} - (ay_x)_x, \\ \frac{1}{m}\Big(y_{xxx}(0) + \alpha\,y(0) - a\,y_x(0) - f(\eta)\Big),\ -\frac{1}{M}\Big(y_{xxx}(1) - a\,y_x(1)\Big)\Big]\,. \tag{2.4}$$

Each solution $t \mapsto U(t) = (y(\cdot,t),\, z(\cdot,t),\, y_t(0,t),\, y_t(1,t))$ of the equation $\frac{dU}{dt} + \mathbf{A}\,U = 0$ is such that y is, at least formally, solution of (1.9), as one can easily check. We study the maximal monotonicity of $\mathbf{A}$. To this end, we shall assume, in addition to (1.14):

$$f \text{ is continuous and non increasing on } \mathbb{R}\,. \tag{2.5}$$

Theorem 1. *The operator $\mathbf{A}$ is maximal monotone in $\mathbf{H}$.*

Proof: Monotonicity: integration by parts yields

$$\langle \mathbf{A}U - \mathbf{A}\tilde{U},\, U-\tilde{U}\rangle = -\Big(f(\eta) - f(\tilde{\eta})\Big)\,(\eta-\tilde{\eta}) \tag{2.6}$$

which is positive by virtue of (2.5).

Maximality: one has to solve the equation $U + \mathbf{A}U = U_0$ for $U = (y, z, \eta, \xi) \in D(\mathbf{A})$ with a given $U_0 = (y_0, z_0, \eta_0, \xi_0) \in \mathbf{H}$. This is equivalent to the resolution of the system

$$\left\{\begin{array}{ll} y - z & = y_0 \\ z + y_{xxxx} - (a\, y_x)_x & = z_0 \\ \eta + \dfrac{1}{m}\Big(y_{xxx}(0) + \alpha\, y(0) - a\, y_x(0) - f(\eta)\Big) & = \eta_0 \\ \xi - \dfrac{1}{M}\Big(y_{xxx}(1) - a\, y_x(1)\Big) & = \xi_0\ . \end{array}\right. \tag{2.7}$$

Since $U \in D(\mathbf{A})$, one has $\eta = z(0)$ and $\xi = z(1)$ so $\eta = y(0) - y_0(0)$ and $\xi = y(1) - y_0(1)$ according to the first equation, thus

$$\left\{\begin{array}{ll} z & = y - y_0 \\ y_{xxxx} - (a\, y_x)_x + y & = y_0 + z_0 \\ \dfrac{1}{m}\Big(y_{xxx}(0) + (\alpha{+}m)\, y(0) - a\, y_x(0) - f\Big(y(0){-}y_0(0)\Big)\Big) & = \eta_0 + y_0(0) \\ -\dfrac{1}{M}\, y_{xxx}(1) + \dfrac{1}{M}\, a\, y_x(1) + y(1) & = \xi_0 + y_0(1)\ . \end{array}\right. \tag{2.8}$$

Following a method used in [6], this system can be solved by means of the minimization of the functional J defined on $H^2(0,1)$ by

$$\begin{aligned} J(\psi) \;=\; & \frac{1}{2}\left[\int_0^1 (\psi^2 + \psi_{xx}^2 + a\,\psi_x^2)\, dx \;+\; (\alpha + m)\,\psi^2(0) + M\,\psi^2(1)\right] - \\ & - \int_0^1 \psi(y_0 + z_0)\, dx - m\Big(\eta_0 + y_0(0)\Big) - G\Big(\psi(0) - y_0(0)\Big) - M\Big(\xi_0 + y_0(1)\Big) \end{aligned} \tag{2.9}$$

with G defined by $G(s) = \displaystyle\int_0^s f(t)\, dt$. ∎

From (2.5) we deduce that J is strictly convex, continuous and coercive on $H^2(0,1)$. Consequently, J has a unique minimum $y \in H^2(0,1)$. As in [6], we show that $y \in H^4(0,1)$. Then setting $z = y - y_0$, $\eta = z(0)$ and $\xi = z(1)$, we obtain that $U = (y, z, \eta, \xi) \in D(\mathbf{A})$ and $U + \mathbf{A}U = U_0$.

By application of the theory of maximal monotone operators *(see* [1]*)*, we obtain:

Theorem 2. *The operator* $\mathbf{A}$ *is the infinitesimal generator of a continuous semi-group of contractions* $(S(t))_{t\in\mathbb{R}^+}$ *on* $\mathbf{H}$*. Consequently, for all* $U_0 \in D(\mathbf{A})$*, the equation*

$$\frac{dU}{dt} + \mathbf{A}U = 0\,, \qquad U(0) = U_0 \tag{2.10}$$

has a unique solution U *with the regularity*

$$U \in W^{1,\infty}(\mathbb{R}^+, \mathbf{H}) \cap L^\infty(\mathbb{R}^+, D(\mathbf{A})) \tag{2.11}$$

defined by $t \mapsto U(t) = S(t)\,U_0$*. Moreover,*

$$t \mapsto \|U(t)\| \quad \text{is non increasing.} \;\blacksquare \tag{2.12}$$

Remark. Let us note $U(t) = (y(\cdot,t),\, z(\cdot,t),\, \eta(t),\, \xi(t))$ the solution of the above equation. By virtue of the definition of $\mathbf{A}$, we see that $y_{tt} + y_{xxxx} - (a y_x)_x = 0$ a.e. $(t,x) \in \mathbb{R}^+ \times]0,1[$, $y_{xx}(0,t) = 0$, $m\,\eta'(t) + y_{xxx}(0,t) = f(\eta(t)) - \alpha\, y(0,t) + a\, y_x(0,t)$, with $\eta(t) = y_t(0,t)$. But without more regularity for y, no sense can *a priori* be given to $y_{tt}(0,t)$. A similar conclusion holds at $x = 1$, since $M\xi'(t) - y_{xxx}(1,t) = -a\, y_x(1,t)$, with $\xi(t) = y_t(1,t)$. □

Proposition 1. *For all* $U_0 \in D(\mathbf{A})$*, the function* $E[U_0]\colon\, t \mapsto \frac{1}{2}\,\|U(t)\|^2$ *has a derivative on* $\mathbb{R}^+$ *and*

$$E[U_0]'(t) \;=\; y_t(0,t)\, f(y_t(0,t)) \;\le\; 0\;. \tag{2.13}$$

Proof: According to (2.11), $U(t)$ is regular enough to insure derivability of $E[U_0]$; an integration by parts leads easily to the result. ∎

