PORTUGALIAE MATHEMATICA
Vol. 57 Fasc. 4 – 2000

# A NOTE ON UNIQUENESS OF CAUCHY PROBLEMS ASSOCIATED TO PLANAR HAMILTONIAN SYSTEMS

C. Rebelo *

**Abstract:** We study uniqueness of solutions to Cauchy problems associated to planar Hamiltonian systems when the Hamiltonian is a $\mathcal{C}^1$ function.

## 1 – Introduction

The study of uniqueness of solutions of Cauchy problems associated to ordinary differential equations in $\mathbb{R}^n$ seems to be not an easy problem in general. If we consider the general $n$-order nonautonomous ordinary differential equation

$$y^n = f(t, y, y', ..., y^{n-1}) ,$$

it is well known that when no Lipschitz assumption on $f$, with respect to $y$, $y'$, ..., $y^{n-1}$, is assumed, the Peano phenomenon can arise, that is, Cauchy problems associated to the above equation can admit more than one solution. Many authors studied this argument and sufficient conditions which guarantee the uniqueness of solutions of Cauchy problems associated to a given equation were found (see [1] and the references therein). We concentrate our attention in the particular case of systems of the form

$$(\mathcal{P}) \quad \begin{cases} x' = -\dfrac{\partial \mathcal{H}}{\partial y}(x, y) \\[2mm] y' = \dfrac{\partial \mathcal{H}}{\partial x}(x, y) \end{cases}$$

*Received*: December 7, 1998; *Revised*: March 30, 1999.
*AMS Subject Classification*: 34A12.
*Keywords and Phrases*: Uniqueness; Hamiltonian systems.
* Supported by FCT, Praxis XXI, FEDER and contract Praxis 2/2.1/MAT/125/94.

where we assume that $\mathcal{H}$ is a $\mathcal{C}^1$ function defined in $\mathbb{R}^2$. Note that even for systems of this form, the Peano phenomenon can arise (see Remark 1 in Section 2). In this paper we give a simple and quite general condition under which uniqueness of solutions to Cauchy problems associated to $(\mathcal{P})$ is guaranteed. Recall the well known result (for the proof see [3])

**Lemma 1.** *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function and consider the autonomous equation*

$$x' = f(x) \ .$$

*We have that the Peano phenomenon can arise only at those values of $\bar{x}$ such that $f(\bar{x}) = 0$.*

Our main theorem (see Theorem 1 below) gives an anologous condition in order to obtain uniqueness of solutions in the case of system $(\mathcal{P})$.

## 2 – Main results

In this section we state and prove a result (Theorem 1) which is in some way the corresponding of Lemma 1 for system $(\mathcal{P})$, and obtain a corollary which is useful in many situations. Theorem 1 was obtained in [2, Remark 2.2] for the case $\mathcal{H}(x,y) = -y^2/2 - G(x)$ with $G\colon \mathbb{R} \to \mathbb{R}$ a $\mathcal{C}^1$ function.

**Theorem 1.** *Local uniqueness of Cauchy problems associated to $(\mathcal{P})$ holds provided that the initial value $(x_0, y_0)$ is not an equilibrium, that is, that $\nabla\mathcal{H}(x_0, y_0) \neq (0,0)$.*

**Proof:** Fix $(x_0, y_0) \in \mathbb{R}^2$ such that $\nabla\mathcal{H}(x_0, y_0) \neq (0,0)$. The hamiltonian structure of system $(\mathcal{P})$ guarantees that solutions $(x,y) = (x(t), y(t))$, which pass on the point $(x_0, y_0)$, satisfy the energy identity

$$\mathcal{H}(x,y) = \mathcal{H}(x_0, y_0) \ . \tag{1}$$

By the assumption we made, we have that either $\frac{\partial\mathcal{H}}{\partial x}$ or $\frac{\partial\mathcal{H}}{\partial y}$ does not vanish at $(x_0, y_0)$. We assume, without loss of generality, that $\frac{\partial\mathcal{H}}{\partial x}(x_0, y_0) \neq 0$. Thus, by the implicit function theorem, we have that (1) defines $x$ as a function of $y$ in a neighbourhood of $(x_0, y_0)$, more precisely, there exists a neighbourhood $U = ]x_1, x_2[ \times ]y_1, y_2[$ of $(x_0, y_0)$ and a $\mathcal{C}^1$ function $x\colon ]y_1, y_2[ \to ]x_1, x_2[$ such that $x(y_0) = x_0$ and

$$(x,y) \in U \ \text{ and } \ (x,y) \text{ satisfies } (1) \quad \Longleftrightarrow \quad x = x(y) \ .$$

At this point we conclude that if $(x(t), y(t))$ is a solution of $(\mathcal{P})$ in $U$, we must have $x(t) = x(y(t))$. Moreover, $y(t)$ will be a solution of the Cauchy problem

$$(2) \qquad \begin{cases} y' = \dfrac{\partial \mathcal{H}}{\partial x}(x(y), y) \\ y(0) = y_0 \end{cases} .$$

Notice that, as we assumed $\mathcal{H}$ of class $\mathcal{C}^1$ and recalling that $x(\cdot)$ is a $\mathcal{C}^1$ function, we have that $\frac{\partial \mathcal{H}}{\partial x}(x(\cdot), \cdot)$ is continuous and thus, applying Lemma 1 taking into account that $\frac{\partial \mathcal{H}}{\partial x}(x(y_0), y_0) \neq 0$, we conclude that (2) has an unique solution in an open neighbourhood of $y_0$ where $\frac{\partial \mathcal{H}}{\partial x}(x(\cdot), \cdot)$ does not vanish. The uniqueness of solution of $(\mathcal{P})$ in an open neighbourhood of $(x_0, y_0)$ contained in $U$ immediately follows. ∎

In many cases the set of equilibria reduces to a point. This is the situation mentioned in the following corollary.

