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# MULTIPLE SEMICLASSICAL SOLUTIONS OF THE SCHRÖDINGER EQUATION INVOLVING A CRITICAL SOBOLEV EXPONENT

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**Abstract:** We prove the existence of multiple solutions of the Schrödinger equation involving a critical Sobolev exponent. We use the Lusternik–Schnirelman theory of critical points.

## 1 – Introduction

The main purpose of this work is to investigate the existence of multiple solutions of the equation

$$(1_\epsilon) \qquad -\epsilon^2 \Delta u + a(x)\, u \;=\; |u|^{2^*-2}\, u \quad \text{ in } \mathbb{R}^N$$

for $\epsilon > 0$ small, where $2^* = \frac{2N}{N-2}$, $N \geq 3$.

Solutions of equation $(1_\epsilon)$ corresponding to a small parameter $\epsilon > 0$ are referred to in the existing literature as semiclassical solutions [1], [2], [11], [13], [14], [15], [16]. Problem $(1_\epsilon)$ arises in the search for standing waves for the nonlinear Schrödinger equation

$$i\, h\, \frac{\partial \psi}{\partial t} \;=\; -h^2 \Delta \psi + U(x)\, \psi - |\psi|^{p-2}\, \psi \quad \text{ in } \mathbb{R}^N ,$$

where $h$ is the Planck constant, $p > 2$ if $N = 1, 2$ and $2 < p \leq 2^*$ if $N \geq 3$. Standing waves of this equation are solutions of the form $\psi(t,x) = \exp(-i\,\lambda\, h^{-1}\, t)\, u(x)$,

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$\lambda \in \mathbb{R}$, where $u$ is a real-valued function satisfying $(1_\epsilon)$ with $a(x) = U(x) + \lambda$ and $h = \epsilon^2$. Obviously, the equation $(1_\epsilon)$ corresponds to the case $p = 2^*$. The first result on the existence of semiclassical solutions was obtained by Floer–Weinstein in [11] via the Lyapunov–Schmidt method in the case $N = 1$. This result was extended by Oh [14], [15] to higher dimensions, in the subcritical case $2 < p < 2^*$. Some related results can be found in the papers [19], [20], [9] and [11]. It is well known that the existence of multiple solutions for the Dirichlet problem for (1) on bounded domains depends on the topology of this domain (see for example [4], [6]). In the case of problem $(1_\epsilon)$ a similar role is played by the graph topology of coefficient $a$. This phenomenon also occurs for the Dirichlet problem on bounded domains [6]. The effect of the graph topology of the coefficients on the existence of multiple solutions in the subcritical case was investigated in the papers [9] and [13] and in [18] for the Dirichlet problem in bounded domains. The aim of this paper is to relate the number of solutions of problem $(1_\epsilon)$ with $\operatorname{cat} a^{-1}(0)$. It is well known that if $a(x) = \text{Constant} \neq 0$, problem $(1_\epsilon)$ has no solution by the Pohozaev identity. A similar situation occurs also for the Dirichlet problem for $(1_\epsilon)$ on bounded starshaped domains if $a(x) = \text{Constant} \geq 0$. However, in the case $a(x) \neq \text{Constant}$ there are existence results for $(1_\epsilon)$, with $\epsilon = 1$, (see for example [3]) and for the Dirichlet problem on bounded domains [18] under some structural assumptions on $a(x)$. For further bibligraphical references on the effect of the coefficient $a(x)$ on the existence and nonexistence of solutions, we refer to the papers [3] and [18].

Throughout this paper we use standard notation and terminology. In a given Banach space $X$, we denote by "$\to$" a strong convergence and by "$\rightharpoonup$" a weak convergence. Let $F \in C^1(X, \mathbb{R})$. A sequence $\{u_n\}$ is said to be the Palais–Smale sequence for $F$ at a level $c$ ($(PS)_c$-sequence for short) if $F(u_n) \to c$ and $F'(u_n) \to 0$ in $X^*$ as $n \to \infty$. We say that $F$ satisfies the Palais–Smale condition at level $c$ ($(PS)_c$ condition for short) if every $(PS)_c$ sequence is relatively compact in $X$.

