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MONOTONE ITERATIVE METHOD FOR DIFFERENTIAL EQUATIONS WITH PIECEWISE CONSTANT ARGUMENTS

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Abstract: In this paper we employ the method of upper and lower solutions coupled with the monotone iterative technique to obtain results of existence and approximation of solutions for periodic boundary value problems of differential equations with piecewise constant arguments.

1 – Introduction

Differential equations with piecewise constant arguments (EPCA for short) are originated in [1, 5]. They are closely related to impulse and loaded equations and, especially, to difference equations of a discrete argument. These equations have the structure of continuous dynamical systems within intervals of certain length. Continuity of a solution at a point joining any two consecutive intervals implies recursion relations for the solution at such points. Many oscillatory properties of EPCA were mentioned, for example, see [1, 4, 5] and the references cited therein. In this paper we discuss the periodic boundary value problem (PBVP for short) of the EPCA

$$x'(t) = f\Big(t, x(t), x([t-1])\Big), \quad t \in J \;, \tag{1}$$

$$x(0) = x(T) \;, \tag{2}$$

where $J=[0,T]$, $f \in C(J\times \mathbb{R}^2, \mathbb{R})$ and $[\,\cdot\,]$ designates the greatest integer function.

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Let $\widetilde{T} = \begin{cases} [T]+1, & T \neq [T], \\ T, & T = [T], \end{cases}$ and Ω denote the class of all functions $x\colon J \cup \{-1\} \to \mathbb{R}$ satisfying that

- **(i)** $x(-1) = x(0)$;
- **(ii)** $x(t)$ is continuous for $t \in J$;
- **(iii)** $x'(t)$ exists and is continuous on the intervals $[n, n+1)$ $(n=0,1,...,\widetilde{T}-2)$ and $[\widetilde{T}-1, T)$.

A function $x : J \to \mathbb{R}$ is said to be a solution of (1) and (2), if $x \in \Omega$ and satisfies (1) and (2) with $x'(t) = x'_+(t)$ on $t = 1,2,...,\widetilde{T}-1$. A function $\alpha \in \Omega$ is said to be a lower solution of (1) and (2) , if it satisfies

$$\alpha'(t) \le f\Big(t, \alpha(t), \alpha([t-1])\Big), \quad t \in J,$$
$$\alpha(0) \le \alpha(T).$$

An upper solution for (1) and (2) is defined analogously by reversing the above inequalities.

We employ the method of upper and lower solutions coupled with the monotone iterative technique (see [2, 3] and the references therein) to establish the results of existence and approximation of solutions for PBVP of EPCA.

2 – A comparison result

For the successful employment of the monotone iterative technique we need a certain comparison theorem. In this section, a gerenal comparison theorem is developed.

Theorem 1. *Suppose that $m \in \Omega$ such that*

$$m'(t) \le -M_1\, m(t) - M_2\, m([t-1]), \quad t \in J, \tag{3}$$

$$m(0) \le m(T), \tag{4}$$

where $M_1 > 0$, $M_2 \ge 0$ are constants such that

$$1 - \frac{M_2}{M_1}\, \widetilde{T}\, e^{M_1}\, (e^{M_1} - 1) \ge 0. \tag{5}$$

Then $m(t) \le 0$ for all $t \in J$.

Proof: Set $p(t) = m(t)\, e^{M_1 t}$. Then the inequality (3) reduces to

$$p'(t) \;\le\; -M_2\, p([t-1])\, e^{M_1(t-[t-1])} \;. \tag{6}$$

Hence

$$p(t) \;\le\; p(n-1) - \frac{M_2}{M_1}\, p(n-2)\, (e^{M_1(t-n+2)} - e^{M_1}) \tag{7}$$

for $t \in [n-1, n)$, $n = 1, 2, ..., \widetilde{T}-1$, and for $t \in [\widetilde{T}-1, T]$

$$p(t) \;\le\; p(\widetilde{T}-1) - \frac{M_2}{M_1}\, p(\widetilde{T}-2) \left(e^{M_1(t-\widetilde{T}+2)} - e^{M_1}\right) \;. \tag{8}$$

