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FORBIDDEN PERIODS IN DELAY DIFFERENTIAL EQUATIONS

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Abstract: We study the one-parameter family of scalar differential-difference equations

$$x'(t) = f\Big(\alpha, x(t), x(t-1)\Big) ,$$

and establish some results on the period of periodic solutions and its relations with the parameter α.

1 – Introduction

The primary goal of this article is to establish conditions for the non-existence of certain nonconstant periodic orbits of scalar differential-delay equations of the form

$$x'(t) = f\Big(\alpha, x(t), x(t-1)\Big) , \tag{1}$$

where $f\colon \mathbb{R}^3 \to \mathbb{R}$ is a continuous map and α is a real parameter. The solution of the initial value problem associated with the above equation, which requires that

$$x(t) = \phi(t) , \quad -1 \le t \le 0 , \tag{2}$$

where ϕ is given in $\mathcal{C}$, the Banach space of continuous maps $\phi : [-1,0] \to \mathbb{R}$ equipped with the uniform convergence norm, will be denoted by $x(\cdot,\phi)$. It is said to be *periodic* (of period T) if $x(t,\phi) = x(t+T,\phi)$ for all $t \in \mathbb{R}$. Thus, periodic solutions of Eq. (1) are of class C^1. When $f(\alpha,a,a) = 0$, $x(t) \equiv a$ is a

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constant solution, said to be an "equilibrium" or a "trivial solution" of Eq. (1) (observe that trivial solutions are periodic of any period T). When this happens for $a = 0$, the corresponding equilibrium is the *null equilibrium* or *null solution*. The shorter locution "T-periodic" is often used to abbreviate "periodic of period T". We assume, from now on, that $f(\alpha, 0, 0) = 0$. Note, incidentally, that if $x(t)$ is T-periodic for some $T > 0$, then, automatically, it is also $n\,T$-periodic for any $n \in \mathbb{Z}$, where $\mathbb{Z}$ denotes the set of integers. In particular, a T-periodic map is also $(-T)$-periodic. The subject of this work has been frequently investigated ([1]–[4], [6]–[14]), due to its importance both in pure and applied mathematics.

2 – Notations and definitions

Given a solution $x(\cdot, \phi)$ of (1)–(2), we define $x_t(\cdot, \phi) \in \mathcal{C}$ by $x_t(\theta, \phi) = x(t+\theta, \phi)$. Then, $\Gamma = \{x_t(\cdot, \phi),\ t \geq 0\}$ is a continuous curve in $\mathcal{C}$ and is called the *positive orbit* of ϕ. This curve is closed (in $\mathcal{C}$) if and only if $x(\cdot, \phi)$ is a periodic solution ([4]). Observe that in terms of $x_t(\cdot, \phi)$, Eqs. (1)–(2) can be written as

$$x_t'(\theta) = f\Big(\alpha, x_t(\theta), x_{t-1}(\theta)\Big), \qquad \theta \in [-1, 0], \quad t+\theta \geq 0 , \tag{3}$$

$$x_0 = \phi . \tag{4}$$

The orbit of ϕ identifies in a natural way the sequence of maps in $\mathcal{C}$, $\{x_n(\cdot, \phi)\}_0^\infty$. Since $\phi = x_0(\cdot, \phi)$ it is simpler to think of ϕ as fixed and denote the solution $x(\cdot, \phi)$ simply by x, and thus identify it with the sequence $\{x_n\}_0^\infty$. It is clear that a solution x is k-periodic (where k is a positive integer) iff $\{x_n\}$ is k-periodic, i.e., iff $x_k = x_0$. Note, incidentally, that x satisfies the family of integral equations

$$x_n(\theta) = x_n(0) + \int_0^\theta f\Big(\alpha, x_n(s), x_{n-1}(s)\Big)\, ds\,, \qquad n = 1, 2, ... \,. \tag{5}$$

In view of the above, we can also say that a k-periodic solution of Eq. (1) is a finite sequence $\{x_n\}_o^{k-1}$ satisfying (5), or else, satisfying the ODE system

$$\begin{aligned}
&x_1'(\theta) = f\Big(\alpha, x_1(\theta), x_0(\theta)\Big) \\
&x_2'(\theta) = f\Big(\alpha, x_2(\theta), x_1(\theta)\Big) \\
&\quad \vdots \\
&x_{k-1}'(\theta) = f\Big(\alpha, x_{k-1}(\theta), x_{k-2}(\theta)\Big) \\
&x_0'(\theta) = f\Big(\alpha, x_0(\theta), x_{k-1}(\theta)\Big), \qquad \theta \in [-1, 0] ,
\end{aligned} \tag{6}$$

subject to the boundary conditions

(7) $$x_n(-1) = x_{n-1}(0), \quad n = 1, 2, ..., k-1, \qquad x_0(-1) = x_{k-1}(0) .$$