By $\mathcal{D}^{1,2}(\mathbb{R}^N)$ we denote the Sobolev space obtained as the closure of $C_\circ^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|^2 = \int_{\mathbb{R}^N} |\nabla u(x)|^2 \, dx \ .$$

By $B(y, R)$ we always denote an open ball in $\mathbb{R}^N$ centered at $y$ of radius $R$.

## 2 – Preliminaries

Throughout this paper we assume that the potential $a(x)$ satisfies the following two conditions:

$(A_1)$ $a(x) \geq 0$ on $\mathbb{R}^N$ and the set $M = \{x \in \mathbb{R}^N;\ a(x) = 0\}$ is nonempty and bounded.

$(A_2)$ There exist two constants $p_1 < \frac{N}{2}$ and $p_2 > \frac{N}{2}$ (with $p_2 < 3$ if $N = 3$) such that $a \in L^p(\mathbb{R}^N)$ for each $p \in [p_1, p_2]$.

Benci–Cerami [3] established the existence of a positive solution of the equation $(1_\epsilon)$, with $\epsilon = 1$, and with $a$ satisfying $(A_2)$ and $\|a\|_{\frac{N}{2}} \leq S(2^{\frac{N}{2}} - 1)$. Here $S$ denotes the best Sobolev constant for a continuous embedding of $\mathcal{D}^{1,2}(\mathbb{R}^N)$ into $L^{2^*}(\mathbb{R}^N)$, that is,

$$S \;=\; \inf\left\{\int_{\mathbb{R}^N} |\nabla u(x)|^2\,dx;\ \|u\|_{2^*} = 1\right\}\,.$$

In this paper we examine the effect of topology of the set $M$ on the number of solutions of $(1_\epsilon)$.

We set for $\delta > 0$ small

$$M_\delta \;=\; \left\{x \in \mathbb{R}^N;\ \operatorname{dist}(x, M) \leq \delta\right\}$$

and

$$\Sigma \;=\; \left\{u \in \mathcal{D}^{1,2}(\mathbb{R}^N);\ \int_{\mathbb{R}^N} |u(x)|^{2^*}\,dx = 1\right\}\,.$$

It is well known that the positive solutions which are radially symmetric about some point in $\mathbb{R}^N$ of the equation

$$-\Delta u = |u|^{2^*-2}\,u \quad \text{in } \mathbb{R}^N \tag{2}$$

have form

$$U_{\lambda,y}(x) = \frac{\Big[N\,(N-2)\,\lambda\Big]^{\frac{N-2}{4}}}{\Big(\lambda + |x-y|^2\Big)^{\frac{N-2}{2}}}\,, \qquad \lambda > 0,\quad y \in \mathbb{R}^N\,,$$

with

$$\|U_{\lambda,y}\|_{2^*} = S^{\frac{N-2}{4}} \quad \text{and} \quad \|\nabla U_{\lambda,y}\|_2^2 = S^{\frac{N}{2}}\,.$$

Let $\bar{U}_{\lambda,y}(x) = S^{-\frac{N-2}{4}}\,U_{\lambda,y}(x)$. Then

$$\|\bar{U}_{\lambda,y}\|_{2^*} = 1 \quad \text{and} \quad \|\nabla \bar{U}_{\lambda,y}\|_2 = S\,.$$

We define the following functionals on $\mathcal{D}^{1,2}(\mathbb{R}^N)$:

$$J_\epsilon(u) = \epsilon^2 \int_{\mathbb{R}^N} \Big(|\nabla u|^2 + a(x)\, u^2\Big)\, dx \ ,$$

$$I_\epsilon(u) = \frac{1}{2} \int_{\mathbb{R}^N} \Big(\epsilon^2\, |\nabla u|^2 + a(x)\, u^2\Big)\, dx - \frac{1}{2^*} \int_{\mathbb{R}^N} |u|^{2^*} dx$$

and

$$I_\epsilon^\infty(u) = \frac{\epsilon^2}{2} \int_{\mathbb{R}^N} |\nabla u|^2\, dx - \frac{1}{2^*} \int_{\mathbb{R}^N} |u|^{2^*} dx \ .$$

To determine the energy levels of the functional $I_\epsilon$ for which the Palais–Smale condition holds, we need the following result due to Benci–Cerami [3].