Finally, let us mention that system (1.10) can be solved in the same fashion by considering the Hilbert space

$$\mathbf{H} = \Big\{ U = (y,z,\eta,\xi) \in H^2(0,1) \times L^2(0,1) \times \mathbb{R} \times \mathbb{R} \;/\; y_x(0) = 0 \Big\} \tag{2.14}$$

endowed with the same norm as for problem (1.9)

$$\|U\|^2 \;=\; \int_0^1 (y_{xx}^2 + a\, y_x^2 + z^2)\, dx + \alpha\, y^2(0) + m\,\eta^2 + M\xi^2 \tag{2.15}$$

and the unbounded operator $\mathbf{A}$ defined by

$$\begin{aligned} D(\mathbf{A}) \;=\; \Big\{ U = (y,z,\eta,\xi) \in \mathbf{H} \;/\; & y \in H^4(0,1),\; z \in H^2(0,1) \\ & y_{xx}(1) = 0,\; z_x(0) = 0,\; \eta = z(0),\; \xi = z(1) \Big\} \end{aligned} \tag{2.16}$$

$$\text{(2.17)} \qquad \mathbf{A}U = \Big(-z,\ y_{xxxx}-(ay_x)_x,\\ \frac{1}{m}\Big(y_{xxx}(0)+\alpha\, y(0)-f(\eta)\Big),\ -\frac{1}{M}\Big(y_{xxx}(1)-a\,y_x(1)\Big)\Big) \ .$$

Under the hypothesis (2.5), theorem 2 can be proved in the same way as it has been done for system (1.9).

3 – Stabilization

Strong stabilization of systems (1.9) and (1.10) can be established by using LaSalle's invariance principle, which shows that the ω-limit set of any orbit $\{U(t),\ t\geq 0\}$, $U_0\in D(\mathbf{A})$, is reduced to $\{0\}$. Let us summarize the process: after checking the precompactness of the orbits, an argument of weak-strong closure of the graph of $\mathbf{A}$ shows that the ω-limit points belong to $D(\mathbf{A})$. LaSalle's principle applied to the Lyapunov function $\|U(t)\|^2$ leads to the consideration of solutions with constant energy. A multiplier method shows that the null function is the only solution satisfying this property.

In addition to (1.14) and (2.5), we shall assume

$$\text{(3.1)} \qquad s f(s)\neq 0 \quad \text{for all}\ \ s>0\ .$$

So f can vanish on an interval $]S,0]$ with $S\geq -\infty$. One can envisage the symmetric situation with respect to 0 for f: setting in this latter case $w=-y$ and $g(s)=-f(-s)$, we recover assumption (3.1).

Let us first consider system (1.9). We assume in addition to (1.2):

$$\text{(3.2)} \qquad a\in H^2(0,1) \quad \text{and} \quad a''\leq 0 \ \text{ a.e. on } [0,1]\ .$$

Notice that this property is satisfied whenever a is affine, which is a reasonable hypothesis from a physical point of view (case of an homogeneous beam).

Theorem 3. *For all initial data U_0 in $D(\mathbf{A})$, the solution U of (2.10) satisfies*

$$\|U(t)\| \underset{t\to\infty}{\longrightarrow} 0\ .$$

The proof is divided into several steps: in Propositions 2 and 3, we establish that the ω-limit sets are non empty and recall some results from dynamical systems theory while in Propositions 4 and 5, we study some solutions of constant energy. Proof of theorem 3 follows then from La Salle's principle.

Proposition 2. *The canonical embedding from $D(\mathbf{A})$, endowed with the graph norm, into $\mathbf{H}$ is compact.*

Proof: One can easily check that the graph topology on $D(\mathbf{A})$ is the restriction of the canonical topology of $H^4(0,1) \times H^2(0,1) \times \mathbb{R}^2$. The result is then a consequence of the compactness of the embedding from this latter space into $H^2(0,1) \times L^2(0,1) \times \mathbb{R}^2$ which has a norm equivalent to the norm on $\mathbf{H}$. ∎

Proposition 3. *For all $U_0 \in D(\mathbf{A})$, the set $\omega(U_0) = \{W \in \mathbf{H} \;/\; \exists\,(t_n) \underset{n\to\infty}{\longrightarrow} \infty$ such that $S(t_n) \underset{t\to\infty}{\longrightarrow} W\}$ is nonempty, included in $D(\mathbf{A})$, invariant under the action of $(S(t))_{t\geq 0}$ and satisfies: $\operatorname{dist}(S(t)\,U_0,\, \omega(U_0)) \underset{t\to\infty}{\longrightarrow} 0$. Moreover, $\|S(t)\,W\|$ is constant on $\mathbb{R}^+$ for all $W \in \omega(U_0)$.*

Proof: It follows from (2.11) that the orbit $\{U(t),\ t \geq 0\}$ is bounded for the graph norm so by Proposition 2, $\omega(U_0)$ is nonempty and by a weak-strong closure argument of the graph of $\mathbf{A}$, included in $D(\mathbf{A})$ *(see* [1], prop. 2.5)*.* The other properties are consequences from classical dynamical systems theory *(see* [4], theorem 2.1.7*).* ∎

Proposition 4. *Let $U\colon t \mapsto (y(\cdot,t),\, z(\cdot,t),\, \eta(t),\, \xi(t))$ be a solution of (2.10), with the regularity of theorem 2, satisfying $\eta(t) = y_t(0,t) = 0\ \forall\, t \geq 0$. Then y is identically 0.*

Proof: Since $y_t(0,t) = 0$, y is solution of the system

$$\begin{cases} y_{tt} + y_{xxxx} - (a\,y_x)_x &= 0\\ y_{xx}(1,t) &= 0\\ M\xi'(t) - y_{xxx}(1,t) &= -a(1)\,y_x(1,t)\\ y_{xx}(0,t) &= 0\\ y_{xxx}(0,t) &= -\alpha\,y(0,t) + a(0)\,y_x(0,t) \end{cases}$$

and moreover $\alpha\,y(0,t) = C_0$ is constant. For all $T > 0$ we then have $0 = \displaystyle\int_0^T\!\!\int_0^1 (y_{xxxx} + y_{tt} - (a\,y_x)_x)\,dx\,dt$ which yields: $0 = \displaystyle\int_0^1 (y_t(x,T) - y_t(x,0))\,dx +$ $M y_t(1,T) - M y_t(1,0) + C_0\,T$ hence $C_0 = 0$ since it is immediate that $\displaystyle\int_0^1 (y_t(x,T) -$

$y_t(x,0))\,dx + My_t(1,T) - My_t(1,0)$ is bounded independently of T. Thus,

$$
\tag{3.3}
\begin{cases}
y_{tt} + y_{xxxx} - (ay_x)_x &= 0 \\
y_{xx}(1,t) &= 0 \\
M\xi'(t) - y_{xxx}(1,t) &= -a(1)\, y_x(1,t) \\
y_{xx}(0,t) &= 0 \\
y(0,t) &= 0 \\
y_{xxx}(0,t) &= a(0)\, y_x(0,t)\ .
\end{cases}
$$