Equations of this kind are known as "cyclic ordinary differential equations". They are often studied ([3], [9]–[12]). It is important to observe that (6) allows us to study solutions with integral periods in a k-dimensional space, instead of the natural infinite dimensional phase space $\mathcal{C}$, mentioned in Section 1. Observe, nevertheless, that the boundary conditions (7) are not the usual ones for ODE's (namely, the pointwise initial value problem (IVP)):

(8) $$x_o(0) = \xi_o , \quad x_1(0) = \xi_1 , \quad ..., \quad x_{k-1}(0) = \xi_{k-1} , \quad \xi_j \in \mathbb{R}, \quad j = 0, 1, ..., k-1 .$$

However, if $f(\alpha, x, y)$ is locally Lipschitz (see [5]), then, once a solution of (6)–(7) is found, it is the unique solution through this *found* functional initial condition x_o ($=\phi$, if we recall our previous notation convention). It is also the unique solution of the corresponding IVP (6)–(8). This property shows that the uniqueness of the functional initial value problem (3)–(4) corresponds to uniqueness of solution of either the related boundary conditions (7) or the related ordinary IVP (8), and conversely.

An analysis of the above arguments shows that the restriction $\theta \in [-1, 0]$ is superfluous if we let $x_n(t) =: x(t+n)$, $t \in \mathbb{R}$. In this case, x_n is just a phase shift of $x(t)$, thus just another solution of Eq. (1), which shares the same orbit as $x(t)$. We shall freely make use of this dual interpretation.

3 – Results

In this section we shall use (6)–(7) and (6)–(8) in order to establish some important features of the phenomenon of existence of k-periodic solutions of Eq. (1).

The following theorem is immediate and is included here just for reference.

Theorem 1. *Equation (1) has no nontrivial 1-periodic solution.*

Proof: If it had one, then there would exist a nonconstant solution of the autonomous scalar ODE $x'(t) = f(\alpha, x(t), x(t))$ satisfying the corresponding boundary condition $x(-1) = x(0)$, an obvious impossiblity (unless f were trivial). ∎

The next theorem has been proved before (see [12], [14]). We offer here a simple proof of it, whose extension to the general k-periodic case furnishes a useful characterization of trivial solutions of Eq. (1) (Theorem 5).

Theorem 2. *If the IVP (6)–(8) has a unique solution when $k = 2$, then Eq. (1) cannot have a nontrivial 2-periodic solution.*

Proof: Suppose $k = 2$. If it had one such solution, say x, then we would have a corresponding, nontrivial, solution pair (x_0, x_1) of (6)–(7). By the intermediate value theorem of calculus there would exist $b \in [-1, 0]$ such that $x_o(b) = x_1(b) =: \xi$. But then, (x_o, x_1) would also satisfy the IVP

$$x_0(b) = \xi\,, \quad x_1(b) = \xi \tag{9}$$

attached to (6). Let y be the solution of the ODE $y'(t) = f(\alpha, y(t), y(t))$ such that $y(b) = \xi$. Then, (y, y) is a solution of (6)–(9). The uniqueness of this solution implies that $x_0 = x_1 = y$. As a consequence, $x(t)$ is in fact a nonconstant 1-periodic solution of Eq. (1), an impossibility by Theorem 1. ∎

The above argument can be easily adapted to the general case of k-periodic solutions:

Theorem 3. *Suppose the solution of (6)–(8) is unique. Then, a necessary and sufficient condition for a k-periodic solution $x(t)$ of Eq. (1) to be trivial is the existence for it of a point $x(t^*)$ of period 1, that is, $x(t^*) = x(t^* - i)$, $i = 1, 2, ..., k-1$.*

Proof: In a similar way as in the proof of the above theorem, it is enough to see that the unique solution of the IVP $y'(t) = f(\alpha, y(t), y(t))$, $y(t^*) = x(t^*)$ yields that $x_i(t) = y(t)$, $t^* - 1 \le t \le t^*$, $i = 0, 1, ..., k-1$ as the unique possible solution of the corresponding problem (6)–(8). ∎

As a consequence of Theorem 3 we obtain the following.