**Theorem 1.** *Let $\{u_n\}$ be a $(PS)_c$-sequence for the functional $I_\epsilon$. Then there exist a number $k \in \mathbb{N}$, $k$ sequences of points $\{y_n^j\} \subset \mathbb{R}^N$, $j = 1, ..., k$, $k$ sequences of positive numbers $\{\sigma_n^j\}$, $j=1, ..., k$ and $k+1$ sequences of functions $\{u_n^j\} \subset \mathcal{D}^{1,2}(\mathbb{R}^N)$, $j = 0, 1, ..., k$, such that, up to a subsequence,*

$$u_n(x) = u_n^\circ(x) + \sum_{j=1}^k \frac{1}{(\sigma_n^j)^{\frac{N-2}{2}}}\, u_n^j \left(\frac{x - y_n^j}{\sigma_n^j}\right) , \tag{3}$$

$$u_n^j(x) \to u^j(x) \quad \text{in } \mathcal{D}^{1,2}(\mathbb{R}^N), \quad j = 0, ..., k \ , \tag{4}$$

*as $n \to \infty$, where $u^\circ$ is a solution of equation $(1_\epsilon)$, $u^j$, $j=1, ..., k$ are solutions of the equation*

$$-\epsilon^2 \Delta u = |u|^{2^*-2} u \quad \text{in } \mathbb{R}^N \ , \tag{5}$$

*and if $y_n^j \to \bar y^j$ as $n \to \infty$, then either $\sigma_n^j \to \infty$ or $\sigma_n^j \to 0$ as $n \to \infty$. Furthermore, we have*

$$\|u_n\|^2 \longrightarrow \sum_{j=0}^k \|u^j\|^2$$

*and*

$$I_\epsilon(u_n) \longrightarrow I_\epsilon(u^\circ) + \sum_{j=1}^k I_\epsilon^\infty(u^j)$$

*as $n \to \infty$.* ∎

Since for every nontrivial solution $u$ of $(1_\epsilon)$, $I_\epsilon(u) > \frac{\epsilon^N}{N} S^{\frac{N}{2}}$, for every positive solution $u$ of (2), $I_\epsilon^\infty(u) > \frac{\epsilon^N}{N} S^{\frac{N}{2}}$ and for every solution $u$ of (5) which changes sign we have $I_\epsilon^\infty(u) \geq \frac{2\,\epsilon^N}{N} S^{\frac{N}{2}}$, we deduce from Theorem 1:

**Corollary 1.** *Let $\{u_n\} \subset \mathcal{D}^{1,2}(\mathbb{R}^N)$ be a $(PS)_c$-sequence for $I_\epsilon$ with $\frac{\epsilon^N}{N} S^{\frac{N}{2}} < c < \frac{2\,\epsilon^N}{N} S^{\frac{N}{2}}$. Then $\{u_n\}$ is relatively compact in $\mathcal{D}^{1,2}(\mathbb{R}^N)$.* ∎

**Corollary 2.** *The functional $J_\epsilon|_\Sigma$ satisfies the $(PS)_c$-condition for $c \in (\epsilon^2 S,\, 2^{\frac{2}{N}} \epsilon^2 S)$.* ∎

The proof of the following lemma can be found in [3] (see formulae (3.7), (3.9) and (3.19) there).

**Lemma 1.** *We have*

(i) $$\lim_{\lambda\to 0} \int_{\mathbb{R}^N} a(x)\, \bar U_{\lambda,y}(x)^2\, dx \;=\; 0 \qquad \text{for every } \; y \in \mathbb{R}^N \;,$$

(ii) $$\lim_{\lambda\to \infty} \int_{\mathbb{R}^N} a(x)\, \bar U_{\lambda,y}(x)^2\, dx \;=\; 0 \qquad \text{for every } \; y \in \mathbb{R}^N \;,$$

*and*

(iii) $$\lim_{|y|\to \infty} \int_{\mathbb{R}^N} a(x)\, \bar U_{\lambda,y}(x)^2\, dx \;=\; 0 \qquad \text{for every } \; \lambda > 0 \;.$$ ∎