We now use a multiplier method to deduce that $y \equiv 0$, namely the multiplier $(x-1)\, y_x$. An integration by parts leads to

$$
\tag{3.4}
\int_0^T\!\!\int_0^1 (x-1)\, y_x\, y_{tt}(x,t)\, dx\, dt = \int_0^1 \Big[(x-1)\, y_x\, y_t\Big]_0^T dx + \frac{1}{2}\int_0^T\!\!\int_0^1 y_t^2(x,t)\, dx\, dt
$$

$$
\tag{3.5}
\begin{aligned}
\int_0^T\!\!\int_0^1 (x-1)\, y_x (a\, y_x)_x(x,t)\, dx\, dt \;=\; & \frac{1}{2}\int_0^T a(0)\, y_x^2(0,t)\, dt \\
& + \frac{1}{2}\int_0^T\!\!\int_0^1 \psi(x)\, y_x^2(x,t)\, dx\, dt
\end{aligned}
$$

where $\psi(x) = (x-1)\, a'(x) - a(x)$. Let us notice that $\psi'(x) = (x-1)\, a''(x)$, which is nonnegative by (3.2). Thus,

$$
\tag{3.6}
\psi(x) \le \psi(1) = -a(1) < 0 \quad \text{for all } \ x \in [0,1]\ .
$$

Finally,

$$
\tag{3.7}
\begin{aligned}
\int_0^T\!\!\int_0^1 (x-1)\, y_x\, y_{xxxx}(x,t)\, dx\, dt \;=\; & \int_0^T a(0)\, y_x^2(0,t)\, dt \\
& + \frac{3}{2}\int_0^T\!\!\int_0^1 y_{xx}^2(x,t)\, dx\, dt\ .
\end{aligned}
$$

Putting together (3.4) to (3.7), the equality

$$
\int_0^T\!\!\int_0^1 (x-1)\, y_x \Big(y_{tt} + y_{xxxx} - (a\, y_x)_x\Big)\, dx\, dt \;=\; 0
$$

yields

$$
\tag{3.8}
\begin{aligned}
0 \;=\; & \int_0^1 \Big[(x-1)\, y_x\, y_t\Big]_0^T dx \;+\; \frac{1}{2}\int_0^T a(0)\, y_x^2(0,t)\, dt \;+ \\
& + \frac{1}{2}\int_0^T\!\!\int_0^1 y_t^2(x,t)\, dx\, dt \;+\; \frac{3}{2}\int_0^T\!\!\int_0^1 y_{xx}^2(x,t)\, dx\, dt \;-\; \frac{1}{2}\int_0^T\!\!\int_0^1 \psi(x)\, y_x^2(x,t)\, dx\, dt\ .
\end{aligned}
$$

From (2.11), it is clear that $\int_0^1 [(x-1)\, y_x\, y_t]_0^T\, dx$ is bounded independently of T. Thanks to (3.6), we then have

$$\int_0^T a(0)\, y_x^2(0,t)\, dt, \quad \int_0^T\!\!\int_0^1 y_t^2, \quad \int_0^T\!\!\int_0^1 y_{xx}^2 \quad \text{and} \quad \int_0^T\!\!\int_0^1 y_x^2 \tag{3.9}$$

are bounded independently of T.

To conclude, we use the following immediate identities:

$$\int_0^1\!\!\int_0^T y\, y_{tt}(x,t)\, dt\, dx = \int_0^1 [y\, y_t(x,t)]_0^T\, dx - \int_0^T\!\!\int_0^1 y_t^2(x,t)\, dx\, dt\ , \tag{3.10}$$

$$\int_0^1\!\!\int_0^T y\, y_{xxxx}(x,t)\, dt\, dx = \int_0^T [y\, y_{xxx}(x,t)]_0^1 dt + \int_0^T\!\!\int_0^1 y_{xx}^2(x,t)\, dx\, dt\ , \tag{3.11}$$

$$\int_0^1\!\!\int_0^T y(a\, y_x)_x(x,t)\, dt\, dx = \int_0^T a(1)\, y\, y_x(1,t)\, dt - \int_0^T\!\!\int_0^1 a(x)\, y_x^2(x,t)\, dx\, dt\ . \tag{3.12}$$

Putting together (3.10) to (3.12), the equality

$$\int_0^T\!\!\int_0^1 y\Big(y_{tt} + y_{xxxx} - (a\, y_x)_x\Big)\, dx\, dt \ = \ 0$$

leads to

$$\begin{aligned} 0 \ = \ &\int_0^1 [y\, y_t(x,t)]_0^T\, dx + \int_0^T [y\, y_{xxx}(x,t)]_0^1\, dt - \int_0^T a(1)\, y\, y_x(1,t)\, dt \ - \\ &- \int_0^T\!\!\int_0^1 y_t^2(x,t)\, dx\, dt + \int_0^T\!\!\int_0^1 y_{xx}^2(x,t)\, dx\, dt + \int_0^T\!\!\int_0^1 a(x)\, y_x^2(x,t)\, dx\, dt\ . \end{aligned} \tag{3.13}$$

Taking into account the boundary condition at $x = 1$, we have

$$\begin{aligned} \int_0^T \Big(y\, y_{xxx}(1,t) - a(1)\, y\, y_x(1,t)\Big)\, dt \ &= \int_0^T M y\, \xi'(t)\, (1,t)\, dt \\ &= [My\, y_t(1,t)]_0^T - M\int_0^T y_t^2(1,t)\, dt \end{aligned}$$

so (3.13) becomes

$$\begin{aligned} 0 \ = \ &\int_0^1 [y\, y_t(x,t)]_0^T\, dx \ + \ [My\, y_t(1,t)]_0^T \ - \ M\int_0^T y_t^2(1,t)\, dt \ - \\ &- \int_0^T\!\!\int_0^1 y_t^2(x,t)\, dx\, dt \ + \int_0^T\!\!\int_0^1 y_{xx}^2(x,t)\, dx\, dt \ + \int_0^T\!\!\int_0^1 a(x)\, y_x^2(x,t)\, dx\, dt\ . \end{aligned} \tag{3.14}$$

But (2.11) insures also the boundedness of $\displaystyle\int_0^1 [y\, y_t(x,t)]_0^T\, dx + [My\, y_t(1,t)]_0^T$ independently of T, so by (3.9), we deduce from (3.14) that

$$M \int_0^T y_t^2(1,t)\, dt \tag{3.15}$$

is bounded independently of T and so is $\displaystyle\int_0^T \|U(t)\|\, dt$. But as (2.13) implies that $\|U(t)\|$ is constant, we deduce that $\|U(t)\| = 0$, and Proposition 4 is proved. ∎

Proposition 5. *Let $U\colon\, t \mapsto (y(\cdot,t),\, z(\cdot,t),\, \eta(t),\, \xi(t))$ be a solution of (2.10), with the regularity provided by theorem 2, satisfying: $\forall\, t \geq 0\;\; \eta(t) = y_t(0,t) \leq 0$ and $f(y_t(0,t)) = 0$. Then $y_t(0,t) \underset{t\to+\infty}{\longrightarrow} 0$.*

