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EXTENDED COMPOSITION OPERATORS IN WEIGHTED SPACES

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Abstract: Let X and Y be Hausdorff completely regular spaces and βX the Stone-Čech compactification of X. For locally convex spaces E and F consisting of continuous functions respectively on X and Y and whose topologies are generated by seminorms that are weighted analogues of the suprimum norm, we give necessary and sufficient conditions for a linear mapping $T : E \to F$ to be an extended composition operator. This means that there exists some map $\varphi : Y \to \beta X$ so that $T(f) = C_\varphi(f)$ $(:= \tilde{f} \circ \varphi)$, $f \in E$. Here $\tilde{f}$ stands for the Stone extension of f. We also characterize those maps φ for which C_φ satisfies one of the following conditions: **1**) C_φ is continuous; **2**) C_φ maps some 0-neighbourhood into a bounded set; **3**) C_φ maps some 0-neighbourhood into an equicontinuous, a compact or a weakly compact set; **4**) C_φ maps any bounded set into an equicontinuous, a compact or a weakly compact set.

1 – Introduction and preliminaries

Let X be a Hausdorff completely regular space and $\mathbb{K}$ the real or complex field. Denote by $\mathcal{F}(X)$ (resp. $B(X)$, $C(X)$) the algebra of all $\mathbb{K}$-valued (resp. bounded $\mathbb{K}$-valued, continuous $\mathbb{K}$-valued) functions on X and put $C_b(X) := C(X) \cap B(X)$. We will say that $u \in \mathcal{F}(X)$ vanishes at infinity if, for every $\epsilon > 0$, some compact $K \subset X$ exists with $|u(x)| < \epsilon$ for all $x \notin K$. The set of all $f \in C(X)$ vanishing at infinity is denoted by $C_0(X)$. A Nachbin family on X is any collection V of non negative upper semicontinuous (in short u.s.c.) real functions such that:

1) for every $x \in X$, there exists some $v \in V$ with $v(x) \neq 0$, and

2) for every $v_1, v_2 \in V$ and $\lambda > 0$, there exists $v \in V$ so that $\lambda v_i \leq v$, $i=1,2$.

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The weighted spaces associated with V are:

$$CV(X) := \left\{ f \in C(X) \colon \ \sup_{t \in X} v(t)\,|f(t)| < +\infty, \ \ \forall\, v \in V \right\}$$

and

$$CV_0(X) := \left\{ f \in C(X) \colon \ fv \text{ vanishes at infinity for every } v \in V \right\} .$$

These spaces are supplied with the weighted topology τ_V generated by the seminorms

$$f \mapsto |f|_v := \sup_{t \in X} v(t)\,|f(t)|, \quad v \in V .$$

The most used Nachbin families on X are:

$$Z := \left\{ \text{non negative constant functions} \right\} ,$$

$$K := \left\{ \text{positive multiples of characteristic functions of compact subsets of } X \right\} ,$$

$$S_0^+ := \left\{ \text{non negative u.s.c. functions on } X \text{ vanishing at infinity} \right\} ,$$

$$F := \left\{ \text{positive multiples of characteristic functions of finite subsets of } X \right\} .$$

The corresponding weighted spaces are respectively $CZ(X) = C_b(X)$ and $CZ_0(X) = C_0(X)$ with the uniform convergence topology σ, $CK(X) = CK_0(X) = C(X)$ with the compact open topology τ_c, $CS_0^+(X) = C(S_0^+)_0(X) = C_b(X)$ with the strict topology β and $CF(X) = CF_0(X) = C(X)$ with the pointwise convergence one τ_s. For further details concerning the weighted spaces, see [1] and [9].

In this paper we deal with extended composition operators from arbitrary subspaces E of a weighted space $CV(X)$ into another weighted space $CU(Y)$. Historically, the composition operators on Hilbert spaces $L^2(X)$, $X = \mathbb{R}$, $\mathbb{Z}$ or $[0,1]$, appeared originally in the work [3] of B.O. Koopmann in connection with classical mechanics. A survey on such operators on spaces of type L^2 and H^2 was given by E.A. Nordgren [7]. Then the composition operators were generalized in several directions. In [2] H. Kamowitz gave a characterization of compact composition operators on the algebra $C(X)$ for X compact. While R.K. Singh and W.H. Summers studied in [12] the composition operators on general weighted spaces of type $CV(X)$ and $CV_0(X)$. There, composition operators were characterized among the linear mappings T from $CV(X)$ or $CV_0(X)$ into itself. It was also given necessary and sufficient conditions on $\varphi \colon X \to X$ so that $C_\varphi \colon f \mapsto f \circ \varphi$ maps continuously $CV(X)$ or $CV_0(X)$ into itself. In [4] the authors consider the special weighted space $CK(X)$, but allow T to land in $CK(Y)$ for some other Hausdorff completely regular space Y. They then gave necessary and sufficient conditions on $\varphi\colon Y \to X$ under which the composition operator $C_\varphi : f \mapsto f \circ \varphi$ is locally (weakly) compact, i.e. C_φ maps every bounded subset of $CK(X)$ into a (weakly) compact subset of $CK(Y)$. Here, we not only consider composition operators $T\colon E \to CU(Y)$ from a general subspace E of $CV(X)$ into a general weighted space $CU(Y)$, but also we allow the map φ to take values outside of X, namely in the Stone-Čech compactification βX of X. In this way we recover, even in case $E = CU(Y)$, operators which fail to be composition ones in the classical sense (see Example 2.2). We then characterize those (continuous) linear operators which turn out to be extended composition ones. Moreover, we provide necessary and sufficient conditions on $\varphi\colon Y \to \beta X$ so that the induced extended composition operator C_φ satisfies one of the following conditions:

