K_W DOES NOT IMPLY K_W

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Abstract: We prove that the cyclic monotonically normal space T of M.E. Rudin is a K_W-space which is not a K_W-space. This answers a question in [3]. In order to do this, we first prove that if a space X has D*(R; ≤) then X is a K_W-space (it is well known that X is also a K_1-space; this does not necessarily mean that X is a K_1-W-space.).

Theorem 1. If a space (X, τ) has the property D*(R; ≤) then X is a K_W-space.

Proof: By Theorem 10 of [4] (i.e. D*(R; ≤) if and only if D*(R; ≤; cch)), let γ: C*(F) → C*(X) be a monotone extender such that γ(a_F) = a_X for each a ∈ R. Then for each U ∈ τ|F, let

\[ \mu(U) = \bigcup \left\{ (f)^{-1}([-\infty, 1]) \mid f \in C(F, [-2, 2]), \ f(F-U) \subset \{2\} \right\}, \]

\[ v(U) = \bigcup \left\{ (f)^{-1}([-1, \infty]) \mid f \in C(F, [-2, 2]), \ f(F-U) \subset \{-2\} \right\}, \]

\[ k(U) = \mu(U) \cup v(U). \]

If U ∈ τ|F and z ∈ U, then there exists f ∈ C(F, [-2, 2]) such that f(z) = -2 and f(F-U) ⊂ \{2\}. Since γ is an extender, we get that F ∩ μ(U) = U; similarly, F ∩ v(U) = U. Hence, F ∩ k(U) = U, for each U ∈ τ|F. Clearly, k(F) = X and k(∅) = ∅, because γ(±2_F) = ±2_X.

It is obvious that k(U) ⊂ k(V) whenever U ⊂ V.

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Next, we prove that if $U \cup V = F$ then $k(U) \cup k(V) = X$ (W log, let us assume that $U \neq F \neq V$). Let $x \in X$ and suppose that $x \notin \mu(U)$. Then, for each $f \in C(F, [-2, 2])$ such that $f(F - U) = 2$, we get that $\gamma(f)(x) \geq 1$. Pick $h \in C(F, [-2, 2])$ such that $h(F - V) = -2$ and $h(F - U) = 2$ (recall that $F$ is normal). It follows that $\gamma(h)(x) \geq 1$, which implies that $x \in v(V)$. Similarly, if $x \notin v(V)$ then $x \in \mu(U)$. Consequently, we get that $x \in k(U) \cup k(V)$, as required.

Finally, we prove that, for each $U \in \tau F$, $k(U) \cap F = \overline{U}$. Suppose there is $p \in F$ such that $p \in k(U)$ and $p \notin U$. Pick $h : F \to [-2, 2]$ such that $h(U) = -2$ and $h(p) = 2$. Then $h \leq f$ for all $f : F \to [-2, 2]$ such that $f(F - U) = 2$, which implies that $\mu(U) \subset \gamma(h)^{-1}([-\infty, 1])$. Since $\gamma(h)^{-1}([-\infty, 1]) \cap \gamma(h)^{-1}(1, \infty] = \emptyset$ and $p \in \gamma(h)^{-1}(1, \infty]$, we get that $p \notin \overline{\mu(U)}$. Similarly, $p \notin \overline{v(U)}$, and this proves that $p \notin \overline{k(U)}$. Therefore, $k(U) \cap F = \overline{U}$. This completes the proof. 

**Theorem 2.** If a space $(X, \tau)$ is a $K_W^*$-space then, for each closed subspace $F$ of $X$ there exists a function $k : \tau F \to \tau$ such that

(i) $F \cap k(U) = U$, for each $U \in \tau F$, $k(F) = (X)$, $k(\emptyset) = \emptyset$;

(ii) $k(U) \subset k(V)$ whenever $U \subset V$;

(iv) $U, V \in \tau F$, $\overline{U \cap V} = U \cap V$ implies $k(U \cap V) = k(U) \cap k(V)$;

Proof: Let $v : \tau F \to \tau$ be a $K_W^*$-function and define $k : \tau F \to \tau$ by

$$k(U) = U \cup \left( X - \left[ F \cup \overline{v(F - U)} \right] \right).$$

From the proof of Theorem 4 of [3], we immediately get that $k$ satisfies (i), (ii) and (iv). To verify (iii), note that

$$k(U) \cap k(V) = \left[ U \cup \left( X - \left[ F \cup \overline{v(F - U)} \right] \right) \right] \cap \left[ V \cup \left( X - \left[ F \cup \overline{v(F - V)} \right] \right) \right]
= (U \cap V) \cup \left( X - \left[ F \cup \overline{v(F - U)} \right] \right) \cap \left( X - \left[ F \cup \overline{v(F - V)} \right] \right)
= (U \cap V) \cup \left( X - \left[ F \cup \overline{v(F - U)} \cup \overline{v(F - V)} \right] \right)$$
We conjecture that the converse of Theorem 2 is false and we have not been able to find a characterization of $K_W^*$-spaces analogous to the characterization of $K_W$-spaces which appears in Theorem 4 of [3].

**Theorem 3.** There is a $K_W$-space $T$ which is not a $K_W^*$-space.

**Proof:** The space $T$ is the space described by M.E. Rudin in [6]. We already know from Theorem 1 (recall that monotonically normal spaces have $D^*(\mathbb{R}; \leq)$) that $T$ is a $K_W$-space.

Assuming that $T$ is a $K_W^*$-space, let $k : \tau|F \rightarrow \tau$ satisfy the conditions of Theorem 3(b). Since the sets $U_{xi}$ and $U_{rx_i}$ defined on p. 305 of [6] are easily seen to be clopen, then we get that

$$
\bigcap_{i<3} U_{xi} = \bigcap_{i<3} U_{xi}^c \quad \text{and} \quad k \left( \bigcap_{i<3} U_{xi} \cap Y \right) = \bigcap_{i<3} k(U_{xi} \cap Y)
$$

and, similarly,

$$
k \left( U_{rx,i} \cap U_{rx,(j-1)} \cap Y \right) = k(U_{rx,i} \cap Y) \cap k(U_{rx,(j-1)} \cap Y).
$$

Consequently, M.E. Rudin’s argument, verbatim, also proves that the above $k$ cannot exist, a contradiction. This completes the proof. □
REFERENCES


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