Corollary 1. *Under the hypothesis of the theorem, a necessary and sufficient condition for a k-periodic solution $x(t)$ of Eq. (1) to be trivial is the existence of an m-periodic point t^* such that k and m are relatively prime.*

Proof: If k and m are relatively prime, then $x(t^*)$ being at the same time k- and m-periodic implies that it is indeed 1-periodic, and the result follows. ∎

The following lemma is obvious and will be used repeatedly in the sequel.

Lemma 1. *Let $x(t)$ be a k-periodic map. Then $g(t) =: x(t) + x(t-1) + \cdots + x(t-k+1)$ is 1-periodic.* ∎

It is interesting to note that g describes, in a certain sense, important features of the map x, as is the case $g = 0$. When $x(t) + x(t-1) + ... + x(t-k+1) = g(t)$, we say that x is *g-harmonic*. If $g = 0$, we simply say that x is *harmonic*. Other sums of $x(t-i)'s$ may lead to various interesting results, like the one in the next theorem. We say a T-periodic map $x(t)$ is *odd-harmonic* if $x(t) = -x(t-\frac{T}{2})$.

Theorem 4. *Suppose $x(t)$ is a 4-periodic solution of Eq. (1) in its linear form,*

$$(*) \qquad x'(t) = \alpha\, x(t) + \beta\, x(t-1)\ ,$$

where $\beta \neq 0$. Then, x is odd-harmonic.

Proof: If we take $y(t) =: x(t) + x(t-2)$, it easily follows that y is a 2-periodic solution of $(*)$. But, the unique such solution is $x(t) \equiv 0$, so that $x(t) = -x(t-2)$ and x is odd-harmonic, as we wished to prove. ∎

Likewise, if $x(t)$ is k-periodic, $j_1 < j_2 < ... < j_m \leq k$, $\{x_{j_1+p}, x_{j_2+p}, ..., x_{j_m+p}\} = \{x_{j_1}, x_{j_2}, ..., x_{j_m}\}$, $p < k$, then $y(t) =: x(t-j_1) + x(t-j_2) + ... + x(t-j_m)$ is p-periodic. In particular, if f is linear and $x(t)$ is a solution of Eq. (1), then $y(t)$ is a p-periodic solution of it as well.

In the case of an equation of the form

$$(10) \qquad x'(t) = f\Big(\alpha, x(t-1)\Big)\ ,$$

we can provide an alternative proof of Theorem 2, without the uniqueness of solution hypothesis, a proof that gives geometrical insight into the impossibility of existence of 2-periodic solutions. Indeed, let F be a primitive of f with respect to x and put $\Phi(u,v) =: F(u) - F(v)$. For every $c \in \mathbb{R}$ the level curve $\gamma = \{(u,v) : \Phi(u,v) = c\}$ defines v as a function of u and conversely. We take x_0 as the solution of the ODE $y'(t) = f(\alpha, v(y(t)))$ (or of $y'(t) = f(\alpha, u(y(t)))$) and put $x_1(t) =: v(x_0(t))$ $(x_1(t) =: u(x_0(t)))$. It follows that $\{x_i\}_0^1$ is a solution of (6) (with $k=2$) since $x_0'(t) = f(\alpha, x_1(t))$ and $x_1'(t) = \frac{d}{du} v(x_0(t))\, x_0'(t) = \frac{f(\alpha, x_o(t))}{f(\alpha, v(x_o(t))} \cdot f(\alpha, v(x_0(t)) = f(\alpha, x_0(t))$ (the converse situation is similar). This shows that Eq. (10) has a nontrivial 2-periodic solution only if $\Phi(u,v)$ has a nonsingular closed level curve γ. Thus, if Eq. (10) had a 2-periodic solution $x(t)$, this solution would just be a parametrization of γ, $0 \leq t \leq 2$. Note, however, that $\Phi(u,u) = 0$, so that γ cannot cross the diagonal of $\mathbb{R}^2$ unless $c = 0$, a case which leads to a constant solution $x = a$, where a is a zero of f. As a consequence, $x(t) \neq x(t+1)$ for any $t \in \mathbb{R}$, which precludes the existence of any nontrivial 2-periodic solution for Eq. (10). Note that γ is a projection into $\mathbb{R}^2$ of the orbit Γ of x.

We can also use the (uniqueness) argument of the above remark in order to prove that Eq. 10 cannot have nontrivial 3-periodic solutions of certain specific types. Recall that a map x is said to be *even* when $x(t) = x(-t)$ and *odd* when $x(t) = -x(-t)$.