1. C_φ is continuous,

2. C_φ maps some 0-neighbourhood into a bounded set,

3. C_φ maps some 0-neighbourhood into an equicontinuous, a compact or a weakly compact set, or

4. C_φ maps any bounded set into an equicontinuous, a compact or a weakly compact set.

Henceforth, unless the contrary is stated, all subspaces of $CV(X)$ in consideration will be supplied with the topology induced by τ_V. For every such subspace E and every $v \in V$, we will denote by $B_v(E)$ the set $\{f \in E\colon |f|_v \leq 1\}$ and by $\operatorname{coz}(E)$ the set of all elements of X at which at least one $f \in E$ does not vanish; here E may be any subset of $C(X)$. We will also consider the following sets:

$$\mathbb{F}(E) := \Big\{x \in \beta X\colon\ \tilde{f}(x) \neq \infty, \text{ for every } f \in E\Big\}\ ,$$

$$M^*(E) := \Big\{x \in \mathbb{F}(E)\colon\ \tilde{f}(x) \neq 0 \text{ for some } f \in E\Big\}\ ,$$

$$M^+(E) := \Big\{x \in M^*(E)\colon\ \delta_x, \text{ the evaluation at } x, \text{ is bounded on bounded subsets of } E\Big\}\ .$$

If $v \in V$ is given, using Lemma 9 of [8], one can extend v by upper semicontinuity to $M^*(E)$ by putting

$$\tilde{v}(t) := \inf\Big\{w(t);\, w\colon\ M^*(E) \to \mathbb{R}^+ \text{ u.s.c. and } w_{|X} \geq v\Big\}\ .$$

This extension is minimal and verifies, for every $f \in E$,

$$|f|_v = \sup_{t \in M^*(E)} \tilde{v}(t)\,|\tilde{f}(t)| \;.$$

Denote by $\widetilde{V}$ the collection of all $\tilde{v}$, when v runs over V, and consider the set

$$M(E) := \Big\{ x \in M^*(E)\colon\;\; \tilde{v}(x) \neq 0 \;\text{ for some } v \in V \Big\} \;.$$

Whenever E is either a (complex) selfadjoint algebra and a $C_b(X)$-module, the sets $M^*(E)$, $M^+(E)$ and $M(E)$ are nothing but the algebraic, the bounded and the continuous characters (i.e. non zero multiplicative functionals) on E respectively [8]. They may differ from each other as shows the following example: let X be the real line, $C_0^+(X)$ the set of all $\mathbb{R}^+$-valued continuous functions on X vanishing at infinity and δ the continuous function defined on X by $\delta(x)=1$ for $x \in \mathbb{N}$, $\delta(x)=0$ for $x \leq 1/2$ or $n + 1/2n \leq x \leq n+1 - 1/2(n+1)$ for some $n \in \mathbb{N}$ and δ piecewise linear. For every $v \in C_0^+(X)$, put $u_v := \max(v,\delta)$ and consider the Nachbin family $U := \{\lambda\, u_v\colon\; v \in C_0^+(X),\;\; \lambda > 0\}$. We then have $CU(X) = C_b(X)$ algebraically and the topology τ_U is stronger than β but coarser then σ. It is clear that every $x \in \overline{\mathbb{N}}^{\beta X}$ defines a continuous character on $CU(X)$ while the evaluation at every point of $\overline{\{n - 1/2n,\; n \in \mathbb{N}\}}^{\beta X} \setminus \mathbb{R}$ defines a discontinuous one. Hence $M(E) \neq M^*(E)$. To get a case where the three sets are all different, take the product of the foregoing example and $C_b(\mathbb{R})$ with the compact open topology. This is a subalgebra of $CV(Y)$, where Y is the disjoint union of $\mathbb{R}$ and $\mathbb{R}$, and V the product $U \times K$ (see [8]).

One may think that E can be considered as a topological subspace of $CU(M^*(E))$ for some Nachbin family U on $M^*(E)$, and then some of our results derive from known ones. This is not true, since otherwise every character of E should be continuous.

In all what follows, X and Y will stand for Hausdorff completely regular spaces, V and U Nachbin families respectively on X and Y and E a vector subspace of $C(X)$. For a linear map $T\colon E \to CU(Y)$, set $Y_{T,E} := \operatorname{coz}(T(E))$ and $T_a := R_a \circ T$, where $R_a\colon\; CU(Y) \to CU_a(Y_{T,E})$ denotes the restriction map and U_a the relative Nachbin family induced by U on $Y_{T,E}$. Whenever $Y_{T,E}$ coincides with the whole of Y we say that $T(E)$ is essential.

2 – Extended composition operators

In this section we define the extended composition operators, give examples, characterize the maps $\varphi : Y \to \beta X$ so that $C_\varphi : E \to CU(Y)$ is a continuous extended composition operator and exhibit among the linear mappings $T\colon E \to CU(Y)$ those which are extended composition operators.

Definition 1. A linear map T from E into $\mathcal{F}(Y)$ is called an extended composition operator (in short *e.c.o.*) if there exists a map φ from Y into βX such that, for every $f \in E$, $T(f) = \tilde{f} \circ \varphi$. We will then write $T = C_\varphi$. □

Notice that φ must map Y into $\mathbb{F}(E)$, since $\tilde{f} \circ \varphi$ is a (finite) function. For such a map, let us denote by Y_a, Y_b and Y_c respectively the sets $\varphi^{-1}(M^*(E))$, $\varphi^{-1}(M^+(E))$ and $\varphi^{-1}(M(E))$; and by φ_a, φ_b and φ_c the restriction of φ to Y_a, Y_b and Y_c respectively. Actually Y_a is nothing but $\mathrm{coz}(C_\varphi(E)) = Y_{C_\varphi,E}$.

Examples 2.