We choose $\rho > 0$ such that $M_\delta \subset B(0, \frac{\rho}{2})$, $\rho = \rho(\delta)$. Let

$$\chi(x) \;=\; \begin{cases} x & \text{for } |x| \le \rho, \\ \dfrac{\rho\, x}{|x|} & \text{for } |x| \ge \rho \;. \end{cases}$$

We define a "barycenter" $\beta\colon\, \Sigma \to \mathbb{R}^N$ by

$$\beta(u) \;=\; \int_{\mathbb{R}^N} \chi(x)\, |u(x)|^{2^*}\, dx$$

and set

$$\gamma(u) \;=\; \int_{\mathbb{R}^N} \Big| \chi(x) - \beta(u) \Big|\, |u(x)|^{2^*}\, dx \;.$$

The functional $\gamma$ measures the concentration of a function $u$ near its barycenter.

With the aid of $\bar U_{\lambda,y}$ we define a mapping $\Phi_{\lambda,y}\colon\, \mathbb{R}^N \to \Sigma$ by $\Phi_{\lambda,y}(\cdot) = \bar U_{\lambda,y}(\cdot)$. We note that

$$\begin{aligned} \beta(\Phi_{\lambda,y}) \;&=\; \int_{\mathbb{R}^N} \chi(x)\, \Phi_{\lambda,y}(x)^{2^*}\, dx \\ &=\; y + \int_{\mathbb{R}^N} \Big( \chi(\lambda\, z + y) - y \Big)\, \bar U_{1,0}(z)^{2^*}\, dz \qquad (6) \\ &=\; y + o(1) \end{aligned}$$

as $\lambda \to 0$. Let

$$V = V(\lambda_1, \lambda_2, \rho) = \left\{ (y, \lambda) \in \mathbb{R}^N \times \mathbb{R};\ |y| < \frac{\rho}{2},\ \lambda_1 < \lambda < \lambda_2 \right\} .$$

It follows from Lemma 1 that for every $\epsilon > 0$ there exist $\lambda_1 = \lambda_1(\epsilon)$ and $\lambda_2 = \lambda_2(\epsilon)$, with $\lambda_1 < \lambda_2$, such that

$$\text{(7)} \quad \sup\left\{ \epsilon^2 \int_{\mathbb{R}^N} |\nabla \Phi_{\lambda,y}|^2 \, dx + \int_{\mathbb{R}^N} a(x)\, \Phi^2_{\lambda,y} \, dx;\ (y, \lambda) \in V \right\} < \epsilon^2 (S + h(\epsilon)) ,$$

where $h(\epsilon) \to 0$ as $\epsilon \to 0$.

To examine the behaviour of $\gamma \circ \Phi_{\lambda,y}$ as $\lambda \to 0$ we need the following estimate.

**Lemma 2.** *Let $0 < 2\epsilon < \rho$ and $x \in B(y, \epsilon)$. Then*

$$|\chi(x) - \chi(y)| \le 2\,|x - y| + 2\,\epsilon .$$

**Proof:** We distinguish three cases: (i) $|y| \ge \rho + \epsilon$, (ii) $|y| \le \rho - \epsilon$ and (iii) $\rho - \epsilon \le |y| \le \rho + \epsilon$.

Case (**i**). Since $|x| \ge |y| - |x-y| \ge \rho$, we have

$$|\chi(x) - \chi(y)| = \rho \left| \frac{x}{|x|} - \frac{y}{|y|} \right| = \rho \, \frac{\big| x|y| - y|x| \big|}{|x|\,|y|} = \rho \, \frac{\big| x|y| - y|y| + y|y| - y|x| \big|}{|x|\,|y|} \le$$

$$\le \frac{\rho}{|x|} \Big( |x - y| + \big| |y| - |x| \big| \Big) \le 2\,|x - y| .$$

Case (**ii**). We have $|x| \le |x-y| + |y| \le \rho - \epsilon + \epsilon = \rho$ and $|\chi(x) - \chi(y)| = |x-y|$.