Proof: According to the hypothesis, y is solution of the system

$$\begin{cases} y_{tt} + y_{xxxx} - (a\, y_x)_x &= 0 \\ y_{xx}(1,t) &= 0 \\ M\xi'(t) - y_{xxx}(1,t) &= -a(1)\, y_x(1,t) \\ y_{xx}(0,t) &= 0 \\ m\,\eta'(t) + y_{xxx}(0,t) &= -\alpha\, y(0,t) + a(0)\, y_x(0,t) \\ y_t(0,t) &\leq 0\ . \end{cases}$$

The function $t \mapsto y(0,t)$ is therefore nonincreasing and bounded according to (2.11). So $l = \lim\limits_{t\to\infty} y(0,t)$ exists. As in the proof of Proposition 4, the equality

$$\int_0^T\!\!\int_0^1 \Big(y_{xxxx} + y_{tt} - (a\, y_x)_x\Big)\, dx\, dt \;=\; 0$$

leads to

$$\int_0^1 [y_t(x,t)]_0^T\, dx \;+\; [My_t(1,t)]_0^T \;+\; [m\, y_t(0,t)]_0^T \;+ \int_0^T \alpha\, y(0,t)\, dt \;\;=\;\; 0\ .$$

The first three terms on the left hand side of this equality are bounded independently of T, and so is $\displaystyle\int_0^T \alpha\, y(0,t)\, dt$, which is only compatible with $l = 0$. Consequently, $y(0,t) \geq 0$ and $\displaystyle\int_0^{+\infty} \alpha\, y(0,t)\, dt < +\infty$.

Next, $\displaystyle\int_0^T y_t^2(0,t)\, dt \;=\; [y_t(0,t)\, y(0,t)]_0^T - \int_0^T y(0,t)\, \eta'(t)\, dt.$

As (2.11) shows that $\eta'(t)$ and $[y_t(0,t)\,y(0,t)]_0^T$ are bounded, then the same is true for $\displaystyle\int_0^T y_t^2(0,t)\,dt$ i.e.

$$\int_0^{+\infty} y_t^2(0,t)\,dt < +\infty \ . \tag{3.16}$$

Let us now consider a positive sequence $t_n \to +\infty$ and the sequence of functions $(v_n)_{n\in\mathbb{N}}$ defined on $[0,1]$ by $v_n(t) = y_t(0,t+t_n) = \eta(t+t_n)$. As η and η' are bounded on $\mathbb{R}^+$, we deduce that the sequence $(v_n)_{n\in\mathbb{N}}$ is bounded in $H^1(0,1)$. So we can extract a subsequence, still denoted $(v_n)_{n\in\mathbb{N}}$, uniformly convergent on $[0,1]$ to a continuous function v, thanks to the compactness of the injection $H^1(0,1) \to C^0[0,1]$. By the dominated convergence theorem, we have

$$\int_0^1 v^2(t)\,dt \ = \lim_{n\to+\infty} \int_0^1 v_n^2(t)\,dt \ = \lim_{n\to+\infty} \int_{t_n}^{t_n+1} y_t^2(0,t)\,dt \ ,$$

the latter integral being zero by virtue of (3.16). As a result, $v \equiv 0$ on [0,1]. The sequence $(t_n)_{n\in\mathbb{N}}$ being arbitrary, one can easily deduce that

$$y_t(0,t) \underset{t\to+\infty}{\longrightarrow} 0 \ . \ \blacksquare \tag{3.17}$$

Let us now achieve the proof of Theorem 3. According to Proposition 3, it is sufficient to establish that $\omega(U_0)$ is reduced to $\{0\}$. For any $W_0 \in \omega(U_0)$ let $W :\ t \mapsto S(t)\,W_0 = (y(\cdot,t), z(\cdot,t), \eta(t), \xi(t))$ be the solution of (2.10) such that $W(0) = W_0$. According to Proposition 3, $\|W(t)\|$ is constant, i.e. $y_t(0,t)\,f(y_t(0,t)) = 0$ by (2.13) which implies by (3.1): $y_t(0,t) \le 0 \quad \forall\, t \ge 0$. So (3.17) holds by Proposition 5.

Next, we choose $W_1 \in \omega(W_0)$, a sequence $t_n \to +\infty$ such that $S(t_n)(W_0) \to W_1$ and set $W_1(t) = S(t)\,W_1$, denoted by $(y_1(\cdot,t), z_1(\cdot,t), \eta_1(t), \xi_1(t))$. Since $S(t)$ is continuous on $\mathbf{H}$ for all $t \ge 0$, we have

$$W(t+t_n) = S(t)\,W(t_n) = S(t)\,(S(t_n)\,W_0) \underset{n\to+\infty}{\longrightarrow} S(t)\,W_1 = W_1(t) \ , \tag{3.18}$$

and in particular,

$$\eta(t+t_n) = y_t(0,t+t_n) \underset{n\to+\infty}{\longrightarrow} \eta_1(t) \ . \tag{3.19}$$

The comparison with (3.17) gives $\eta_1(t) = 0$ for all $t \ge 0$. Thus, $\eta_1 \equiv 0$ and by Proposition 4, we deduce that $W_1(t) = 0 \ \forall\, t \ge 0$, in particular, $W_1 = 0$.

To conclude, on the one hand we have: $\|W(t_n)\|$ is constant and equal to $\|W(0)\| = \|W_0\|$ and on the other hand, $\|W(t+t_n)\|$ converges to $\|W_1\| = 0$ as $n \to +\infty$. Thus, $W_0 = 0$, and the proof of Theorem 3 is now complete. ∎

For the second hybrid system (1.10), we shall assume that

$$a \text{ is affine on } [0,1] \,. \tag{3.20}$$

It is possible, but rather artificial, to relax this last assumption: some careful further computations would lead to the same conclusion if we had assumed instead of (3.20): $a \in H^3(0,1)$, $a'' \geq 0$, $a''' \leq 0$ a.e. and $a'(0) + a(0) > 0$.

Under hypothesis (3.20) and within the framework of (2.14)–(2.17), we can prove theorem 3 using the same approach as with operator (2.17). In fact, it is sufficient to prove Proposition 4, which is equivalent to showing that 0 is the only solution of the system

$$\begin{cases} y_{tt} + y_{xxxx} - (a\, y_x)_x = 0 \\ y_{xx}(1,t) = 0 \\ M\xi'(t) - y_{xxx}(1,t) = -a(1)\, y_x(1,t) \\ y_x(0,t) = 0 \\ y(0,t) = 0 \\ y_{xxx}(0,t) = 0 \end{cases} \tag{3.21}$$

because we can first easily prove that y_0 is a constant which is necessarily 0.