1. Every composition operator (i.e. induced by a map $\varphi\colon Y \to X$) is an *e.c.o.*.
2. Consider $X = Y = \{0\} \cup [1,\infty)$ with the relative topology induced by $\mathbb{R}$, $V = U = Z$ and $E = CV_0(X)$ ($= C_0(X)$ with the topology of the uniform norm). If $p \in \beta X \backslash X$ is fixed, then put
$$T(f)(t) \;=\; \begin{cases} f(t)\,, & t \in [1,\infty), \\ \tilde{f}(p)\,, & t = 0\ . \end{cases}$$
Then T is a continuous linear map from E into $CU(Y)$ and $T(fg) = T(f)\,T(g)$ for every $f, g \in E$. If we set $\varphi(t) = t$ for $t \neq 0$ and $\varphi(0) = p$, we get that $T = C_\varphi$ and then T is an extended composition operator. It is shown in [12] that T is not a composition operator (example 3.6).
3. Set $X = Y = \mathbb{R}$, $U = K$ and V the system of those weights of the form $\max(v, \sin)$, $v \in S_0^+$. Put then $T\colon CV(\mathbb{R}) \to CK(\mathbb{R})$ assigning to any f the constant function with value $\tilde{f}(t_0)$; t_0 being a cluster point in $\beta\,\mathbb{R}$ of the set $\{2\,n\,\pi + \frac{\pi}{2},\ n \in \mathbb{N}\}$. Then T is a continuous *e.c.o.* induced by the constant map $\varphi(t) = t_0$.
4. The same example as in 3. with t_0 a cluster point of the set $\{2\,n\,\pi,\ n \in \mathbb{N}\}$. Here T is bounded but not continuous.
5. Take $X = Y = \mathbb{R}$, $E = C_0(E)$ and $T : E \to \mathcal{F}(\mathbb{R})$ defined by $T(f) = f$ on $\mathbb{R}^+$ and 0 elsewhere. Then T is an *e.c.o.* induced by the map $\varphi\colon \mathbb{R} \to \beta\,\mathbb{R}$ defined by $\varphi(t) = t$ for $t > 0$ and $\varphi(t) = p$ for some $p \in \beta\,\mathbb{R} \backslash \mathbb{R}$ elsewhere. Here the range of T is not contained in $C(Y)$. □

We now give a characterization of continuous composition operators between weighted spaces. To avoid trivialities, we will assume that $Y_a \neq \emptyset$ in the following result.

Theorem 3. *Let $E \subset CV(X)$ be a $C_b(X)$-module and $\varphi\colon Y \to \mathbb{F}(E)$ a map. If $C_\varphi(E) \subset C(Y)$, then Y_a is open and φ_a is continuous. Moreover the following are equivalent:*

1. *C_φ maps continuously E into $CU(Y)$.*
2. *$C_\varphi(E) \subset C(Y)$ and $U_a \leq \widetilde{V} \circ \varphi_a$ on Y_a.*

Proof: Let $t_0 \in Y_a$ and Ω a neighbourhood of $\varphi(t_0)$ in βX. There is some $f \in E$ and $g \in C(\beta X)$ with $\tilde{f}(\varphi(t_0)) = 1$, $g(\varphi(t_0)) = 1$, $0 \leq g \leq 1$ and $\operatorname{supp}(g) \subset \Omega$. Put $h = fg$, we still denote the restriction of g to X by g. Then $h \in E$ and one has: $\tilde{h}(\varphi(t_0)) = \tilde{f}g(\varphi(t_0)) = 1$. But $\tilde{h} \circ \varphi$ is continuous. Hence $G := \{y \in Y : \frac{1}{2} < |\tilde{h} \circ \varphi(y)| < \frac{3}{2}\}$ is open and contains t_0. But $G \subset Y_a$ and $\varphi(G) \subset \Omega$, hence Y_a is open and φ_a continuous.

For the equivalence of 1. and 2., notice that φ_a is continuous in both cases.

1.$\Rightarrow$2.: Fix $y_0 \in Y_a$. There exists some $g \in E$ with $\tilde{g}(\varphi(y_0)) = 1$. The continuity of C_φ gives

$$\forall\, u \in U,\ \ \exists\, v \in V\colon \quad |\tilde{f} \circ \varphi|_u \leq |f|_v, \quad f \in E\ .$$

In particular

$$u(y_0)\,|\tilde{f}(\varphi(y_0))| \;\leq\; \sup_{t \in X} v(t)\,|f(t)|\,, \qquad f \in E\ .$$

For every $n \in \mathbb{N}$, consider an open subset $\Omega_{n,1}$ of βX such that $\Omega_{n,1} \cap M^*(E) = \{x \in M^*(E)\colon \tilde{v}(x) < \tilde{v}(\varphi(y_0)) + 1/n\}$ and put $\Omega_n := \Omega_{n,1} \cap \{x \in \beta X\colon 1 - 1/n < |\tilde{g}(x)| < 1 + 1/n\}$. Then Ω_n is a neighbourhood of $\varphi(y_0)$ in βX. Choose $f_n \in C(\beta X)$ with $0 \leq f_n \leq 1$, $f_n(\varphi(y_0)) = 1$ and $\operatorname{supp} f_n \subset \Omega_n$. Since $h_n := gf_n$ belongs to E, one has

$$u(y_0)\,|\tilde{h}_n(\varphi(y_0))| \;\leq\; \sup_{t \in X} v(t)\,|h_n(t)|\,, \qquad n \in \mathbb{N}\ ,$$

or

$$u(y_0)\,\Big|\tilde{g}_n(\varphi(y_0))\, f_n(\varphi(y_0))\Big| \;\leq\; \sup_{t \in \Omega_n} \tilde{v}(t)\,|h_n(t)|\,, \qquad n \in \mathbb{N}\ .$$

Whence

$$u(y_0) \;\leq\; \Big[\tilde{v}(\varphi(y_0)) + 1/n\Big]\,(1 + 1/n)\ .$$

Passing to the limit, we get $u(y_0) \leq \tilde{v} \circ \varphi(y_0)$.

2. $\Rightarrow$ 1.: Assume that $U_a \le \widetilde{V} \circ \varphi_a$. Then for every $u \in U$, there is some $v \in V$ such that $u \le \tilde{v} \circ \varphi$ on Y_a. Hence, for every $f \in E$, $\tilde{f} \circ \varphi$ is continuous and one has

$$\begin{aligned} |\tilde{f} \circ \varphi|_u &= \sup_{y \in Y} u(y)\, |\tilde{f}(\varphi(y))| \\ &= \sup_{y \in Y_a} u(y)\, |\tilde{f}(\varphi(y))| \\ &\le \sup_{y \in Y_a} \tilde{v}(\varphi(y))\, |\tilde{f}(\varphi(y))| \ \le\ |f|_v \ . \end{aligned}$$

This gives at once $\tilde{f} \circ \varphi \in CU(Y)$ and the continuity of C_φ. ∎

Remark 4.