Case (**iii**). In this case $\rho - 2\,\epsilon \le |x| \le \rho + 2\,\epsilon$. If $|x| \le \rho$ and $|y| \le \rho$, then $|\chi(x) - \chi(y)| = |x - y|$. If $|x| \ge \rho$ and $|y| \ge \rho$, we show as in the case (i) that $|\chi(x) - \chi(y)| \le 2\,|x - y|$. If $|x| \le \rho$ and $|y| \ge \rho$, then

$$|\chi(x) - \chi(y)| = \left| x - \rho \frac{y}{|y|} \right| = \frac{\big| x|y| - \rho\, y \big|}{|y|} \le \frac{\big| x|y| - y|y| + |y| y - \rho\, y \big|}{|y|} \le$$

$$\le |x - y| + \big| |y| - \rho \big| \le |x - y| + \epsilon .$$

Finally, if $|x| \ge \rho$ and $|y| \le \rho$, then

$$|\chi(x) - \chi(y)| = \left| \rho \frac{x}{|x|} - y \right| = \frac{\big| x\rho - y|x| \big|}{|x|} \le \frac{|x\rho - \rho\, y| + \big| \rho\, y - y|x| \big|}{|x|} \le$$

$$\le \rho \, \frac{|x - y|}{|x|} + \frac{|y|}{|x|} \, (|x| - \rho) \le |x - y| + 2\,\epsilon . \ \blacksquare$$

**Lemma 3.** *We have* $\lim_{\lambda\to 0}\gamma\circ\Phi_{\lambda,y}=0$ *uniformly for* $|y|\le\frac{\rho}{2}$.

**Proof:** Let $0<2\epsilon<\rho$. We commence by observing that

$$\begin{aligned}\int_{\mathbb{R}^N-B(0,\epsilon)}\Phi_{\lambda,0}^{2^*}(x)\,dx &= C_N\int_{\mathbb{R}^N-B(0,\epsilon)}\frac{\lambda^{\frac{N}{2}}}{(\lambda+|x|^2)^N}\,dx\\ &= C_N\int_{|x|\ge\frac{\epsilon}{\sqrt{\lambda}}}\frac{1}{(1+|x|^2)^N}\,dx \longrightarrow 0\end{aligned}\tag{8}$$

as $\lambda\to 0$. We write

$$\int_{\mathbb{R}^N}\Big|\chi(x)-\beta\circ\Phi_{\lambda,y}\Big|\,\Phi_{\lambda,y}(x)^{2^*}\,dx =$$

$$=\int_{B(y,\epsilon)}\Big|\chi(x)-\beta\circ\Phi_{\lambda,y}\Big|\,\Phi_{\lambda,y}(x)^{2^*}\,dx+\int_{\mathbb{R}^N-B(y,\epsilon)}\Big|\chi(x)-\beta\circ\Phi_{\lambda,y}\Big|\,\Phi_{\lambda,y}(x)^{2^*}\,dx\;.$$

We deduce from (8) that

$$\lim_{\lambda\to 0}\int_{\mathbb{R}^N-B(y,\epsilon)}\Big|\chi(x)-\beta\circ\Phi_{\lambda,y}\Big|\,\Phi_{\lambda,y}(x)^{2^*}\,dx = 0\;.$$

The integral over $B(y,\epsilon)$ can be estimated using Lemma 2 and (6) as follows

$$\int_{B(y,\epsilon)}\Big|\chi(x)-\beta\circ\Phi_{\lambda,y}\Big|\,\Phi_{\lambda,y}(x)^{2^*}\,dx \le$$

$$\begin{aligned}&\le\int_{B(y,\epsilon)}\Big|\chi(x)-\chi(y)\Big|\,\Phi_{\lambda,y}(x)^{2^*}\,dx+\int_{B(y,\epsilon)}\Big|\chi(y)-\beta\circ\Phi_{\lambda,y}\Big|\,\Phi_{\lambda,y}(x)^{2^*}\,dx\\ &\le 2\int_{B(y,\epsilon)}|x-y|\,\Phi_{\lambda,y}(x)^{2^*}\,dx+2\,\epsilon\,S+o(1)\;.\end{aligned}$$