Fix any $T > 0$. First of all, with $\phi(x) = x - 1$, the equality

$$\int_0^T\!\!\int_0^1 \phi(x)\, y_x \Big(y_{xxxx} + y_{tt} - (a\, y_x)_x \Big)\, dx\, dt \;=\; 0$$

leads, after integration by parts, to

$$\begin{aligned} 0 \;=\; & \int_0^1 [\phi(x)\, y_x\, y_t(x,t)]_0^T\, dx \;+\; \frac{1}{2}\int_0^T\!\!\int_0^1 y_t^2 \;- \\ & -\; \frac{1}{2}\int_0^T y_{xx}^2(0,t)\, dt \;+\; \frac{3}{2}\int_0^T\!\!\int_0^1 y_{xx}^2 \;-\; \frac{1}{2}\int_0^T\!\!\int_0^1 (a'\phi - a\,\phi')\, y_x^2 \,. \end{aligned} \tag{3.22}$$

Since a is affine, $a'\phi - a\,\phi' \;=\; -a(1)$. Furthermore, according to (2.11), there exists a constant C depending only on $U(0)$ such that

$$\forall\, t \geq 0\,, \quad \|y_x(\cdot,t)\|_{L^2(0,1)} \text{ and } \|y_t(\cdot,t)\|_{L^2(0,1)} \leq C\,,$$

so that by (3.22),

$$\frac{1}{2}\int_0^T\!\!\int_0^1 y_t^2 \;+\; \frac{3}{2}\int_0^T\!\!\int_0^1 y_{xx}^2 \;+\; \frac{a(1)}{2}\int_0^T\!\!\int_0^1 y_x^2 \;\le\; \frac{1}{2}\int_0^T y_{xx}^2(0,t)\,dt \;+\; C\;. \tag{3.23}$$

Next, the equality

$$\int_0^T\!\!\int_0^1 y_{xxx}\Big(y_{xxxx} + y_{tt} - (ay_x)_x\Big)\,dx\,dt \;=\; 0$$

leads, after integration by parts, to

$$\begin{aligned} 0 \;=\; &\frac{1}{2}\int_0^T y_{xxx}^2(1,t)\,dt \;+\; \int_0^1 [y_{xxx}\,y_t(x,t)]_0^T\,dx \;+\; \frac{1}{2}\int_0^T y_{xt}^2(1,t)\,dt \\ &+\; \frac{1}{2}\int_0^T a(0)\,y_{xx}^2(0,t)\,dt \;+\; \frac{3}{2}\int_0^T\!\!\int_0^1 a' y_{xx}^2\;, \end{aligned} \tag{3.24}$$

with a' being constant. According to (2.11),

$$\mathbf{A}U = -\frac{dU}{dt} \;\in\; L^\infty\Big((0,+\infty),\, H^2(0,1)\times L^2(0,1)\times\mathbb{R}^2\Big)\;.$$

In particular, we deduce that $y_{xxxx} \in L^\infty((0,+\infty), L^2(0,1))$. The equality

$$y_{xxx}(x,t) \;=\; y_{xxx}(0,t) \;+\; \int_0^x y_{xxxx}(s,t)\,ds$$

implies

$$y_{xxx}^2(x,t) \;\le\; \int_0^1 y_{xxxx}^2(s,t)\,ds\;,$$

thus there exists a constant C, depending only on $U(0)$ such that

$$\forall\,(x,t)\,,\qquad y_{xxx}^2(x,t) \le C\;. \tag{3.25}$$

Therefore (3.24) and (3.25) imply in particular

$$\frac{1}{2}\int_0^T a(0)\,y_{xx}^2(0,t)\,dt \;\le\; -\frac{3}{2}\int_0^T\!\!\int_0^1 a' y_{xx}^2 + C\;. \tag{3.26}$$

The comparison between (3.23) and (3.26) yields

$$\frac{1}{2}\int_0^T\!\!\int_0^1 y_t^2 \;+\; \frac{3}{2}\int_0^T\!\!\int_0^1 y_{xx}^2 \;+\; \frac{a(1)}{2}\int_0^T\!\!\int_0^1 y_x^2 \;\le\; \frac{-3}{2\,a(0)}\int_0^T\!\!\int_0^1 a' y_{xx}^2 + C\;. \tag{3.27}$$

We have $a(0) + a' = a(1)$ and (3.27) gives

$$\frac{1}{2}\int_0^T\!\!\int_0^1 y_t^2 \;+\; \frac{3\,a(1)}{2\,a(0)}\int_0^T\!\!\int_0^1 y_{xx}^2 \;+\; a(1)\int_0^T\!\!\int_0^1 y_x^2 \;\le\; C\;. \tag{3.28}$$

Consequently,

$$\int_0^T\!\!\int_0^1 y_t^2 + \int_0^T\!\!\int_0^1 y_{xx}^2 + \int_0^T\!\!\int_0^1 y_x^2 \;\le\; C\;. \tag{3.29}$$

Finally, the equality

$$\int_0^T\!\!\int_0^1 y\Big(y_{xxxx} + y_{tt} - (a\,y_x)_x\Big) \;=\; 0$$

combined with (3.29) gives easily

$$\int_0^T y_t^2(1,t)\,dt \;\le\; C\;.$$

We deduce that

$$\int_0^T E(t)\,dt \;\le\; C$$

independently of T and we conclude as in (3.15).

4 – Stabilization under a relaxed hypothesis on monotonicity

In this last section, we prove that we can get rid of global monotonicity for f to stabilize both systems (1.9) and (1.10), provided that f is more regular than in (2.5) and is locally monotone in a neighbourhood of 0. The method used here can also be adapted to the hybrid system studied in [2], where stabilization with no global monotonicity was established by a technique based on some Riemann's invariants for a wave equation, which does not seem to be applicable to the beam equation.

Moreover, the monotonicity of the operator $\mathbf{A}$ followed from the monotonicity of f, so the well-posedness may not hold and no contraction property can *a priori* be expected.

Let us describe this last section. First, we prove using an a priori estimate that the systems are well posed for a large class of feedback laws. Then, we show that despite any contraction property, $\|\mathbf{A}U(t)\|$ is bounded thanks to an energy inequality for $\frac{dU}{dt}$ obtained by means of a weak formulation of a non autonomous equation of which $\frac{dU}{dt}$ is solution.