Theorem 5. *Suppose that the solution of (6)–(8) when $k = 3$ is unique in the case of Eq. (10). Then, this equation cannot have a nontrivial 3-periodic solution which is either even or odd.*

Proof: Let x be an even 3-periodic solution of Eq. (10) and let $\{x_n\}_0^2$ be the corresponding solution of (6)–(8) with $k = 3$. Then, as it is easy to check, we must have: $x_0(t) = x_2(-1-t)$ and $x_1(t) = x_1(-1-t)$ for t in $[-1, 0]$. The boundary conditions (7) automatically become

$$x_0(-1) = x_2(0)\,, \quad x_1(-1) = x_0(0)\,, \quad x_2(-1) = x_1(0)\ . \tag{11}$$

It also follows that:

$$x_0'(t) = -x_2'(-1-t) = -f(\alpha, x_1(-1-t)) = -f(\alpha, x_1(t)) = -x_2'(t)\ ,$$
$$x_1'(t) = -x_1'(-1-t) = -f(\alpha, x_0(-1-t)) = -f(\alpha, x_2(t)) = -x_0'(t)\ .$$

Hence,

$$x_0(t) + x_2(t) = c\,, \quad x_1(t) + x_0(t) = d \tag{12}$$

where c and d are constants. Let $x_0(-1) = a$ and $x_0(0) = b$. Then, using (11) and (12) at $t = -1$ and at $t = 0$ we get

$$a + b = c\,, \quad 2\,b = d\,, \quad a + b = d\ ,$$

which readily implies that $c = d$ and $a = b$. Hence, $\{x_0, x_1, x_2\}$ satisfies the initial condition

$$x_0(0) = x_1(0) = x_2(0) = a\ .$$

Let y be the solution of the ODE $y'(t) = f(\alpha, y(t))$ such that $y(0) = a$. Since (y, y, y) is a solution of (6)–(8) that satisfies the above initial condition, and $f(\alpha, \cdot)$ is locally Lipschitz, we must have $x_0 = x_1 = x_2 = y$ as long as these maps are defined, due to the uniqueness of the solution. But then, $x(t)$ is 1-periodic, and, by Theorem 1, it must be constant.

Suppose now that $x(t)$ is an odd 3-periodic solution of Eq. (10). It is easily seen that $x_0(t) = -x_2(-1-t)$ and $x_1(t) = -x_1(-1-t)$, $t \in [-1, 0]$. The boundary conditions (7) combined with the constraint of glueing together the three parts $x_0(\theta)$, $x_1(\theta)$, $x_2(\theta)$, $\theta \in [-1, 0]$ to get the odd solution $x(t)$ imply

$$x_0(0) = -x_1(-1) = x_1(-1) = -x_1(0) = -x_0(0)\,, \quad x_0(-1) = -x_2(0) = x_2(0)\ .$$

From these equalities we derive $x_0(0) = x_1(0) = x_2(0) = 0$. The result follows just as above, with $a = 0$, and this finishes the proof of the theorem. ∎

We have:

Theorem 6. *If the map $f(\alpha,\cdot)$ is not odd, then Eq. (10) cannot have non-trivial 3-periodic solutions, which are odd-harmonic.*

Proof: Suppose that there exists one such solution $x(t)$. Then, the sequence $\{x_k\}_0^2$ would satisfy:

$$x_0(t) = -x_1(t+\tfrac{1}{2}),\quad x_1(t) = -x_2(t+\tfrac{1}{2}),\quad x_2(t) = -x_0(t+\tfrac{1}{2})\,, \qquad t\in[-1,-\tfrac{1}{2}]\,,$$
$$x_0(t) = -x_2(t-\tfrac{1}{2}),\quad x_1(t) = -x_0(t-\tfrac{1}{2}),\quad x_2(t) = -x_1(t-\tfrac{1}{2})\,, \qquad t\in[-\tfrac{1}{2},0]\,.$$

As a consequence of this, we have (using Eqs. (6) with $k=3$):

$$\begin{aligned} x_0'(t) &= -x_1'(t+\tfrac{1}{2}) = f(\alpha, x_0(t+\tfrac{1}{2})) \\ &= -f(\alpha,-x_2(t)) = f(\alpha,x_2(t)), \qquad \text{for } \ t\in[-1,-\tfrac{1}{2}] \end{aligned}$$

$$\begin{aligned} x_0'(t) &= -x_2'(t-\tfrac{1}{2}) = -f(\alpha, x_1(t-\tfrac{1}{2})) \\ &= -f(\alpha,-x_2(t)) = f(\alpha,x_2(t)), \qquad \text{for } \ t\in[-\tfrac{1}{2},0]\,. \end{aligned}$$