1. Example 5 above shows that, for a given $\varphi : Y \to \mathbb{F}(E)$, the inclusion $C_\varphi(E) \subset C(Y)$ need not hold although Y_a is open and φ_a continuous.
2. A consequence of the condition 2 of Theorem 3 is that, in order C_φ to be continuous, φ must map Y_a into $M(E)$. Hence $Y_c = Y_a$ and $\varphi_c = \varphi_a$. Furthermore the implication 2. $\Rightarrow$ 1. holds although E fails to be a $C_b(X)$-module. □

It is known that if T is a continuous *unital* algebra morphism from $CK(X)$ into $CK(Y)$, then there is a continuous map φ from Y into X such that $T = C_\varphi$ (see [5]). Actually the condition *unital* is not redundant for T to be induced by a map φ. The multiplication by the characteristic function of a closed and open subset of a disconnected space X provides a continuous algebra morphism from $CK(X)$ into itself. But such a morphism is not induced by any self map $\varphi\colon X \to X$. The following lemma yields an extended version of the result of [5]. Here we drop the continuity condition and take general subspaces of $C(X)$.

Lemma 5. *Let E be a selfadjoint subalgebra of $C(X)$ which is also a $C_b(X)$-module and T a linear map from E into $C(Y)$. Assume in addition that either $M^*(E) \ne \mathbb{F}(E)$ or $T(E)$ essential. Then T is an e.c.o. if and only if it is an algebra morphism.*

Proof: The necessity is trivial in both cases. For the sufficiency, assume first that some $x_0 \in \mathbb{F}(E)\backslash M^*(E)$ exists and let $y \in Y$ be given. If $y \notin Y_{T,E}$, then set $\varphi(y) = x_0$. Otherwise $\delta_y \circ T$ is a character of E and by Corollary 14 of [8], there is a (unique) $x \in M^*(E)$ such that $\delta_y \circ T = \delta_x$. Put then $\varphi(y) = x$ and obtain $T(f)(y) = \tilde{f}(\varphi(y))$ for every $y \in Y$ and $f \in E$. Now, if $T(E)$ is essential, then $Y_{T,E} = Y$ and we argue as above. ∎

If none of the conditions $M^*(E) \neq \mathbb{F}(E)$ and $T(E)$ *is essential* is fulfilled, then T may fail to be an *e.c.o.*. The example given in Remark 4 yields such a situation.

Despite its algebraic aspect, Lemma 5 provides, as an application, a characterization of continuous extended composition operators between some weighted spaces. This includes in particular Theorem 3.3 and Theorem 3.4 of [12]. Moreover our proof does not utilize the approximation theory as in [12]. In the following, let us write E_b to mean $E \cap B(X)$.

Proposition 6. *Let $E \subset CV(X)$ be a vector space, $T\colon E \to CU(Y)$ a linear mapping and $F \subset E$ an algebra which is either selfadjoint and a $C_b(X)$-module.*

1. *Suppose that $Y_{T,F} = Y_{T,E}$. If $M^*(F) \subset \mathbb{F}(E)$ or T continuous and $M(F) \subset \mathbb{F}(E)$, then T_a is an e.c.o. if and only if $T(fg) = T(f)\,T(g)$ for every $f \in E$, $g \in F$ with $fg \in F$. If in addition $M^*(E) \neq \mathbb{F}(E)$, the conclusion still holds for T instead of T_a.*
2. *If T is continuous and F dense in E, then T_a is an e.c.o. if and only if $T(fg) = T(f)\,T(g)$ for every $f, g \in F$.*

Proof: It is clear that if T_a or T is an *e.c.o.*, then $T(fg) = T(f)\,T(g)$ even for every $f, g \in E$. For the converse, by Lemma 5, there exists $\varphi_1\colon Y_{T,F} \to M^*(F)$ such that $T(f)(y) = \tilde{f} \circ \varphi_1(y)$ for every $y \in Y_{T,F}$ and $f \in F$.

1. If $f \in E$ and $y \in Y_{T,E}$ are given, then there exists $g \in F$ such that $\tilde{g}(\varphi_1(y)) = 1$. Choose $h \in C(\beta X)$ such that $h(\varphi_1(y)) = 1$, $0 \leq h \leq 1$ and $\operatorname{supp} h \subset \{x \in \beta X : |\tilde{f}(x) - \tilde{f}(\varphi_1(y))| < 1\}$. The function fhg belongs to F and since $\varphi_1(y) \in \mathbb{F}(E)$, we get $T(fhg)(y) = \widetilde{fhg}(\varphi_1(y)) = \tilde{f}(\varphi_1(y))$ on one hand, and $T(fhg)(y) = T(f)(y)\,T(hg)(y) = T(f)(y)$ on the other hand. Since y is arbitrary in $Y_{T,F} = Y_{T,E}$, $T_a(f) = \tilde{f} \circ \varphi_1$. Now, if T is assumed to be continuous, then the map φ_1 takes its values in $M(F)$ and we only need that the latter be contained in $\mathbb{F}(E)$. Furthermore if $M^*(E) \neq \mathbb{F}(E)$, we extend φ_1 to the whole of Y by putting $\varphi(y) = x_0$ for a fixed $x_0 \in \mathbb{F}(E) \backslash M^*(E)$ and every $y \notin Y_{T,E}$. This gives $T = C_\varphi$.

2. Since F is dense in E, $M(F) = M(E)$. Moreover the continuity of T yields $Y_{T,F} = Y_{T,E}$. To conclude using 1., we only need to show that $T(fg) = T(f)\,T(g)$ for every $f \in E$, $g \in F$ with $fg \in F$. Let then $f \in E$, $g \in F$ be so that $fg \in F$ and $y \in Y_{T,E}$. There exists a net $(f_i)_{i \in I} \subset F$ converging to f in E and some $h \in C(\beta X)$ with $h(\varphi_1(y)) = 1$, $0 \leq h \leq 1$ and $\operatorname{supp} h \subset \Big\{x \in \beta X\colon \max\Big(|\tilde{f}(x) - \tilde{f}(\varphi_1(y))|, |\tilde{g}(x) - \tilde{g}(\varphi_1(y))|\Big) < 1\Big\}$. The function fhg belongs to F and $hg \in F$ is bounded so that $(f_i hg)_{i \in I}$ converges to fhg in E. Hence $\lim\limits_i T(f_i hg) = T(fhg)$. In particular $\lim_i T(f_i hg)(y) = T(fhg)(y)$. But $T(f_i hg) = T(f_i)T(hg)$ for every $i \in I$. Then $T(fhg)(y) = \lim_i T(f_i)(y)\, T(hg)(y)$. The continuity of T again gives $T(fhg)(y) = T(f)(y)\, T(hg)(y) = T(f)(y)\, T(g)(y)$. Since y is arbitrary in $Y_{T,E}$, we get $T(f)\, T(g) = T(fg)$. ∎

Remark 7.

