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K_W DOES NOT IMPLY K_W^*

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Abstract: We prove that the cyclic monotonically normal space T of M.E. Rudin is a K_W-space which is not a K_W^*-space. This answers a question in [3]. In order to do this, we first prove that if a space X has $D^*(\mathbb{R};\leq)$ then X is a K_W-space (it is well known that X is also a K_1-space; this does not necessarily mean that X is a K_{1W}-space.).

Theorem 1. *If a space (X,τ) has the property $D^*(\mathbb{R};\leq)$ then X is a K_W-space.*

Proof: By Theorem 10 of [4] (i.e. $D^*(\mathbb{R};\leq)$ if and only if $D^*(\mathbb{R};\leq;cch)$), let $\gamma\colon C^*(F)\to C^*(X)$ be a monotone extender such that $\gamma(a_F)=a_X$ for each $a\in\mathbb{R}$. Then for each $U\in\tau|F$, let

$$\mu(U) \;=\; \bigcup\Big\{\gamma(f)^{-1}(]-\infty,1[) \;\mid\; f\in C(F,[-2,2]),\;\; f(F-U)\subset\{2\}\Big\}\;,$$

$$v(U) \;=\; \bigcup\Big\{\gamma(f)^{-1}(]-1,\infty[) \;\mid\; f\in C(F,[-2,2]),\;\; f(F-U)\subset\{-2\}\Big\}\;,$$

$$k(U) \;=\; \mu(U)\cup v(U)\;.$$

If $U\in\tau|F$ and $z\in U$, then there exists $f\in C(F,[-2,2])$ such that $f(z)=-2$ and $f(F-U)\subset\{2\}$. Since γ is an extender, we get that $F\cap\mu(U)=U$; similarly, $F\cap v(U)=U$. Hence, $F\cap k(U)=U$, for each $U\in\tau|F$. Clearly, $k(F)=X$ and $k(\emptyset)=\emptyset$, because $\gamma(\pm 2_F)=\pm 2_X$.

It is obvious that $k(U)\subset k(V)$ whenever $U\subset V$.

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Next, we prove that if $U\cup V=F$ then $k(U)\cup k(V)=X$ ($\underline{W\log}$, let us assume that $U\neq F\neq V$). Let $x\in X$ and suppose that $x\notin\mu(U)$. Then, for each $f\in C(F,[-2,2])$ such that $f(F-U)=2$, we get that $\gamma(f)(x)\geq 1$. Pick $h\in C(F,[-2,2])$ such that $h(F-V)=-2$ and $h(F-U)=2$ (recall that F is normal). It follows that $\gamma(h)(x)\geq 1$, which implies that $x\in v(V)$. Similarly, if $x\notin v(V)$ then $x\in\mu(U)$. Consequently, we get that $x\in k(U)\cup k(V)$, as required.

Finally, we prove that, for each $U\in\tau|F$, $\overline{k(U)}\cap F=\overline{U}$: Suppose there is $p\in F$ such that $p\in\overline{k(U)}$ and $p\notin\overline{U}$. Pick $h\colon F\to[-2,2]$ such that $h(\overline{U})=-2$ and $h(p)=2$. Then $h\leq f$ for all $f\colon F\to[-2,2]$ such that $f(F-U)=2$, which implies that $\mu(U)\subset\gamma(h)^{-1}(]-\infty,1[)$. Since $\gamma(h)^{-1}(]-\infty,1[)\cap\gamma(h)^{-1}(]1,\infty[)=\emptyset$ and $p\in\gamma(h)^{-1}(]1,\infty[)$, we get that $p\notin\overline{\mu(U)}$. Similarly, $p\notin\overline{v(U)}$, and this proves that $p\notin\overline{k(U)}$. Therefore, $\overline{k\,U}\cap F=\overline{U}$. This completes the proof. ∎

Theorem 2. *If a space (X,τ) is a K_W^*-space then, for each closed subspace F of X there exists a function $k\colon\tau|F\to\tau$ such that*

(**i**) $F\cap k(U)=U$, *for each* $U\in\tau|F$, $k(F)=(X)$, $k(\emptyset)=\emptyset$;

(**ii**) $k(U)\subset k(V)$ *whenever* $U\subset V$;

(**iv**) $U,V\in\tau|F$, $\overline{U\cap V}=\overline{U}\cap\overline{V}$ *implies* $k(U\cap V)=k(U)\cap k(V)$;

(**iv**) $\overline{k(U)}\cap F=\overline{U}$.