Since $\epsilon>0$ is arbitrary, $\lim_{\lambda\to 0}\gamma(\Phi_{\lambda,y})=0$. Due to the compactness of $\{y\colon |y|\le\frac{\rho}{2}\}$ this convergence can be made uniform on this set. ∎

We now define a set $\Sigma_\epsilon\subset\Sigma$ by

$$\Sigma_\epsilon=\Big\{u\in\Sigma;\;\; S<\epsilon^{-2}J_\epsilon(u)<S+h(\epsilon),\;\;(\beta(u),\gamma(u))\in V\Big\}\;,$$

where $V$ has been chosen so that (7) holds. According to Lemma 3 we can modify $\lambda_1(\epsilon)$ and $\lambda_2(\epsilon)$ so that $\Sigma_\epsilon\ne\emptyset$ for each $\epsilon>0$ small.

**Proposition 3.** *We have*

$$\lim_{\epsilon\to 0}\sup_{u\in\Sigma_\epsilon}\inf_{y\in M_\delta}\Big[\beta(u)-\beta(\Phi_{\lambda,y})\Big]=0\;.\tag{9}$$

**Proof:** Let $\{\epsilon_n\}$ be a sequence of positive numbers such that $\epsilon_n \to 0$. For every $n$ there exists $u_n \in \Sigma_{\epsilon_n}$ such that

$$\inf_{y\in M_\delta}\Big[\beta(u_n)-\beta(\Phi_{\epsilon_n,y})\Big] \;=\; \sup_{u\in\Sigma_{\epsilon_n}}\inf_{y\in M_\delta}\Big[\beta(u)-\beta(\Phi_{\epsilon_n,y})\Big]+o(1)\ .$$

In order to prove (9) it is sufficient to find a sequence $\{y_n\}\subset M_\delta$ such that

$$\lim_{n\to\infty}\Big[\beta(u_n)-\beta(\Phi_{\epsilon_n,y_n})\Big]=0\ . \tag{10}$$

Since $\{u_n\}\subset\Sigma_{\epsilon_n}$ we have

$$\epsilon_n^2 S \;\le\; \epsilon_n^2\int_{\mathbb{R}^N}|\nabla u_n|^2\,dx \;\le\; \epsilon_n^2\int_{\mathbb{R}^N}|\nabla u_n|^2\,dx+\int_{\mathbb{R}^N}a(x)\,u_n^2\,dx \;\le\; \epsilon_n^2(S+h(\epsilon_n))\ .$$

Hence

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|\nabla u_n|^2\,dx \;=\; S \quad\text{and}\quad \lim_{n\to\infty}\int_{\mathbb{R}^N}a(x)\,u_n^2\,dx \;=\; 0\ . \tag{11}$$

It then follows from Corollary 2.11 in [3] that there exist a sequence of points $\{y_n\}\subset\mathbb{R}^N$, a sequence $\{\delta_n\}\subset(0,\infty)$ and a sequence of functions $\{w_n\}\subset\mathcal{D}^{1,2}(\mathbb{R}^N)$ such that $w_n\to 0$ as $n\to\infty$ and

$$u_n(x)=w_n(x)+\Phi_{\delta_n,y_n}(x)\qquad\text{on }\ \mathbb{R}^N\ .$$

We claim that (i) $\delta_n\to 0$ and (ii) $\{y_n\}$ is bounded. We begin by showing that $\{\delta_n\}$ is bounded. In the contrary case we may assume that $\delta_n\to\infty$ as $n\to\infty$. Since $w_n\to 0$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$, we have

$$\beta(u_n) \;=\; \beta(\Phi_{\delta_n,y_n})+o(1)\ . \tag{12}$$

Indeed, (12) follows from the following relation

$$\begin{aligned}
\beta(u_n) &= \int_{\mathbb{R}^N}\chi(x)\,|u_n|^{2^*}\,dx\\
&= \int_{\mathbb{R}^N}\chi(x)\,|w_n+\Phi_{\delta_n,y_n}|^{2^*}\,dx\\
&= \int_{\mathbb{R}^N}\chi(x)\,\Phi_{\delta_n,y_n}^{2^*}\,dx \;+\; O\Big(\int_{\mathbb{R}^N}|w_n|^{2^*-1}\,\Phi_{\delta_n,y_n}\,dx\Big)\\
&= \int_{\mathbb{R}^N}\chi(x)\,\Phi_{\delta_n,y_n}^{2^*}\,dx \;+\; O\Big(\|w_n\|_{2^*}^{2^*-1}\,\|\Phi_{\delta_n,y_n}\|_{2^*}\Big)\\
&= \int_{\mathbb{R}^N}\chi(x)\,\Phi_{\delta_n,y_n}^{2^*}\,dx \;+\; o(1)\ .
\end{aligned}$$