1. If F happens to be contained in $CV_0(X)$, then F_b is automatically dense in F. On the other hand, if $T(E)$ is essential, then $T = T_a$. In particular Theorem 3.3 and Theorem 3.4 of [12] are obtained by taking respectively in 2. of the proposition above $E = CV_0(X)$ and $F = CV_0(X) \cap C_b(X)$ and in 1. $E = CV(X)$ and $F = CV_0(X) \cap C_b(X)$. Recall that in [12] $CV_0(X)$ is assumed to be essential and that $M(CV_0(X))$ always is contained in X.

2. An algebra morphism $T\colon CK(X) \to CK(Y)$ need not be an *e.c.o.*. However, there is some open subset Y_T of Y and a continuous map $\varphi_a : Y_T \to X$ such that $T_a = C_{\varphi_a}$. Moreover, in the light of Theorem 3, T is continuous if and only if for every compact $K \subset Y$, $\varphi_a(K \cap Y_T)$ is relatively compact in X.

3. If $E \subset CV(X)$ is a vector space such that E_b is dense in E, then E_b is even large in E and $M^+(E) = M^+(E_b)$. Indeed if B is a bounded set of E and $A := \{\Lambda_n(f),\, f \in B,\, n \in \mathbb{N}\}$, then A is bounded in E_b and $B \subset \overline{A}$. Here for every $f \in B$ and $n \in \mathbb{N}$, $\Lambda_n(f)$ denotes the bounded function defined by $\Lambda_n(f)(x) = f(x)$ if $|f(x)| \le n$ and $\Lambda_n(f)(x) = n\,\dfrac{f(x)}{|f(x)|}$ otherwise. Now let $x \in M^+(E)$ be given. We have to show that δ_x does not vanish identically on E_b. If $f \in E$ is such that $\delta_x(f) = 1$ and $(x_i)_i$ is a net in X converging to x, then assuming with no loss of generality that $|f(x_i)| \le 2$ for every i, we have $\delta_x(\Lambda_2(f)) = \lim_i \Lambda_2(f)(x_i) = \lim_i f(x_i) = \delta_x(f) = 1$. Hence $M^+(E) \subset M^+(E_b)$. Next, assume that $x \in M^+(E_b)$ and B is a bounded set of E. Consider the set A as above. There exists some $M > 0$ so that $|\widetilde{\Lambda_n(f)}(x)| < M$. We claim that $|\tilde{f}(x)| \le M$. If not, there will exist a neighbourhood O of x with $|\tilde{f}(t)| > M$ for every $t \in O$. Hence, for $n > M$ and $t \in X \cap O$, we have $|\Lambda_n(f)(t)| > M$. This leads to the contradiction $|\widetilde{\Lambda_n(f)}(x)| \ge M$. □

Now, if C_φ has to land in a (smaller) subspace of $CU(Y)$ instead of the whole space $CU(Y)$, the behaviour of φ turns out to be affected. To see this, we need the following lemma containing analogous of Lemma 3.1 of [11] and Lemma 2, p. 69 of [6].

Lemma 8. *Let $E \subset CV(X)$ be a vector subspace which is a $C_b(X)$-module. If K is a compact subset of $M^*(E)$ and C a closed one such that $K \cap C = \emptyset$, then*

1. *There exists an open subset Ω of $M^*(E)$ such that $K \subset \Omega$ and, for every $v \in V$, $\tilde{v}$ is bounded on Ω.*
2. *There exists $f \in E$ with $\tilde{f} = 1$ on K and $\tilde{f} = 0$ on C.*

Proof: **1**. Let $x \in M^*(E)$ and $f_x \in E$ be so that $\widetilde{f_x}(x) = 1$. Put $\Omega_x := \{|\widetilde{f_x}| > 1/2\}$. By compacity, there exist $x_1, ..., x_n$ such that $K \subset \Omega := \bigcup_{i=1}^n \Omega_{x_i}$. Then Ω is the required open set.

2. We first associate with every $g \in C(X)$, a positive bounded and continuous function $\Gamma(g)$ defined on X by $\Gamma(g)(x) := |g(x)|$ if $|g(x)| \le 1$ and $\Gamma(g)(x) := \dfrac{1}{|g(x)|}$ otherwise. Now let $x \in M^*(E) \backslash C$ and $g \in E$ so that $\tilde{g}(x) = 1$ and consider $g_x \in C(\beta X)$ with $g_x(x) = 1$, $0 \le g_x \le 1$ and $g_x = 0$ identically on C. The function $k_x := g g_x$ belongs to E and so does $h_x := k_x \overline{k_x}\, \Gamma(k_x^2)$. Moreover $0 \le h_x \le 1$, $\widetilde{h_x}(x) = 1$ and h_x vanishes identically on C. Now, by compacity of K, there exist $h_{x_1}, h_{x_2}, ..., h_{x_m}$ in E such that $K \subset \bigcup_{i=1}^m \{h_{x_i} > 1/2\}$. The function $h := \sum_{n=1}^m h_{x_n}$ belongs to E and satisfies $h(t) > 1/2$ for every $t \in K$. Now $f := 2\,h\,\Gamma(2h)$ enjoys the required conditions. ∎

For a non negative function u on Y and $\epsilon > 0$, set $N_u := \{y \in Y : u(y) > 0\}$ and $N(u, \epsilon) := \{y \in Y : u(y) \ge \epsilon\}$. Consider next as in [8] the following algebras:

$$C_\ell U_{(0)}(Y) := \Big\{ f \in CU_{(0)}(Y) : \ \forall u \in U, \ \exists u' \in U \ \text{ with } |f|\, u \le u' \Big\} ,$$

$$C_A U_{(0)}(Y) := \Big\{ f \in CU_{(0)}(Y) : \ f \ \text{ is bounded on each } N_u, \ u \in U \Big\} ,$$

$$C_{uA} U_{(0)}(Y) := CU_{(0)}(Y) \cap C_b(Y) .$$

The proof of the following proposition is easy and then omitted:

Proposition 9. *Let $\varphi : Y \to \mathbb{F}(E)$ be a map such that C_φ maps continuously E into $CU(Y)$. Then*

1. *$C_\varphi(E) \subset C_\ell U(Y)$ if and only if for every $f \in E$ and $u \in U$, there exists $u' \in U$ such that $|C_\varphi(f)| \le u'/u$ on N_u.*
2. *$C_\varphi(E) \subset C_A U(Y)$ if and only if for every $u \in U$, $\varphi(N_u)$ is E-bounding (i.e. $E \subset B(\varphi(N_u))$).*
3. *$C_\varphi(E) \subset C_{uA} U(Y)$ if and only if $\varphi(Y)$ is E-bounding.* ∎

For the corresponding subspaces of $CV_0(X)$, combine Proposition 9 and Proposition 10 below.