From now on, we make the following assumptions:

$$\begin{aligned} &\text{(i)}\;\; f\in W^{2,\infty}_{\rm loc}(\mathbb{R})\;,\\ &\text{(ii)}\;\; s\,f(s)<0 \quad \text{for all}\;\; s\neq 0\;,\\ &\text{(iii)}\;\; \exists\,\varepsilon{>}0\,/\;\; f'(s)\le 0\;\; \text{for all}\; s\in[-\varepsilon,\varepsilon]\;. \end{aligned} \tag{4.1}$$

All notations of section 2 will be kept and only system (1.9) will be studied. *Mutatis mutandis*, the same results can be obtained for system (1.10). We still work in the Hilbert space $\mathbf{H}$ defined by (2.1) and (2.2) and more or less the same operator $\mathbf{A}$, but considered as a lipschitz perturbed maximal monotone operator. More precisely, let $\mathbf{A}_0$ be the operator with domain defined by (2.3) and by

$$\begin{aligned}(4.2)\qquad \mathbf{A}_0(U) = \Big[&-z,\ y_{xxxx}-(a\,y_x)_x,\\ &\frac{1}{m}\Big(y_{xxx}(0)+\alpha\,y(0)-a\,y_x(0)\Big),\ -\frac{1}{M}\Big(y_{xxx}(1)-a\,y_x(1)\Big)\Big]\ .\end{aligned}$$

This operator corresponds to the choice $f\equiv 0$ in (2.4). In particular,

$$\text{(4.3)}\qquad \mathbf{A}_0 \ \text{ is maximal monotone on } \ \mathbf{H}$$

and (2.6) yields

$$\text{(4.4)}\qquad \langle \mathbf{A}_0 U, U\rangle = 0\ .$$

The absence of f in (4.2) is compensated by the introduction of the operator $\mathbf{B}$ defined on $\mathbf{H}$ by

$$\text{(4.5)}\qquad \mathbf{B}U = \Big(0,\, 0,\, -\frac{1}{m}\,f(\eta),\, 0\Big)\ .$$

Now, set

$$\text{(4.6)}\qquad \mathbf{A} = \mathbf{A}_0 + \mathbf{B}$$

with domain $D(\mathbf{A})$ defined by (2.3), which is not necessarily monotone.

We consider the evolution equation

$$\text{(4.7)}\qquad \frac{dU}{dt} + \mathbf{A}U = 0\,, \qquad U(0) = U_0\ ,$$

which gives, as in section 2, at least formal solutions for system (1.9).

For the existence of solutions for equation (4.7), we recall the following result (see [1] or [3]):

Theorem 4. *Let $\mathbf{H}$ be a Hilbert space, $\mathbf{A}_1$ a maximal monotone operator on $\mathbf{H}$ and $\mathbf{L}$ a lipschitzian operator defined on $\mathbf{H}$. Then for all $U_0\in D(\mathbf{A}_1)$, there exists a unique function $U\colon\ [0,+\infty[\ \to \mathbf{H}$ such that:*

$$\text{(4.8)}\qquad \begin{aligned}&\text{(i)}\quad U(0)=U_0\ ,\\ &\text{(ii)}\quad \forall\, t\geq 0,\ \ U(t)\in D(\mathbf{A}_1)\ ,\\ &\text{(iii)}\quad \forall\, T>0,\ \ U\in W^{1,\infty}((0,T),\mathbf{H})\ ,\\ &\text{(iv)}\quad \frac{dU}{dt}(t)+\mathbf{A}_1 U(t)+\mathbf{L}\,U(t)=0 \qquad \text{a.e. } t\in[0,+\infty[\ .\ \blacksquare\end{aligned}$$

We can then state:

Theorem 5. *The operator* $\mathbf{A}$ *is the infinitesimal generator of a semigroup* $(S(t))_{t\in\mathbb{R}^+}$ *of continuous operators on* $\mathbf{H}$. *Therefore, for all* $U_0 \in D(\mathbf{A})$, *equation (4.7) admits a unique solution* $t \mapsto U(t)$ *satisfying*

$$\forall T>0, \quad U \in W^{1,\infty}((0,T),\mathbf{H}) \cap L^\infty((0,T),D(\mathbf{A})) \tag{4.9}$$

given by $U(t) = S(t)\,U_0$. *Moreover,*

$$t \mapsto \|U(t)\| \quad \textit{is non increasing}\ . \tag{4.10}$$

Proof: As in [2], it is based on an *a priori* estimate: given any $U_0 \in D(\mathbf{A})$, a straightforward computation shows that every solution $U = (y, z, \eta, \xi)$ of (4.7) satisfies

$$\left\langle \frac{dU}{dt}, U \right\rangle = \langle -\mathbf{A}_0 U, U\rangle - \langle \mathbf{B}U, U\rangle = \frac{1}{m}\, f(\eta)\,\eta \le 0$$

so $t \mapsto \frac{1}{2}\|U(t)\|^2$ is non increasing. In particular,

$$|\eta(t)| \le \frac{1}{\sqrt{m}}\, \|U(t)\| \le K = \frac{1}{\sqrt{m}}\, \|U_0\|\ .$$

Consequently, the knowledge of f on the interval $I_K = [-K, K]$ is sufficient to solve (4.7). In other words, given any function $\bar f$ coinciding with f on I_K and the operator $\overline{\mathbf{B}}$ defined on $\mathbf{H}$ by $\overline{\mathbf{B}}U = (0, 0, -\frac{1}{m}\bar f(\eta), 0)$, then any solution $\overline{U}$ of

$$\frac{d\overline{U}}{dt} + \mathbf{A}_0 \overline{U} + \overline{\mathbf{B}}\,\overline{U} = 0\,, \qquad \overline{U}(0) = U_0\ , \tag{4.11}$$

is a solution of (4.7) and *vice versa*.

From (4.1)(i), there exists a constant $c_K \ge 0$ such that the function $f_K\colon\, s \mapsto f(s) - c_K s$ is decreasing on I_K. Let us consider any function $\bar f_K$ continuous and decreasing on $\mathbb{R}$ extending f_K and set $\bar f(s) = \bar f_K(s) + c_K s$, which by construction coincides with f on I_K. With that choice of $\bar f$, we may now consider equation (4.10). In fact, $\overline{\mathbf{B}} = \mathbf{B}_1 + \mathbf{L}$, where

$$\mathbf{B}_1 U = \left(0,\, 0,\, \frac{1}{m}\,\overline{f}_K(\eta),\, 0\right)\,, \tag{4.12}$$

$$\mathbf{L}\,U = \left(0,\, 0,\, \frac{1}{m}\, c_K\,\eta,\, 0\right)\,. \tag{4.13}$$

The operator $\mathbf{L}$ is lipschitz on $\mathbf{H}$ and it is easy to see that $\mathbf{B}_1$ is maximal monotone on $\mathbf{H}$; thus, $\mathbf{A}_0+\overline{\mathbf{B}} = (\mathbf{A}_0+\mathbf{B}_1)+\mathbf{L}$ and according to [1], corollary 2.7, p. 36, the operator $\mathbf{A}_1 = \mathbf{A}_0 + \mathbf{B}_1$, with domain $D(\mathbf{A})$, is maximal monotone on $\mathbf{H}$. The existence, uniqueness and regularity of a solution for the equation (4.10) follows from theorem 4. ∎

The main result of this paper is the following:

Theorem 6. *Under assumptions (4.1), for all U_0 in $D(\mathbf{A})$, the solution U of (4.7) satisfies*

$$\|U(t)\| \underset{t\to\infty}{\longrightarrow} 0 \ .$$

Let us first analyse the situation: (4.9) gives only *local in time* estimations of the solution $\{U(t),\ t\geq 0\}$ for the graph norm of $\mathbf{A}$, so the orbits may not be precompact for the topology of $\mathbf{H}$; consequently, the ω-limit sets may be empty or even if they are not, not included in $D(\mathbf{A})$ (this lack of regularity does not allow to make any use of a method based on multipliers for the resolution of the problem of uniqueness that occurs, even if La Salle's principle could be employed).