Therefore we may assume that

$$\beta(\Phi_{\delta_n,y_n}) \subset B\Big(0,\frac{\rho}{2}\Big) . \tag{13}$$

We now observe that for each $R>0$ we have

$$\lim_{n\to\infty}\int_{B(0,R)}\Phi^{2^*}_{\delta_n,y_n}\,dx = 0 ,$$

since $\lim_{n\to\infty}\delta_n=\infty$. Using this and (13) we can write the following inequalities

$$\begin{aligned}
\gamma\circ\Phi_{\delta_n,y_n} &= \int_{\mathbb{R}^N}\Big|\chi(x)-\beta\circ\Phi_{\delta_n,y_n}\Big|\,\Phi_{\delta_n,y_n}(x)^{2^*}\,dx\\
&\ge \int_{\mathbb{R}^N}|\chi(x)|\,\Phi_{\delta_n,y_n}(x)^{2^*}\,dx - |\beta\circ\Phi_{\delta_n,y_n}|\\
&\ge \int_{\mathbb{R}^N}|\chi(x)|\,\Phi_{\delta_n,y_n}(x)^{2^*}\,dx - \frac{\rho}{2}\\
&\ge \rho\int_{\mathbb{R}^N-B(0,\rho)}\Phi^{2^*}_{\delta_n,y_n}\,dx - \frac{\rho}{2}+o(1)\\
&= \rho\int_{\mathbb{R}^N}\Phi^{2^*}_{\delta_n,y_n}\,dx - \frac{\rho}{2}+o(1)\\
&= \frac{\rho}{2}+o(1) .
\end{aligned}\tag{14}$$

On the other hand we have

$$\gamma(u_n) = \gamma(\Phi_{\delta_n,y_n})+o(1) ,$$

because $w_n\to 0$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. Since $u_n\in\Sigma_{\epsilon_n}$ we have that

$$\lambda_1(\epsilon_n)<\gamma(u_n)<\lambda_2(\epsilon_n) \tag{15}$$

with $\lambda_i(\epsilon_n)\to 0$, $i=1,2$, as $\epsilon_n\to\infty$. This contradicts the estimate (14) and therefore $\{\delta_n\}$ is bounded. It remains to show that $\delta_n\to 0$. In the contrary case we may assume that $\delta_n\to\bar\delta>0$ as $n\to\infty$. Then we must have that $|y_n|\to\infty$ as $n\to\infty$, since otherwise $\Phi_{\delta_n,y_n}$ would converge strongly in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ and so would $u_n$. Consequently $J_\epsilon$ subject to the constraint $\Sigma$ would have minimizer which is impossible by Proposition 2.2 in [3]. We now observe that for every $R>0$, the fact that $\lim_{n\to\infty}|y_n|=\infty$, implies that $\lim_{n\to\infty}\int_{B(0,R)}\Phi^{2^*}_{\delta_n,y_n}dx=0$. Consequently one can easily show that the estimate (14) must be valid giving the contradiction with the fact that $u_n$ satisfies (15). The proof of the claim (ii) is similar and it is omitted. We now choose subsequences of $\{\delta_n\}$ and $\{\epsilon_n\}$ so that