Proposition 10. *Let $\varphi\colon Y \to \mathbb{F}(E)$ be such that C_φ maps E continuously in $CU(Y)$. Consider the following assertions.*

1. $C_\varphi(E) \subset CU_0(Y)$.
2. $\forall\, u \in U,\ \varepsilon > 0$ *and* $K \subset M^*(E)$ *compact,* $\varphi^{-1}(K) \cap N(u,\varepsilon)$ *is relatively compact in* Y.
3. $\forall\, u \in U,\ \varepsilon > 0$ *and* $v \in V$ *such that* $u \le \tilde{v} \circ \varphi$ *on* Y_a, $\varphi^{-1}(K) \cap N(u,\varepsilon)$ *is relatively compact whenever* K *is a compact subset of* $N(v,\varepsilon)$.

Then 1.$\Rightarrow$2.$\Rightarrow$3. If in addition $E \subset CV_0(X)$, then also 3.$\Rightarrow$1.

Proof: In order to show the implication 1.$\Rightarrow$2., let $u \in U$, $\varepsilon > 0$ and K a compact subset of $M^*(E)$ be given. By Lemma 8, we may find $f \in E$ such that $0 \le f \le 1$ and $\tilde{f}_{|K} = 1$. By assumption, the set $Y_1 := \{y \in Y\colon u(y)\,|\tilde{f} \circ \varphi(y)| \ge \varepsilon\}$ is compact. Since $\varphi^{-1}(K) \cap N(u,\varepsilon)$ is contained in Y_1, it is relatively compact. The assertion 3. derives obviously from 2. Now assume that $E \subset CV_0(X)$ and consider $f \in E$, $u \in U$ and $\varepsilon > 0$. Set $Y_1 := \{y \in Y : u(y)\,|\tilde{f} \circ \varphi(y)| \ge \varepsilon\}$. If $v \in V$ is such that $u \le \tilde{v} \circ \varphi$ on Y_a, then $Y_1 \subset \{y \in Y\colon \varphi(y) \in \{|f|\,v \ge \varepsilon\}\}$. But $f \in CV_0(X)$, then $K := \{|f|\,v \ge \varepsilon\}$ is compact and $K \subset N(v, \varepsilon/M)$, where $M := \sup\{|f(t)|;\ t \in K\}$. By 3., $\varphi^{-1}(K) \cap N(u, \varepsilon/M)$ is relatively compact. The proof is achieved since Y_1 is contained in $\varphi^{-1}(K) \cap N(u,\varepsilon/M)$. ∎

The following proposition gives a further algebraic characterization of extended composition operators. To show it, we need an additional notation. For $Y_1 \subset Y$, set $\Delta_{Y_1} := \{\delta_y,\ y \in Y_1\}$ and let Δ denote the set $\{\delta_x,\ x \in M^*(E)\}$.

Proposition 11. *Let $E \subset C(X)$ be a $C_b(X)$-module and $T : E \to C(Y)$ a linear map. If T is an e.c.o., then $T^*(\Delta_{Y_a}) \subset \Delta$. Conversely if $T^*(\Delta_{Y_{T,E}}) \subset \Delta$ (with $\mathbb{F}(E) \ne M^*(E)$ in case $Y_{T,E} \ne Y$), then $T = C_\varphi$ for some $\varphi\colon Y \to \mathbb{F}(E)$.*

Proof: If $T = C_\varphi$ for some $\varphi\colon Y \to \mathbb{F}(E)$, $f \in E$ and $y \in Y_a$, then $T^*(\delta_y)(f) = \delta_y \circ T(f) = T(f)(y) = \tilde{f}(\varphi(y)) = \delta_{\varphi(y)}(f)$. Since $y \in Y_a$ and $f \in E$ were arbitrary, the necessity follows. For the sufficiency, let $y \in Y_{T,E}$ and $x \in \beta X$ satisfy $T^*(\delta_y) = \delta_x$. The compacity of βX combined with the fact that E is a $C_b(X)$-module shows that x is uniquely determined. Put then $\varphi(y) = x$. For eventual $y \notin Y_{T,E}$, put $\varphi(y) = x_0$, x_0 being a (also eventual) fixed point of $\mathbb{F}(E) \backslash M^*(E)$. We then have $T = C_\varphi$. ∎

3 – Compact extended composition operators

This section deals with the conditions on φ so that C_φ becomes strongly bounded, (locally) equicontinuous, (locally) weakly compact or (locally) compact.

Let $S: A \to B$ be a linear mapping from a locally convex space A into another B. Recall that S is said to be (weakly) compact (resp. locally (weakly) compact) if it maps some 0-neighbourhood (resp. every bounded set) in A into a relatively (weakly) compact subset of B. S is strongly bounded if it maps some 0-neighbourhood in A into a bounded subset of B. If B consists of continuous functions on a topological space Z, then S is equicontinuous (resp. locally equicontinuous) if the image under S of some 0-neighbourhood (resp. of any bounded set) in A is equicontinuous on Z.