Hence, $f(\alpha,-x_2(t)) = -f(\alpha,x_2(t))$ for $t\in[-1,0]$. In a similar way, one can show that $f(\alpha,-x_0(t)) = -f(\alpha,x_0(t))$ and $f(\alpha,-x_1(t)) = -f(\alpha,x_1(t))$ for $t\in[-1,0]$, which indicates that $f(\alpha,\cdot)$ is odd in the range of $x(t)$, in contradiction to the other hypothesis of the theorem. ∎

Let us now give some information about the distribution (as a function of the parameter α) of the period of periodic orbits of the following form of Eq. (1):

$$x'(t) = \alpha\, f\Big(x(t),\, x(t-1)\Big)\,. \tag{13}$$

Theorem 7. *Suppose that Eq. (13) has at $\alpha=\alpha_0$ a $(T-1)$-periodic solution. Then, it has at $\alpha=\alpha_n =: [n(T-1)+1]\,\alpha_0$, $n=0,1,2,\ldots$, a $(\frac{T-1}{n(T-1)+1})$-periodic solution. And, conversely, if Eq. (13) has at α a $\frac{T-1}{n(T-1)+1}$-periodic solution, then it has at $\beta =: \frac{(n-1)(T-1)+1}{n(T-1)+1}\,\alpha$ a $\frac{T-1}{(n-1)(T-1)+1}$-periodic solution, $n=1,2,\ldots$. Moreover, if one of these periods is minimum, so is the other one.*

Proof: Let $y(t)$ be a $(T-1)$-periodic solution of Eq. (13) when $\alpha=\alpha_0$. Put $x(t) = y(Tt)$. Then,

$$\begin{aligned} x'(t) &= T\,y'(Tt) = \alpha_0\,T f\Big(y(tT), y(tT-1)\Big) = \alpha_0\,T f\Big(y(tT), y(Tt-1-T+1)\Big) \\ &= \alpha_0\,T f\Big(y(Tt), y(T(t-1))\Big) = \alpha_0\,T f\Big(x(t), x(t-1)\Big)\,, \end{aligned}$$

so that x is a solution of (13) at $\alpha = \alpha_0 T$. Moreover, we have: $x(t + \frac{T-1}{T}) = y(T(t + \frac{T-1}{T})) = y(Tt + T - 1)) = y(Tt) = x(t)$, so that x is $\frac{T-1}{T}$-periodic. Thus, the result of the theorem holds when $n = 1$. Put now $S - 1 =: \frac{T-1}{T}$. Then, repeating the above argument with T replaced by S, we obtain that Eq. (13) has an $(\frac{S-1}{S})$-periodic solution when $\alpha = S\,T\alpha_0$. But, since $\frac{S-1}{S} = \frac{\frac{T-1}{T}}{\frac{T-1}{T}+1} = \frac{T-1}{2T-1}$ we obtain that $S\,T\alpha_0 = (\frac{T-1}{T} + 1)\,T\alpha_0 = (2T - 1)\,\alpha_0$, which is the result of the theorem when $n = 2$. An induction argument easily finishes this part of the proof. The converse part of the theorem can be proved in a similar way if one takes $x(t) = y\Big(\frac{(n-1)(T-1)+1}{n(T-1)+1}\,t\Big)$. The statement concerning the minimality of the periods is transparent from the technique used in the above proof. ∎

The idea of this proof was known to Cooke (see [10]) who used it in the case of the equation $x'(t) = -\alpha\, f(x(t-1))$ to show that the existence of slowly (see also [2]) oscillating solutions implies the existence of rapidly oscillating (resonant) ones.

Theorem 8. *If the solution of (6)–(8) is unique, then Eq. (13) cannot have nontrivial periodic solutions of period* $\frac{2}{2n-1}$ *for any integer* $n \in \mathbb{Z}$.