Notice that by continuity, the relation (7) is satisfied on each interval $[n-1, n]$ $(n = 1, 2, ..., \widetilde{T}-1)$. In particular, with $N = \dfrac{M_2}{M_1}\, e^{M_1} (e^{M_1} - 1)$, we have for $n = 1, 2, ..., \widetilde{T}-1$,

$$p(n) \;\le\; p(n-1) - N\, p(n-2) \;. \tag{9}$$

If $p(j) > 0$ for every $j = 0, 1, ..., \widetilde{T}-1$, then, by (6), $p(t)$ is strictly decreasing on each interval $[n, n+1)$ $(n = 0, 1, ..., \widetilde{T}-1)$ and on $[\widetilde{T}-1, T]$. This enables, by continuity, to conclude that $p(0) > p(1) ... > p(T)$, contradicting (4). So there exists a $k \in \{0, 1, ..., \widetilde{T}-1\}$ such that $p(k) \le 0$.

Assume that $k \in \{1, 2, ..., \widetilde{T}-1\}$ such that $p(j) > 0$ for $j \in \{0, 1, ..., k-1\}$. This implies, in particular by (4), that $p(T) > 0$. If $k = \widetilde{T}-1$, then by (8)

$$p(T) \;\le\; p(\widetilde{T}-1) - \frac{M_2}{M_1}\, p(\widetilde{T}-2) \left(e^{M_1(T-\widetilde{T}+2)} - e^{M_1}\right) < p(\widetilde{T}-1) \;\le\; 0$$

and we obtain a contradiction. If $k < \widetilde{T}-1$, then by (9)

$$p(k+1) \;\le\; p(k) - N\, p(k-1) \;<\; p(k)$$

and $p(k+1) < 0$. More generally, for $i = 1, ..., \widetilde{T}-k-1$, one can state that

$$p(k+i) \;<\; \Big(1 - (i-1)\, N\Big)\, p(k) \;,$$

which carries that $p(k+i) < 0$ and enables to conclude that

$$\begin{aligned} p(T) &\le p(\widetilde{T}-1) - \frac{M_2}{M_1}\, p(\widetilde{T}-2) \left(e^{M_1(T-\widetilde{T}+2)} - e^{M_1}\right) \\ &\le p(\widetilde{T}-1) - N\, p(\widetilde{T}-2) \;. \end{aligned}$$

Therefore applying (9) continuously we have

$$\begin{aligned} p(T) \;&\le\; \Big(1-(\widetilde{T}-k-2)\,N\Big)\,p(k+1)-N\,p(k)\\ &<\; \Big(1-(\widetilde{T}-k-1)\,N\Big)\,p(k) \end{aligned}$$

and the same contradiction is obtained.

Thus necessarily $p(0)\le 0$. This implies, when $n=1$, that $p(t)\le 0$ for $t\in[0,1]$, since on this interval

$$\begin{aligned} p(t) \;&\le\; p(0)-\frac{M_2}{M_1}\,p(-1)\,\Big(e^{M_1(t+1)}-e^{M_1}\Big)\\ &\le\; p(0)-\frac{M_2}{M_1}\,p(0)\,e^{-M_1}\,\Big(e^{2M_1}-e^{M_1}\Big)\\ &=\; (1-N)\,p(0)\ . \end{aligned}$$

In particular, we obtain that

$$p(1)\;\le\;(1-N)\,p(0)\ . \tag{10}$$

Assume now that for $n\le\widetilde{T}-1$, one has $p(t)\le 0$ for every $t\in[0,n-1]$ and let $t\in[n-1,n]$. By (7), the hypothesis of induction and a continued application of (9) we obtain

$$\begin{aligned} p(t) \;&\le\; p(n-1)-\frac{M_2}{M_1}\,p(n-2)\,\Big(e^{M_1(t-n+2)}-e^{M_1}\Big)\\ &\le\; p(n-1)-N\,p(n-2)\\ &\le\; \Big(1-(n-2)\,N\Big)\,p(1)-N\,p(0)\ , \end{aligned}$$

and by (10) we conclude that

$$p(t)\;\le\;(1-n\,N)\,p(0)\;\le\;0\ .$$

Analogously, if $p(t)\le 0$ for every $t\in[0,\widetilde{T}-1]$, then for $t\in[\widetilde{T}-1,T]$ we obtain from (8) and (9),

$$p(t)\;\le\;(1-\widetilde{T}N)\,p(0)\;\le\;0\ .$$

The proof of Theorem 1 is complete. ∎

If $m \in \Omega$ is a solution of the PBVP

$$m'(t) + M_1\, m(t) + M_2\, m([t-1]) \;=\; 0, \qquad t \in J \;, \tag{11}$$

$$m(0) = m(T) \;, \tag{12}$$

then both $m(t)$ and $-m(t)$ satisfy the inequality (3) and (4). Hence we have

Corollary 1. *Assume that (5) holds. Then the PBVP (11) and (12) has the unique solution $m(t) = 0$.* ∎