Proposition 1. *Let $\varphi: Y \to \mathbb{F}(E)$ be a mapping such that $C_\varphi(E) \subset C(Y)$.*

1. *If $C_\varphi(E) \subset CU(Y)$, then C_φ is strongly bounded if and only if there is some $v \in V$ such that $U_a \leq \{\lambda\, \tilde{v} \circ \varphi_a \colon \lambda > 0\}$.*

2. *If $\varphi_a = \varphi$, then the following assertions are equivalent:*

- **i**. *C_φ is equicontinuous.*
- **ii**. *C_φ is locally equicontinuous.*
- **iii**. *φ is locally constant.*

Proof: **1**. We have just to show the necessity. If $C_\varphi(B_v(E))$ is bounded for some $v \in V$, then for every $u \in U$, there is $M > 0$ with $u(t)\, |\tilde{f} \circ \varphi(t)| \leq M$ for every $t \in Y$ and $f \in B_v(E)$. Fix y_0 in Y_a. If $\tilde{v}(\varphi(y_0)) = 0$, consider, for every $n \in \mathbb{N}$, $f_n \in B_v(E)$ such that $\tilde{f}_n(\varphi(y_0)) = n$ and apply to f_n the inequality above to get that $u(y_0) = 0$. Now if $\tilde{v}(\varphi(y_0)) \neq 0$, the same inequality applied to any $f_n \in B_v(E)$ such that $\tilde{f}_n(\varphi(y_0)) = 1/(\tilde{v}(\varphi(y_0)) + 1/n)$ gives $u(y_0) \leq M\, \tilde{v} \circ \varphi(y_0)$, see Lemma 9 of [8].

2. It is clear that i. $\Rightarrow$ ii. In order to deduce iii. from ii., assume that φ is not constant on any neighbourhood of some $y_0 \in Y$. Fix a neighbourhood Ω of $\varphi(y_0)$ in $M^*(E)$ such that for every $v \in V$, $\tilde{v}$ is bounded on Ω. Then for every neighbourhood G of y_0, there is some $y_G \in G \cap \varphi^{-1}(\Omega)$ such that $\varphi(y_G) \neq \varphi(y_0)$. Take $f_G \in E$ with $0 \leq f_G \leq 1$, $\tilde{f}_G(\varphi(y_G)) = 1$, $\tilde{f}_G(\varphi(y_0)) = 0$ and $\operatorname{supp} \tilde{f}_G \subset \Omega$. The set $\{f_G, G \text{ a neighbourhood of } y_0\}$ is bounded in E and then $C_\varphi(B)$ is equicontinuous on Y. But the equality $|\tilde{f}_G(\varphi(y_G)) - \tilde{f}_G(\varphi(y_0))| = 1$ for every G contradicts the convergence of y_G to y_0. The implication iii. $\Rightarrow$ i. derives from the fact that if φ is locally constant, then $C_\varphi(B_v(E))$ is equicontinuous for every $v \in V$. ∎

Notice that an equicontinuous operator need not be even bounded on bounded sets as shows the following example: Take $X = Y = \mathbb{R}$, $V = K$, $U = Z$ or K and $\varphi(t) = x_0$ for a given $x_0 \in \beta\mathbb{R}\backslash\mathbb{R}$ and every $t \in Y$. By Proposition 1, C_φ is equicontinuous from $E := C_b(\mathbb{R})$ into $CU(Y)$. However if $B := \{f_n \colon n \in \mathbb{N}\}$, where $f_n(x) = \min(|x|, n)$ for every $x \in \mathbb{R}$, then B is bounded in E while $C_\varphi(B)$ is not bounded in $CU(Y)$. This example shows that Theorem A of [2] is extendible neither to Schmets algebras $C_{\mathcal{P}}(X)$ (even for compact X) nor to weighted spaces in the present sense. Actually the necessary and sufficient condition for C_φ in Theorem A to be locally compact turns out to be here equivalent to the equicontinuity of C_φ. In order to get the local compacity, we need some more conditions.

We will say that a subset X_1 of a topological space X satisfies the sequences cluster point property ($SCPP$) if every (infinite) sequence of X_1 has a cluster point in X. Every relatively countably compact subspace of X has ($SCPP$). In particular every relatively compact and every relatively sequentially compact subset of X has ($SCPP$).

Proposition 2. *Assume that $\varphi(Y) \subset M^*(E)$. If $C_\varphi \colon E \to CU(Y)$ is locally weakly compact, then for every subset K of Y satisfying (SCPP), $\varphi(K)$ is finite.*

Proof: Assume that $K \subset Y$ satisfies ($SCPP$) and $\varphi(K)$ is infinite. Let $(x_n)_n = (\varphi(y_n))_n$ be a sequence of pairewise disctinct points of $\varphi(K)$. By assumption on K, there is some $y \in Y$ such that $y \in \overline{\{y_n,\, n \geq m\}}$ for every m. The continuity of φ gives $x := \varphi(y) \in \overline{\{x_n,\, n \geq m\}}$ for every m. Let G be a neighbourhood in $M^*(E)$ of $x := \varphi(y)$ such that, for every $v \in V$, $\tilde{v}$ is bounded on G. We assume, with no loss of generality, that $x \neq x_n$, $n \in \mathbb{N}$. We can then find $f_n \in E$ with $0 \leq f_n \leq 1$, $\tilde{f}_n(x) = 1$, $\tilde{f}_n(x_k) = 0$, for $k \leq n$ and $\tilde{f}_n$ vanishes outside of G. Then the set $B := \{f_n \colon n \in \mathbb{N}\}$ is bounded in E and hence $C_\varphi(B)$ is relatively weakly compact. If g is a (weakly) cluster point of $C_\varphi(B)$, then $g \in \overline{\{\tilde{f}_n \circ \varphi,\, n \geq m\}}^{\sigma(CU(Y), CU(Y)')}$, $m \in \mathbb{N}$. Therefore $g(t) \in \overline{\{\tilde{f}_n \circ \varphi(t),\, n \geq m\}}$ for every $m \in \mathbb{N}$ and $t \in Y$. Whereby $g(y) = 1$ and $g(y_m) = 0$ for every m. This is in contradiction with $y \in \overline{\{y_n,\, n \geq m\}}$. ∎

In the following, if A is a subset of $M^*(E)$, we will denote by E_A the locally convex space obtained by endowing E with the topology of pointwise convergence on A. In case $A = M^*(E)$, $M^+(E)$ or $M(E)$, we will write E_{s^*}, E_{s^+} or E_s. We then get

Lemma 3. *Let $A \subset \mathbb{F}(E)$ and $\varphi \colon Y \to \mathbb{F}(E)$ be given. If $C_\varphi \colon E_A \to CU(Y)$ is continuous, then $\varphi(N_u \cap Y_a)$ is a finite subset of A for every $u \in U$. The converse is true provided $U \subset B(Y)$ (or $C_\varphi \colon E \to CU(Y)$ continuous).*