Proof: The cases $n = 0$ and $n = 1$ are straightforward translations of the results of Theorem 2. For the other values of n, if we use Theorem 7 in conjunction with Theorem 2, we see that Eq. (13) cannot have any nontrivial $(T-1)$-periodic solution such that $\frac{T-1}{n(T-1)+1} = 2$ for some $n \in \mathbb{Z}^+$, since this would imply the existence of a nontrivial 1-periodic orbit of this equation. But, $\frac{T-1}{n(T-1)+1} = 2$ iff $T - 1 = -\frac{2}{2n-1}$. The result follows now from the property that an orbit which is S-periodic is automatically $(-S)$-periodic. ∎

The theorem below is a simple corollary of Theorem 1, since the $\frac{1}{n}$-periodic maps are automatically 1-periodic. It is also a corollary of Theorem 7, since we clearly see that Eq. (13) cannot have a nontrivial $(T-1)$-periodic solution such that $n(T-1) + 1 = 0$, i.e., when $T - 1 = \frac{1}{n}$, $n \in \mathbb{Z}$.

Theorem 9. *Eq. (13) cannot have* $\frac{1}{n}$*-periodic solution for any* $n \in \mathbb{Z}$. ∎

As an application, let's consider the scalar case of Wright's equation,

$$x'(t) = -\alpha\, x(t-1)\,[1 + x(t)]\,, \qquad \alpha \neq 0\ . \tag{14}$$

We already know (as shown above) that this equation cannot have 1- and 2-periodic solutions. It also cannot have nontrivial 3- and 4-periodic solutions as shown in [6] by Carvalho and Ladeira (see also Nussbaum [9], for the case of nonexistence of 4-periodic solutions). In Nussbaum [8] it is shown that it has a nonconstant T-periodic solution for each $T > 4$ (at a corresponding convenient value of the parameter α). Hence, the above results imply, in particular, that the range of periods of Wright's equation is dense in $\mathbb{R}$ and every interval $(\frac{4}{2^n+1}, \frac{4}{2^n})$, $n = 1, 2, \ldots$ is in the range of periods of this equation.

An interesting bound relating the period and the Lipschitz constant is given by the following theorem.

Theorem 10. *Let $x(t)$ be a nontrivial T-periodic solution of the scalar delay differential equation $x'(t) = f(x(t-1))$. Assume that f is Lipschitz with constant L on the interval $[\min x(t), \max x(t)]$. Then, $TL \geq 2\pi$.*

Proof: Recall that for every $\phi(t)$ such that $\phi, \phi' \in L^2[0,T]$ and $\int_0^T \phi(t)\,dt = 0$ (ϕ has mean value 0) we have (Wirtinger's inequality)

$$\int_0^T \phi^2(t)\,dt \;\leq\; \left(\frac{T}{2\,\pi}\right)^2 \int_0^T \phi'^2(t)\,dt\ .$$

Let $\phi(t) = x(t) - x(t-h)$ where $h \in (0,T)$. Then ϕ has mean value 0 and we can write:

$$\begin{aligned}
\int_0^T \Big(x(t) - x(t-h)\Big)^2 dt \;&\leq\; \left(\frac{T}{2\pi}\right)^2 \int_0^T \Big(x'(t) - x'(t-h)\Big)^2 dt \\
&\leq\; \left(\frac{T}{2\pi}\right)^2 \int_0^T \Big[f\Big(x(t-1)\Big) - f\Big(x(t-h-1)\Big)\Big]^2 dt \\
&\leq\; \left(\frac{T}{2\pi}\right)^2 \int_0^T \Big[f\Big(x(t)\Big) - f\Big(x(t-h)\Big)\Big]^2 dt \\
&\leq\; \left(L\,\frac{T}{2\pi}\right)^2 \int_0^T \Big(x(t) - x(t-h)\Big)^2 dt\ ,
\end{aligned}$$

so that the result follows immediately. ∎

Of course, if f depends on α as is the case of Eq. (10), then $L = L(\alpha)$. Note, moreover, that this result may also cover equations of the form

$$x'(t) \;=\; g(x(t))\,h(x(t-1))\ , \tag{15}$$

as is the case of Eq. (13), since they can, sometimes, be written in the form $x'(t) = f(x(t-1))$. In fact, if $z(t)$ is a nontrivial solution of $z'(t) = g(z(t))$, then, for each solution $y(t)$ of Eq. (15) we can implicitly define $x(t)$ by means of the equality $z(x(t)) =: y(t)$. It follows that

$$\begin{aligned} y'(t) &= z'(x(t))\,x'(t) \;=\; g(z(x(t)))\,x'(t) \\ &= g(y(t))\,h(y(t-1)) \;=\; g(z(x(t)))\,h(z(x(t-1)))\ , \end{aligned}$$