From the proof of Theorem 1, we can obtain

Corollary 2. *Assume that (5) holds and $m \in \Omega$ satisfying (3) and*

$$m(0) \;\le\; 0 \;.$$

Then $m(t) \le 0$ for all $t \in J$. ∎

3 – Linear PBVPs

In this section, in order to develop the monotone iterative technique for (1) and (2), we consider the linear PBVP

$$x'(t) + M_1\, x(t) + M_2\, x([t-1]) \;=\; \sigma(t), \qquad t \in J \;, \tag{13}$$

$$x(0) = x(T) \;, \tag{14}$$

where M_1, M_2 are constants and $\sigma(t)$ is continuous and bounded on the intervals $[n, n+1)$ $(n = 0, 1, ..., \widetilde{T}-2)$ and $[\widetilde{T}-1, T]$.

Theorem 2. *Assume that $M_1 > M_2 \ge 0$. Then PBVP (13) and (14) has a unique solution.*

Proof: For any $x \in \Omega$, denote $|x| = \max_{t \in J} |x(t)|$. Let

$$w_0 = \max_{t \in J} |\sigma(t)| \,, \qquad w_1 = \frac{w_0}{M_1 - M_2}$$

and define an operator $S\colon \Omega_1 \to \Omega$ as

$$\begin{aligned}
(Sx)(t) \;&=\; \frac{e^{-M_1 t}}{e^{M_1 T} - 1} \int_0^T \Big(\sigma(s) - M_2\, x([s-1])\Big)\, e^{M_1 s}\, ds \\
&\quad +\; e^{-M_1 t} \int_0^t \Big(\sigma(s) - M_2\, x([s-1])\Big)\, e^{M_1 s}\, ds \,, \qquad t \in J \;, \\
(Sx)(-1) \;&=\; (Sx)(0) \;,
\end{aligned}$$

where $\Omega_1 = \{x \in \Omega\colon\; |x| \le w_1,\; x(0) = x(T)\}$.

It is easy to see that Ω_1 is a closed bounded convex set. For any $x \in \Omega_1$, we have for $t \in J$

$$
\begin{aligned}
|S(x)(t)| &\leq \frac{e^{-M_1 t}}{e^{M_1 T}-1}\int_0^T (|\sigma(s)| + |M_2 x([s-1])|)e^{M_1 s}ds \\
&\quad + e^{-M_1 t}\int_0^t (|\sigma(s)| + |M_2 x([s-1])|)e^{M_1 s}ds \\
&\leq \frac{e^{-M_1 t}}{e^{M_1 T}-1}\int_0^T (w_0 + M_2 w_1)e^{M_1 s}ds + e^{-M_1 t}\int_0^t (w_0 + M_2 w_1)e^{M_1 s}ds \\
&= \frac{w_0 + M_2 w_1}{M_1} = w_1,
\end{aligned}
$$

which implies that $|S(x)| \leq w_1$, that is, $S(\Omega_1) \subset \Omega_1$. According to Ascoli–Arzela's theorem one can see that $S\colon \Omega_1 \to \Omega_1$ is compact. Hence there exists a solution of PBVP (13) and (14) by Schauder's fixed point theorem. The uniqueness of solutions of the PBVP (13) and (14) follows from Theorem 1. In fact, suppose that x and y are two distinct solutions of (13) and (14) and let $m(t) = x(t) - y(t)$, then $m(t)$ satisfies (11) and (12). Hence, by Corollary 1 we have $m(t) = 0$. The proof of Theorem 2 is complete. ∎

A function $v \in \Omega$ is said to be a lower solution for (13) and (14) if it satisfies

$$v'(t) + M_1 v(t) + M_2 v([t-1]) \leq \sigma(t), \quad t \in J\,, \tag{15}$$

$$v(0) \leq v(T). \tag{16}$$

An upper solution for (13) and (14) is defined analogously by reversing the inequalities of (15) and (16).