$\frac{\delta_{n_i}}{\epsilon_{n_i}} = o(1)$ as $n_i \to \infty$. So we may replace $\delta_{n_i}$ by $\epsilon_{n_i}$. The new sequence $\{\epsilon_{n_i}\}$ is relabelled again by $\{\epsilon_n\}$. Suppose that $y_n \to \bar{y}$. Let

$$v_n(x) \,=\, \epsilon_n^{\frac{N-2}{2}}\, u_n(\epsilon_n\, x + y_n) \ .$$

Then $v_n \to U_{1,0}$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. Since $\int_{\mathbb{R}^N} |\nabla v_n|^2\, dx \to S$ and

$$\begin{aligned} \epsilon_n^2 S \,&<\, \epsilon_n^2 \int_{\mathbb{R}^N} |\nabla u_n|^2\, dx \,+\, \int_{\mathbb{R}^N} a(x)\, u_n^2\, dx \\ &\quad \epsilon_n^2 \left[ \int_{\mathbb{R}^N} \Big( |\nabla v_n|^2 + a(\epsilon_n\, x + y_n)\, v_n^2 \Big)\, dx \right] \\ &<\, \epsilon_n^2 (S + h(\epsilon_n)) \ , \end{aligned}$$

we deduce that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} a(\epsilon_n\, x + y_n)\, v_n^2\, dx \,=\, 0 \ .$$

This implies that $\int_{\mathbb{R}^N} a(\bar{y})\, U_{1,0}\, dx = 0$ and so $a(\bar{y}) = 0$. This means that $\bar{y} \in M$. Therefore $y_n \in M_\delta$ for large $n$. The relation (10) follows from the fact that $w_n \to 0$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. ∎

## 3 – Main result

We are now in a position to formulate our main result on the existence of multiple solutions in terms of $\mathrm{cat}_{M_\delta} M$.

**Theorem 2.** *For small $\epsilon > 0$ the problem $(1_\epsilon)$ has $\mathrm{cat}_{M_\delta} M$ solutions.*

**Proof:** We fix an $\epsilon > 0$ small. Then $\Phi_{\lambda,y}\colon [\lambda_1, \lambda_2] \times M \to \Sigma_\epsilon$ and by virtue of (6) and Proposition 3, $\beta(\Sigma_\epsilon) \subset M_\delta$. Therefore $\beta \circ \Phi_{\lambda,y}\colon [\lambda_1, \lambda_2] \times M \to [\lambda_1, \lambda_2] \times M_\delta$ and it is easy to check that $\beta \circ \Phi_{\lambda,y}\colon [\lambda_1, \lambda_2] \times M \to [\lambda_1, \lambda_2] \times M_\delta$ is homotopic to the inclusion map $\mathrm{id}\colon [\lambda_1, \lambda_2] \times M \to [\lambda_1, \lambda_2] \times M_\delta$. The functional $J_\epsilon$ satisfies the $(PS)_c$-condition for $c \in (\epsilon^2 S,\, \epsilon^2 (S + h(\epsilon)))$. Hence by the Lusternik–Schnirelman theory of critical points (see [3], [4], [5])

$$\mathrm{cat}(\Sigma_\epsilon) \,\geq\, \mathrm{cat}_{[\lambda_1, \lambda_2] \times M_\delta}([\lambda_1, \lambda_2] \times M) \,=\, \mathrm{cat}_{M_\delta} M \ . \ ∎$$

**Remark.** Using the argument of Lemma 2.7 in [18] one can show that solutions obtained in Theorem 2 are positive. □

In the next result we show that solutions $u_\epsilon$ obtained in Theorem 2 concentrate on $M$ as $\epsilon \to 0$.

**Theorem 3.** *Let $\{u_\epsilon\}$ be solutions from Theorem 2. Then $u_\epsilon \to \bar{U}_{0,\bar{y}}$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ as $\epsilon \to 0$ and $\bar{y} \in M$.*

**Proof:** It follows from (11), that

$$\lim_{\epsilon\to 0} \int_{\mathbb{R}^N} |\nabla u_\epsilon|^2 \, dx \;=\; S \quad\text{ and }\quad \lim_{\epsilon\to 0} \int_{\mathbb{R}^N} a(x)\, u_\epsilon^2 \, dx \;=\; 0 \;.$$

Thus $u_\epsilon = w_\epsilon + \Phi_{\delta_\epsilon, y_\epsilon}$. As in Proposition 3 we show that $\delta_\epsilon \to 0$ and $y_\epsilon \to \bar{y} \in M$ as $\epsilon \to 0$. ∎

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