from where we obtain $x'(t) = h(z(x(t-1)))$, i.e., $x(t)$ is a solution of an equation of the type of Eq. (10) with $f = h \circ z$. Clearly, x is T-periodic if, and only if, y is T-periodic. For instance, in the case of Eq. (13), we have that $z'(t) = 1 + z(t) = g(z(t))$ has $z(t) = e^t - 1$ as nonconstant solution, so that Eq. (14) becomes

$$x'(t) = -\alpha\left(e^{x(t-1)} - 1\right) \quad (=: f(x(t-1))\ . \tag{16}$$

As we mentioned previously, this scalar case of Wright's equation has a T-periodic solution for each $T > 4$ and a suitable value of α. As a consequence of Theorem 10, we see that a solution pair $(\alpha, x(t))$ satisfies a particular constraint. By selecting h very small we can see that $|f(x(t)) - f(x(t-h))| \le \alpha\, e^{\eta}|x(t) - x(t-h)|$, where $\eta \in [\min x(t), \max x(t)]$ is given by the Mean Value Theorem of calculus. Hence, if $|x(t)| \le a$ we may take, in particular, $L = \alpha\, e^a$ and write

$$TL \;=\; |\alpha|\, T e^a \;\ge\; 2\,\pi\ ,$$

which shows that as $\alpha \to 0$ we must have either $T \to \infty$ or $a \to \infty$. In the particular case described by Theorem 7, if we let, say, $\alpha_n = (nT+1)\alpha$ and $T_n = \frac{T}{nT+1}$, the product $\alpha_n T_n = \alpha\, T$ is constant. Clearly, the amplitude of these T_n-periodic solutions is also constant.

As another example, consider the following simple linear equation:

$$x'(t) \;=\; \alpha\, x(t-1)\ , \tag{17}$$

Here, $f(\alpha, x) = \alpha\, x$, $\alpha \ne 0$, has Lipschitz constant $L = L(\alpha) = |\alpha|$, and the next theorem holds (see [7]):

Theorem 11. *Eq. (17) has nonzero periodic solutions with integer period if and only if* $\alpha \in \{\frac{4m-1}{2}\pi :\ m \in \mathbb{Z}\}$. *In this case the minimum period is* $\frac{4}{4m-1}$ *(hence, the minimum integer period is* 4*) and the periodic solutions are given by*

$$x(t) \;=\; \gamma\, \cos\alpha\, t \;+\; \delta\, \sin\alpha\, t\,, \qquad \gamma, \delta \in \mathbb{R}\ . \ \blacksquare$$

If we apply Theorem 11 with $T = 4$ (this implies $m = 0$ is the tightest choice in the corresponding formula for α) and $\alpha = -\frac{\pi}{2}$ (thus $L = \frac{\pi}{2}$), we see that we have $T\alpha = 4\,\frac{\pi}{2} = 2\,\pi$, and the theorem is satisfied in this case.

Note that the reasoning of Theorem 10 remains unchanged if x is a vector in $\mathbb{R}^n$ or in a infinite dimensional Hilbert space H. In case of a Banach space E setting for the equations discussed in this paper, we have the following result, adapted from the one given by Busenberg, Fisher and Martelli in [13].

Theorem 12. *If f is Lipschitz with constant L in the range of a nontrivial T-periodic solution $x(t)$ of $x'(t) = f(x(t))$, where f maps a Banach space E into itself, then $LT \geq 6$.* ∎

Their proof is based on the fact that for any T-periodic differentiable map $y(t) \in E$ (let the norm of E be denoted by $\|\cdot\|$) such that $\|y'(t)\|$ is integrable, we have

$$\int_0^T\int_0^T \|y(t) - y(s)\|\, ds\, dt \;\leq\; \frac{T}{6}\int_0^T\int_0^T \|y'(t) - y'(s)\|\, ds\, dt\ .$$

We can easily adapt the above property to an equation of the form (10). In fact, assume that $x(t)$ is a T-periodic solution of Eq. (9) and that $f(\alpha,\cdot)$ is Lipschitz with constant $L = L(\alpha)$ in the interval $[m, M]$, where $m = \min\{x(t),\ t \in [0,T]\}$, $M = \max\{x(t),\ t \in [0,T]\}$. Then,

$$\begin{aligned}
\int_0^T\int_0^T \|x(t) - x(s)\|\, ds\, dt &\leq \frac{T}{6}\int_0^T\int_0^T \|x'(t) - x'(s)\|\, ds\, dt \\
&= \frac{T}{6}\int_0^T\int_0^T \Big\| f\Big(\alpha, x(t-r)\Big) - f\Big(\alpha, x(s-r)\Big)\Big\|\, ds\, dt \\
&\leq \frac{TL}{6}\int_0^T\int_0^T \Big\| x(t-r) - x(s-r)\Big\|\, ds\, dt \\
&= \frac{TL}{6}\int_0^T\int_0^T \|x(t) - x(s)\|\, ds\, dt\ .
\end{aligned}$$