For any $v, w \in \Omega$ such that $v(t) \leq w(t)$, $t \in J$, denote $[v,w] = \{x \in \Omega\colon v(t) \leq x(t) \leq w(t),\ t \in J\}$.

Theorem 3. *Let v and w be lower and upper solutions of (13) and (14) such that $v \leq w$ on J and assume that (5) is satisfied. Then (13) and (14) has a unique solution $x \in [v,w]$.*

Proof: For any $a_1, a_2 \in \mathbb{R}$, the initial value problem of (13) and

$$x(-1) = a_1\,, \quad x(0) = a_2 \tag{17}$$

can be looked as a family of initial value problems of ordinary differential equations on the successive intervals $[n, n+1]$ $(n = 0, 1, ..., \widetilde{T}-2)$ and $[\widetilde{T}-1, T]$.

Hence the problem (13) and (17) has a unique solution given by

$$x(t)=\begin{cases} a_2\, e^{-M_1 t} + \int_0^t \Big(\sigma(s) - M_2\, a_1\Big)\, e^{-M_1(t-s)}\, ds, & t \in [0,1]\ , \\ x(1)\, e^{-M_1(t-1)} + \int_1^t \Big(\sigma(s) - M_2\, a_2\Big)\, e^{-M_1(t-s)}\, ds, & t \in [1,2]\ , \\ x(2)\, e^{-M_1(t-2)} + \int_2^t \Big(\sigma(s) - M_2\, x(1)\Big)\, e^{-M_1(t-s)}\, ds, & t \in [2,3]\ , \\ \cdots\cdots \\ x(\widetilde{T}-1)\, e^{-M_1(t-\widetilde{T}-1)} + \int_{\widetilde{T}-1}^t \Big(\sigma(s) - M_2\, x(\widetilde{T}-2)\Big)\, e^{-M_1(t-s)}\, ds, & t \in [\widetilde{T}-1, T]\,. \end{cases}$$

For each $a \in [v(0), w(0)]$, denote by $x(\cdot; a)$ the unique solution of (13) with $x(-1) = x(0) = a$. Using Corollary 2, it is easy to see that $v(t) \le x(t; a) \le w(t)$ for $t \in J$ and $x(t; a_1) \le x(t; a_2)$ on J for $a_1, a_2 \in [v(0), w(0)]$ with $a_1 \le a_2$. Hence

$$v(0) \ \le\ v(T) \ \le\ x(T; v(0)) \ \le\ x(T; w(0)) \ \le\ w(T) \ \le\ w(0)\ .$$

Denote sequence $\{v_n(t)\}$ by

$$\begin{cases} v_n(t) = x(t; v_{n-1}(T)), \quad n = 1, 2, \ldots\ , \\ v_0(t) = v(t)\ , \end{cases} \qquad t \in J\ .$$

By induction we can see that the sequence satisfies

$$v_0(t) \ \le\ v_1(t) \ \le\ v_2(t) \ \le\ \cdots \ \le\ v_n(t) \ \le\ \cdots \ \le\ w(t)\ ,$$

and is convergent. Let $y(t) = \lim_{n\to\infty} v_n(t)$. Notice that

$$\begin{cases} v_n(t) = v_n(0)\, e^{-M_1 t} + \int_0^t \Big(\sigma(s) - M_2\, v_n([s-1])\Big)\, e^{-M_1(t-s)}\, ds, \quad t \in J\ , \\ v_n(0) = v_{n-1}(T)\ , \end{cases}$$

by Lebesgue Dominated Convergence Theorem, we know that $y(t)$ is the solution of (13) and (14).

The uniqueness of the solution of (13) and (14) can be obtained from Corollary 1. The proof of Theorem 3 is complete. ∎

4 – Monotone method

We are now in a position to prove the results concerning the extremal solutions of (1) and (2).