We prove that despite the lack of the contraction property of the semigroup, the orbits $\{U(t),\ t\geq 0\}$ are bounded for the graph norm. The method used in the sequel is summarized below:

(**i**) equations (1.9) are formally differentiated with respect to t and it is proved that the evolution problem which is associated to this new system has strong solutions as well as weak solutions in the sense of uniform limits of strong solutions; a functional related to the energy is associated to these two kinds of solutions;

(**ii**) weak solutions satisfy a variational formulation, also satisfied by $\frac{dU}{dt}$;

(**iii**) a uniqueness argument for this formulation allows to associate to $\frac{dU}{dt}$ the functional considered in (i); using Gronwall's lemma, boundedness of this energy is shown and then, the conclusion follows as in section 3.

Let us now detail the proof of theorem 6. $U_0 \in D(\mathbf{A})$, $T>0$ are given as well as the solution $t\mapsto U(t)$ of (4.7) where $U(t)$ is denoted $(y(\cdot,t), y_t(\cdot,t), \eta(t), \xi(t))$, with $\eta(t) = y_t(0,t)$, $\xi(t) = y_t(1,t)$. We set

$$g(t) = f'(y_t(0,t)) \ . \tag{4.14}$$

According to (4.1)(i) and (4.8), we know that

$$g \ \text{ is lipschitz on } \ [0,T] \ . \tag{4.15}$$

Differentiating (1.9) with respect to t, we see that $v = y_t$ satisfies formally

$$\begin{cases} v_{tt} + v_{xxxx} - (a\,v_x)_x &= 0 \\ v_{xx}(1,t) &= 0 \\ M v_{tt}(1,t) - v_{xxx}(1,t) &= -a(1)\,v_x(1,t) \\ v_{xx}(0,t) &= 0 \\ m\,v_{tt}(0,t) + v_{xxx}(0,t) &= v_t(0,t)\,g(t) - \alpha\,v(0,t) + a(0)\,v_x(0,t)\ . \end{cases} \tag{4.16}$$

It must be pointed out that $g(t) = f'(y_t(0,t))$ is a *coefficient* of this system.

As in section 2, we transform (4.16) into an abstract evolution equation by setting $V(t) = (v(\cdot,t), v_t(\cdot,t), v_t(0,t), v_t(1,t))$, which leads to the consideration of

$$\begin{array}{rccc} F: & \mathbb{R}^+ \times \mathbf{H} & \longrightarrow & \mathbf{H} \\ & \big(t,(v,w,\theta,\chi)\big) & \mapsto & \big(0,0,-\frac{1}{m}g(t).\theta,0\big)\ . \end{array} \tag{4.17}$$

As one can easily check up, any solution $t \mapsto V(t) = (v(\cdot,t), w(\cdot,t), \theta(t), \chi(t))$ of the equation

$$\frac{dV}{dt}(t) + \mathbf{A}_0 V(t) + F(t,V(t)) \;=\; 0 \tag{4.18}$$

is such that $(x,t) \mapsto v(x,t)$ satisfies formally (4.16). In fact, (4.18) is the equation obtained by formal differentiation of (4.7). In the sequel, we show that $\frac{dU}{dt}$ is a solution of (4.18) in a certain sense.

It is clear, from (4.15), that F is lipschitz with respect to both variables. Therefore,

Proposition 6.

(**i**) *For all $V_0 \in D(\mathbf{A})$, equation (4.18) has on $[0,T]$ a unique strong solution $t \mapsto V(t) \in W^{1,1}((0,T),\mathbf{H})$ such that $V(0) = V_0$. Moreover, there exists a constant $K > 0$ such that for all $(V_0,W_0) \in D(\mathbf{A})^2$, the corresponding solutions $t \mapsto V(t)$, $W(t)$ satisfy*

$$\|V(t) - W(t)\| \;\le\; K\,\|V_0 - W_0\|\ . \tag{4.19}$$

(**ii**) *There exists a family $(S(t))_{t\ge 0}$ of operators defined and continuous on $\mathbf{H}$ such that for all $V_0 \in D(\mathbf{A}_0)$, $t \mapsto S(t)\,V_0$ is the unique strong solution of (4.18) which satisfies $V(0) = V_0$.*

Proof: (**i**) is a consequence of [7] (theorem 1.6 p. 189); (**ii**) follows from (4.19) and the density of $D(\mathbf{A})$ in $\mathbf{H}$. ∎

Next, we define a functional related to the energy for these two kinds of solutions:

Proposition 7. *For all* $V_0 \in \mathbf{H}$, *set* $E[V_0]\colon\, t \mapsto \frac{1}{2}\|S(t)V_0\|^2$. *Then* $E[V_0]$ *is differentiable on* $[0,T]$ *and denoting* $S(t)V_0 = (v(\cdot,t), w(\cdot,t), \theta(t), \chi(t))$, *one has*

$$E[V_0]'(t) = \theta^2(t)\, g(t) \tag{4.20}$$

Proof: We first establish the result for $V_0 \in D(\mathbf{A})$, which is obtained by an integration by parts as in Proposition 1. When $V_0 \in \mathbf{H}$, $t \mapsto S(t)V_0$ is a uniform limit on $[0,T]$ of a sequence $(V_n(\cdot))_{n\geq 1}$ of strong solutions. Obviously, $E[V_n](t) \underset{n\to\infty}{\longrightarrow} E[V_0](t)$ uniformly on $[0,T]$. Noting $V_n(t) = (v_n(\cdot,t), w_n(\cdot,t), \theta_n(t), \chi_n(t))$, we know that $E[V_n]'(t) = \theta_n^2(t)\, g(t)$. As $\theta_n(t) \underset{n\to\infty}{\longrightarrow} \theta(t)$ uniformly $[0,T]$, we deduce that it is also true for their squares (since $(\theta_n(t))$ is bounded). Proposition 7 follows from an elementary result about sequence of differentiable functions. ∎

Let us consider now a variational formulation of (4.18).

Proposition 8. *For all* $V_0 \in \mathbf{H}$, $V(t) = S(t)V_0$ *satisfies:*

$$\begin{cases} \forall\, W \in D(\mathbf{A}_0^*)\,, \quad t \mapsto \langle V(t), W\rangle \in W^{1,1}((0,T),\mathbb{R}) \ \text{ and} \\ \dfrac{d}{dt}\langle V(t), W\rangle + \langle V(t), \mathbf{A}_0^* W\rangle + \langle F(t, V(t)), W\rangle \;=\; 0 \quad \textit{for a.e. } t\,. \end{cases} \tag{4.21}$$

Proof: The proposition is obvious when $V_0 \in D(\mathbf{A})$. When $V_0 \in \mathbf{H}$, as in Proposition 7, we choose a sequence $(V_n(\cdot))_{n\geq 1}$ of strong solutions which converges uniformly to $V(\cdot)$ on $[0,T]$.