Proof: For every $u \in U$, there is some finite set $S \subset A$ and some $M > 0$ such that

$$u(t)\,|\tilde{f} \circ \varphi(t)| \;\le\; M \sup_{s\in S} |\tilde{f}(s)| \,, \qquad t \in Y \quad \text{and} \quad f \in E \;.$$

This is only possible if $\varphi(N_u \cap Y_a) \subset S$. Otherwise, the inequality above would not hold for any $t \in N_u \cap Y_a$ with $\varphi(t) \notin S$ and any $f \in E$ such that $\tilde{f}(\varphi(t)) = 1$ and $\tilde{f}_{|S} = 0$. For the converse if $U \subset B(Y)$ and $S := \varphi(N_u \cap Y_a)$ is a finite subset of A, then for every $f \in E$, we have $|C_\varphi(f)|_u \le \|u\|_\infty \sup_{s\in S} |\tilde{f}(s)|$. Now, if $C_\varphi : E \to CU(Y)$ is continuous, for $u \in U$, there is some $v \in V$ verifying $u_a \le \tilde{v} \circ \varphi_a$. If $S := \varphi(N_u \cap Y_a) \subset A$ is finite, then

$$\begin{aligned} |\tilde{f} \circ \varphi|_u \;&=\; \sup_{t\in N_u\cap Y_a} u(t)\,|\tilde{f} \circ \varphi(t)| \\ &\le\; \sup_{t\in N_u\cap Y_a} \tilde{v} \circ \varphi(t)\,|\tilde{f} \circ \varphi(t)| \\ &\le\; |\tilde{v}|_S\,|\tilde{f}|_S \;. \end{aligned}$$

Whence the conclusion. ∎

The following result gives a sufficient condition under which C_φ is locally compact.

Proposition 4. *Assume that $\varphi(Y) \subset M^+(E)$, $CU(Y)$ is quasi-complete and either $U \subset B(Y)$ or $C_\varphi \colon E \to CU(Y)$ continuous. If $\varphi(N_u)$ is finite for every $u \in U$, then $C_\varphi \colon E \to CU(Y)$ is locally compact.*

Proof: By Theorem 3, $C_\varphi : E_{s^+} \to CU(Y)$ is continuous. Then C_φ also is continuous from $(E, \sigma(E, E^+))$ into $CU(Y)$, where E^+ is the bounded dual of E. Since E and $(E, \sigma(E, E^+))$ have the same bounded sets, every bounded set B of E is precompact in $(E, \sigma(E, E^+))$. Hence for every bounded subset B of E, $\overline{C_\varphi(B)}$ is compact, for it is precompact and complete. ∎

The condition $\varphi(Y) \subset M^+(E)$ is not superfluous as shows the example after Proposition 1. Notice also that if N_u satisfies $(SCPP)$ for every $u \in U$, then U is automatically contained in $B(Y)$. Indeed if $u \in U$ is not bounded on Y, there is a sequence $(y_n)_n \subset Y$ such that $u(y_n) > \max(n,\, u(y_{n-1}) + 1)$. If y is a cluster point of $\{y_n :\, n \in \mathbb{N}\}$ and $G := \{t \in Y :\, u(t) < u(y) + 1\}$, then $G \cap \{y_n,\, n \ge m\} \ne \emptyset$ for every $m \in \mathbb{N}$. This gives $u(y) \ge m$ for every m, which is a contradiction.

Combining Proposition 2, Proposition 4 and the foregoing remark, we get the main result of this section:

Theorem 5. *Assume that $\varphi(Y) \subset M^+(E)$, $CU(Y)$ is quasi-complete and N_u satisfies $(SCPP)$ for every $u \in U$. Then the following assertions are equivalent:*

1. *C_φ is locally compact from E into $CU(Y)$.*

2. *C_φ is locally weakly compact from E into $CU(Y)$.*

3. *For every $u \in U$, $\varphi(N_u)$ is finite.* ∎

Next, if $H \subset CU(Y)$ and $u \in U$ are given, we will say that uH vanishes at infinity, if for every $\varepsilon > 0$, there exists a compact set $K \subset Y$ outside of which $u|f| < \varepsilon$ for every $f \in H$. It is clear that for every bounded subset H of $CU(Y)$ and every $u \in U$, uH vanishes at infinity whenever U fulfils the following condition: For every $u \in U$, there exists $u' \in U$ such that for every $\varepsilon > 0$, there exists a compact set $K \subset Y$ with $u(t) \le \varepsilon\, u'(t)$ for every $t \notin K$. We then obtain

Theorem 6. *Let $C_\varphi : E \to CU(Y)$ be an e.c.o. Assume that for every $u \in U$ and every bounded subset H of $C_\varphi(E)$, uH vanishes at infinity. Then C_φ is compact provided it is strongly bounded and equicontinuous. The converse is true whenever Y is a $U_{\mathbb{R}}$-space.*

Proof: Let $v \in V$ be such that $H := C_\varphi(B_v)$ is either bounded and equicontinuous. Then the pointwise closure $\overline{H}^{\tau_s}$ of H also is either contained in $CU(Y)$, (τ_s-) bounded and equicontinuous. By Ascoli's theorem $\overline{H}^{\tau_s}$ is compact in the compact open topology τ_c. To conclude, it is sufficient to show that τ_V is coarser that τ_c on $\overline{H}^{\tau_s}$. Let then $f_0 \in \overline{H}^{\tau_s}$, $u \in U$ and $\varepsilon > 0$ be given. By our assumption, there exists a compact $K \subset Y$ out of which $u|f| < \varepsilon/2$ for every $f \in \overline{H}^{\tau_s}$. Set $M := \sup\{u(t),\, t \in K\}$ and $\varepsilon' := \min(\varepsilon/M,\, \varepsilon/2)$. Then the set $\{f \in \overline{H}^{\tau_s} : |f(t) - f_0(t)| \le \varepsilon',\ t \in K\}$ is a τ_c-neighbourhood of f_0 in $\overline{H}^{\tau_s}$. Since it is contained in $\{f \in \overline{H}^{\tau_s} : |f - f_0|_u \le \varepsilon\}$, the result follows. The converse is a consequence of Lemma 2.3 of [10]. ∎

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