Theorem 4. *Let α and β be lower and upper solutions of (1) and (2) such that $\alpha \leq \beta$ on J. Assume that $f \in C[J\times\mathbb{R}^2]$ is such that*

$$f(t,v_1,w_1) - f(t,v_2,w_2) \;\geq\; -M_1(v_1 - v_2) - M_2(w_1 - w_2) \tag{18}$$

for $t\in J$ and $v_i, w_i \in \mathbb{R}$ $(i=1,2)$ with $\alpha(t) \leq v_2 \leq v_1 \leq \beta(t)$, $\alpha([t-1]) \leq w_2 \leq w_1 \leq \beta([t-1])$, where M_1 and M_2 satisfy (5). Then, there exist monotone sequences $\{\alpha_n\}$ and $\{\beta_n\}$ such that $\alpha_n \to \rho$, $\beta_n \to r$ as $n\to\infty$ uniformly and monotonically on J and that $\rho, r \in \Omega$ are minimal and maximal solutions of PBVP (1) and (2) respectively.

Proof: For any $y \in \Omega$ with $\alpha \leq y \leq \beta$, let

$$\sigma(t) \;=\; f\Big(t, y(t), y([t-1])\Big) + M_1\, y(t) + M_2\, y([t-1]) \;,$$

then we have that

$$\begin{aligned}\alpha'(t) + M_1\,\alpha(t) + M_2\,\alpha([t-1]) \;\leq&\\ &\leq\; f\Big(t,\alpha(t),\alpha([t-1])\Big) + M_1\,\alpha(t) + M_2\,\alpha([t-1])\\ &\leq\; f\Big(t, y(t), y([t-1])\Big) + M_1\, y(t) + M_2\, y([t-1]) \;=\; \sigma(t)\end{aligned}$$

and $\beta'(t) + M_1\,\beta(t) + M_2\,\beta([t-1]) \geq \sigma(t)$. As a consequence, α and β are respectively a lower and an upper solution for (13) and (14). Thus PBVP (13) and (14) has a unique solution $x \in [\alpha,\beta]$ by Theorem 3.

Now define a mapping A by $Ay = x$ where for any $y \in \Omega$ with $\alpha \leq y \leq \beta$, x is the unique solution of (13) and (14). First we shall show that $\alpha \leq A\alpha$ and $A\beta \leq \beta$. Set $m(t) = \alpha(t) - A\alpha(t)$. From (13), (14) and the definition of lower solution we know that $m(t)$ satisfies (3) and (4). Hence $m(t) \leq 0$ due to Theorem 1. Similarly we can prove that $A\beta \leq \beta$. Next we shall show that $Ay_1 \leq Ay_2$ for any y_1 and $y_2 \in \Omega$ with $\alpha \leq y_1 \leq y_2 \leq \beta$. Set $m(t) = Ay_1(t) - Ay_2(t)$. Using (13), (14) and (18), we obtain

$$m'(t) \;\leq\; -M_1\, m(t) - M_2\, m([t-1])\,, \qquad m(0) = m(T) \;,$$

which implies, by Theorem 1, that $Ay_1 \leq Ay_2$. It is therefore easy to see that the sequences $\{\alpha_n(t)\}$, $\{\beta_n(t)\}$ with $\alpha_0 = \alpha$ and $\beta_0 = \beta$ can be defined by

$$\alpha_{n+1} = A\alpha_n \quad \text{ and } \quad \beta_{n+1} = A\beta_n$$

and the iterates satisfy

$$\alpha \leq \alpha_1 \leq \cdots \leq \alpha_n \leq \beta_n \leq \cdots \leq \beta_1 \leq \beta$$

on J. It follows, from standard arguments(see [2]), that $\lim\limits_{n\to\infty} \alpha_n(t) = \rho(t)$ and $\lim\limits_{n\to\infty} \beta_n(t) = r(t)$ uniformly on J and $\rho(t)$ and $r(t)$ are solutions of (1) and (2).

Finally, to prove that ρ is the minimal solution on $[\alpha, \beta]$, let x be any solution of (1) and (2) on $[\alpha, \beta]$. It is obvious that $\alpha_0 \leq x$. Now if $\alpha_n \leq x$, one can easily see that $\alpha_{n+1} \leq x$ by considering the function $\phi = \alpha_{n+1} - x$ and applying Theorem 1 again. Thus, passing to the limit, we may conclude that $\rho \leq x$.

The same arguments prove that $x \leq r$. The proof of the Theorem 4 is complete. ∎

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