Next, for all $W \in D(\mathbf{A}^*)$ let us consider $\phi[W]\colon\, t \mapsto \langle V(t), W\rangle$ and $\phi_n[W]\colon$ $t \mapsto \langle V_n(t), W\rangle$. It is clear that $\phi_n[W](t) \underset{n\to\infty}{\longrightarrow} \phi[W](t)$ uniformly on $[0,T]$.

On the other hand, we can easily check that

$$\langle V_n(t), \mathbf{A}_0^* W\rangle + \langle F(t, V_n(t)), W\rangle \underset{n\to\infty}{\longrightarrow} \langle V(t), \mathbf{A}_0^* W\rangle + \langle F(t, V(t)), W\rangle$$

uniformly on $[0,T]$ by the lipschitz property of F. As a result, $(\phi_n[W])_{n\geq 1}$ is a Cauchy sequence in $W^{1,1}((0,T),\mathbb{R})$. Since uniform convergence on $[0,T]$ implies convergence in $L^1(0,T)$, we deduce that: $\phi[W] \in W^{1,1}((0,T),\mathbb{R})$ and (4.21) holds a.e. $t \in [0,T]$. ∎

We consider now the solution $t \mapsto U(t) = (y(\cdot,t), y_t(\cdot,t), \eta(t), \xi(t))$ of (4.7) which was fixed previously.

Proposition 9. *$V(t)=\frac{dU}{dt}(t)$ satisfies the variational formulation (4.21).*

Proof: For all $t\in[0,T]$ and all $W\in D(\mathbf{A}^*)$, one has, according to (4.7),

$$\left\langle \frac{dU}{dt}(t),W\right\rangle + \langle U(t),\mathbf{A}_0^* W\rangle + \langle \mathbf{B}U(t),W\rangle \;=\; 0\ .$$

Obviously, $t\mapsto\langle U(t),\mathbf{A}_0^*W\rangle$ is differentiable, with derivative

$$\left\langle \frac{dU}{dt}(t),\mathbf{A}_0^* W\right\rangle\ .$$

Let us denote by η_W the third component of W; thus,

$$\langle \mathbf{B}U(t),W\rangle = -\frac{1}{m}\,f(\eta(t))\,\eta_W$$

which is clearly differentiable, with derivative

$$-\frac{1}{m}\,\eta'(t)\,g(t)\,\eta_W$$

which is precisely the value of $\langle F(t,V(t)),W\rangle$, since the third component of $V(t)=\frac{dU}{dt}$ is $\eta'(t)$.

Finally, $t\mapsto\langle\frac{dU}{dt}(t),W\rangle$ is differentiable on $[0,T]$ as a sum of differentiable applications, and (4.21) holds. ∎

The last point concerns the uniqueness of the variational formulation:

Proposition 10. *Let V_1 and V_2 be two solutions of (4.21) such that $V_1(0)=V_2(0)$; then $V_1\equiv V_2$.*

For the proof, we can refer to [8], Proposition 3.4, p. 71, where a link between *mild* and *weak* solutions is established. ∎

Now we are ready to prove theorem 6: according to Propositions 9 and 8, $\frac{dU}{dt}(t)=S(t)\,(\frac{dU}{dt}(0))$ and from Proposition 7, $\psi(t)=\frac{1}{2}\,\|\frac{dU}{dt}(t)\|^2$ is differentiable and $\psi'(t)=\eta'^2(t)\,g(t)$. Let us write

$$\begin{aligned}\psi(t)\;&=\;\psi(0)\;+\int_0^t\psi'(\tau)\,d\tau\\ &=\;\psi(0)\;+\int_0^t\mathbf{1}_{\{|\eta|>\varepsilon\}}\,\eta'^2(\tau)\,g(\tau)\,d\tau\;+\int_0^t\mathbf{1}_{\{|\eta|\le\varepsilon\}}\,\eta'^2(\tau)\,g(\tau)\,d\tau\ ,\end{aligned}$$

where $\mathbf{1}_{\{|\eta|\le\varepsilon\}}$ denotes the characteristic function of the set $\{\tau\ /\ |\eta(\tau)|\le\varepsilon\}$, on which $g(\tau)\le 0$ by (4.1)(iii). Consequently,

$$\psi(t)\;\le\;\psi(0)+\int_0^t\mathbf{1}_{\{|\eta|>\varepsilon\}}\,\eta'^2(\tau)\,g(\tau)\,d\tau\ . \tag{4.22}$$

From the definition of the norm on $\mathbf{H}$ (see (2.2)), we have $\eta'^2(\tau) \le 2m\,\psi(\tau)$; on the other hand, we write

$$g(\tau) = f'(\eta(\tau)) = \frac{f'(\eta(\tau))}{f(\eta(\tau))\,\eta(\tau)}\, f(\eta(\tau))\,\eta(\tau)$$

and we introduce

$$c = \sup \left\{ \left| \frac{f'(s)}{f(s)s} \right|,\ s \;/\; \ \varepsilon \le |s| \le \frac{1}{\sqrt{m}}\, \|U_0\| \right\} .$$

Observing that $|\eta(\tau)| \le \frac{1}{\sqrt{m}}\, \|U_0\|$, we deduce that

$$\psi(t) \ \le\ \psi(0) + \ 2\,m\,c \int_0^t \psi(\tau)\, |f(\eta(\tau))\, \eta(\tau)|\ d\tau\ ,$$

and since (2.13) yields

$$\int_0^{+\infty} f(\eta(\tau))\,\eta(\tau)\, d\tau \ <\ +\infty\ ,$$

we deduce from Gronwall's lemma that

$$\psi(t) \ \le\ \psi(0)\, e^{2mc \int_0^t f(\eta(\tau))\,\eta(\tau)\, d\tau}$$

which implies that $\psi(t)$ is bounded independently of t.

Thus, $\|\frac{dU}{dt}(t)\|$ is bounded, $\|\mathbf{B}\,U(t)\|$ is also clearly bounded so the same property holds for $\|\mathbf{A}_0\, U(t)\|$. The orbit $\{U(t),\ t \ge 0\}$ is then bounded for the graph norm and precompact for the topology of $\mathbf{H}$. The existence of ω-limit points for that orbit is guaranteed and these points belong to $D(\mathbf{A})$, because the graph of $\mathbf{A} = \mathbf{A}_0 + \mathbf{B}$ possesses the weak-strong closure property since $\mathbf{B}$ is a continuous operator on $\mathbf{H}$ and $\mathbf{A}_0$ is maximal monotone. We can proceed further and conclude as in section 3. ∎

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