Thus:

Theorem 13. *Suppose $f(\alpha,\cdot) : R \to R$ is Lipschitz with constant $L(\alpha)$. Then if $x(t)$ is a nontrivial T-periodic solution of $x'(t) = f(\alpha, x(t-1))$, we must have $TL(\alpha) \geq 6$.* ∎

A little trick will allow us to use this result also in the case of Wright's equation in its form (14), instead of (16) as done above, and compare the results. In fact, let $x(t)$ be a T-periodic solution of Eq. (14), with amplitude a. Observe that

$$\begin{aligned}
&\Big| x(t)\Big[1 + x(t-1)\Big] - x(t-s)\Big[1 + x(t-s-1)\Big]\Big| \leq \\
&\qquad \leq (a+1)\Big| x(t) - x(t-s)\Big| + a\Big| x(t-1) - x(t-s-1)\Big|\ ,
\end{aligned} \tag{18}$$

a bound that is easily obtained by inserting $x(t-s)\,[1-x(t-1)]-x(t-s)\,[1-x(t-1)]$ into the left hand side of (18), and then using the triangle inequality and $|x(t-s)| \le a$ and $|1+x(t-1)| \le a+1$. Adapting the argument of the proof of Theorem 13 to the present situation, we get

$$\begin{aligned}\int_0^T\!\!\int_0^T \Big|x(t)-x(t-s)\Big|\,ds\,dt &\le \frac{T}{6}\int_0^T\!\!\int_0^T \Big|x'(t)-x'(t-s)\Big|\,ds\,dt\\ &\le \frac{T}{6}\,|\alpha|\,(1+a)\int_0^T\!\!\int_0^T \Big|x(t)-x(t-s)\Big|\,ds\,dt\\ &\quad+\frac{T}{6}\,|\alpha|\,a\int_0^T\!\!\int_0^T \Big|x(t-1)-x(t-1-s)\Big|\,ds\,dt\\ &=\frac{T}{6}\,|\alpha|\,(2\,a+1)\int_0^T\!\!\int_0^T \Big|x(t)-x(t-s)\Big|\,ds\,dt\ ,\end{aligned}$$

from where we obtain the estimate $T\,|\alpha|\,(2\,a+1) \ge 6$. As before, as $\alpha \to 0$ we must have either $T \to \infty$ or $a \to \infty$. The graph in Fig. 1 shows that the bound $e^a\,|\alpha|\,T \ge 2\,\pi$ is better than the bound $(2\,a+1)\,|\alpha|\,T \ge 6$ for $a < 1.35966...$, but is worse after that.

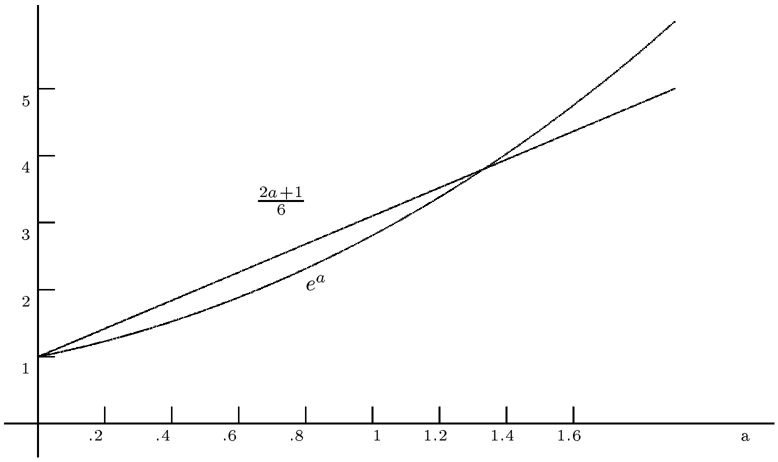

Fig. 1: Comparison between the bounds $e^a\,|\alpha|\,T \ge 2\pi$ and $(2a+1)\,|\alpha|\,T \ge 6$.

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