Proof: Let $v\colon\tau|F\to\tau$ be a K_W^*-function and define $k\colon\tau|F\to\tau$ by

$$k(U)\;=\;U\cup\Big(X-\Big[F\cup\overline{v(F-\overline{U})}\,\Big]\Big)\;.$$

From the proof of Theorem 4 of [3], we immediately get that k satisfies (i), (ii) and (iv). To verify (iii), note that

$$\begin{aligned}
k(U)\cap k(V)\;&=\;\Big[U\cup\Big(X-\Big[F\cup\overline{v(F-\overline{U})}\,\Big]\Big)\Big]\cap\Big[V\cup\Big(X-\Big[F\cup\overline{v(F-\overline{V})}\,\Big]\Big)\Big]\\
&=\;(U\cap V)\cup\Big[\Big(X-\Big[F\cup\overline{v(F-\overline{U})}\,\Big]\Big)\cap\Big(X-\Big[F\cup\overline{v(F-\overline{V})}\,\Big]\Big)\Big]\\
&\qquad\Big(\text{because }\;U\cap\Big(X-\Big[F\cup\overline{v(F-\overline{V})}\,\Big]\Big)=\emptyset=\\
&\qquad\qquad=\Big(X-\Big[F\cup\overline{v(F-\overline{U})}\,\Big]\Big)\cap V,\;\text{ since }\;U\subset F\text{ and }V\subset F\Big)\\
&=\;(U\cap V)\cup\Big(X-\Big[F\cup\overline{v(F-\overline{U})}\,\cup\,\overline{v(F-\overline{V})}\,\Big]\Big)\;=
\end{aligned}$$

$$= (U\cap V)\cup\Big(X-\big[F\cup\overline{v(F-\overline{U})\cup v(F-\overline{V})}\,\big]\Big)$$

$$= (U\cap V)\cup\Big(X-\big[F\cup\overline{v\big((F-\overline{U})\cup(F-\overline{V})\big)}\,\big]\Big)$$

(because v is a K_W^*-function)

$$= (U\cap V)\cup\Big(X-\big[F\cup\overline{v(F-\overline{U}\cap\overline{V})}\,\big]\Big)$$

$$= (U\cap V)\cup\Big(X-\big[F\cup\overline{v(F-\overline{U\cap V})}\,\big]\Big)$$

$$= k(U\cap V)\,,\qquad\text{because } \overline{U\cap V}=\overline{U}\cap\overline{V}\ ,$$

which completes the proof. ∎

We conjecture that the converse of Theorem 2 is false and we have not been able to find a characterization of K_W^*-spaces analogous to the characterization of K_W-spaces which appears in Theorem 4 of [3].

Theorem 3. *There is a K_W-space T which is not a K_W^*-space.*

Proof: The space T is the space described by M.E. Rudin in [6]. We already know from Theorem 1 (recall that monotonically normal spaces have $D^*(\mathbb{R};\le)$) that T is a K_W-space.

Assuming that T is a K_W^*-space, let $k:\tau|F\to\tau$ satisfy the conditions of Theorem 3(b). Since the sets U_{xi} and U_{rxi} defined on p. 305 of [6] are easily seen to be clopen, then we get that

$$\overline{\bigcap_{i<3}U_{xi}}=\bigcap_{i<3}\overline{U_{xi}}\quad\text{and}\quad k\Big(\bigcap_{i<3}U_{xi}\cap Y\Big)=\bigcap_{i<3}k(U_{xi}\cap Y)$$

and, similarly,

$$k\Big(U_{rx_ij}\cap U_{rx_i(j-1)}\cap Y\Big) \;=\; k(U_{rx_ij}\cap Y)\cap k(U_{rx_i(j-1)}\cap Y)\ .$$

Consequently, M.E. Rudin's argument, verbatim, also proves that the above k cannot exist, a contradiction. This completes the proof. ∎

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