**Corollary 1.** *Assume that there exists $(x^*, y^*) \in \mathbb{R}^2$ such that $\mathcal{H}(x, y) \neq \mathcal{H}(x^*, y^*)$ if $(x, y) \neq (x^*, y^*)$ and*

$$\nabla \mathcal{H}(x, y) \neq (0, 0) \quad \text{if } (x, y) \neq (x^*, y^*) .$$

*Then for each $(x_0, y_0) \in \mathbb{R}^2$ there exists a unique solution $(x(t), y(t))$ of $(\mathcal{P})$ which satisfies $(x(0), y(0)) = (x_0, y_0)$, that is global uniqueness is guaranteed.*

**Proof:** Let $(x_0, y_0) \in \mathbb{R}^2$ and consider a solution $(x(t), y(t))$ such that $(x(0), y(0)) = (x_0, y_0)$. If $(x_0, y_0) = (x^*, y^*)$, then by the energy identity (1) and using our assumption, we conclude that $(x(t), y(t)) = (x_0, y_0)$ for each $t$. If $(x_0, y_0) \neq (x^*, y^*)$ then $(x(t), y(t)) \neq (x^*, y^*)$ for each $t$ and hence the preceeding theorem guarantees the uniqueness of the solution. ∎

**Remark 1.** It is interesting to notice that for systems of the form $(\mathcal{P})$ Peano phenomena can arise in many cases. Let us concentrate on system

$$\begin{cases} x' = y \\ y' = -g(x) \end{cases}$$

derived from the second order nonlinear scalar equation $x'' + g(x) = 0$. We have the following result

*Assume that* $g\colon \mathbb{R} \to \mathbb{R}$ *is a continuous function such that* $g(0) = 0$ *and* $g(x) < 0$ *for* $x \in ]0,\epsilon]$, $\epsilon > 0$ *(resp.* $g(x) > 0$ *for* $x \in [-\epsilon, 0[$ *). Then, writing* $G(x) = \int_0^x g(\xi)\, d\xi$, *if* $\int_0^\epsilon \frac{dx}{\sqrt{-G(x)}}$ *converges, Peano phenomenon arises on* $(0,0)$.

This result applies to the case of system $x' = y$, $y' = -\sqrt{|x|}$, and hence Peano phenomenon arises on $(0,0)$. Notice that the corresponding hamiltonian is $\mathcal{H}(x,y) = -y^2/2 - 2/3\,|x|^{1/2}\,x$ and the equilibria set reduces to the origin. In this case, as $\mathcal{H}(x,y) = \mathcal{H}(0,0)$ in the set $\{(x,y)\colon\ y^2/2 + 2/3\,|x|^{1/2}\,x = 0\}$, one of the assumptions of Corollary 1 is not satisfied. We remark that when the equilibrium is a strict maximum of $\mathcal{H}$, then Corollary 1 applies. This is the case in the following example. □

**Example.** Let us consider the autonomous equation, (see [4, Section 3.1]),

$$x'' + g(x) = s\ , \tag{3}$$

where $s \in \mathbb{R}_0^+ = ]0,+\infty[$ and $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and strictly increasing with $g(0) = 0$ and $g(+\infty) = +\infty$. For instance, we could assume that $g(0) = 0$, $g(+\infty) = +\infty$ and $g$ continuously differentiable in $\mathbb{R}_0^+$ with $g'(x) > 0$ for $x > 0$. Notice that under these conditions the lipschitzianity of the associated vector field is not guaranteed. Nevertheless, as we see below, global uniqueness of solutions follows from Corollary 1.

With the above positions, we have that for each $s > 0$ there exists exactly one $\gamma_s$ such that $g(\gamma_s) = s$. Moreover, setting $G(x) = \int_0^x g(\xi)\, d\xi$, the function $G(x) - s\,x$ is continuously differentiable in $\mathbb{R}_0^+$ and attains its absolute minimum in $\gamma_s$. If we consider the system

$$\begin{cases} x' = y \\ y' = -\hat{g}(x) + s \end{cases} \tag{4}$$

where

$$\hat{g}(x) = \begin{cases} g(x) & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}\ ,$$

we have that this system coincides with system $(\mathcal{P})$ if we put $\mathcal{H}(x,y) = -y^2/2 - \hat{G}(x) + s\,x$ where $\hat{G}(x) = \int_0^x \hat{g}(\xi)\, d\xi$ for every $x \in \mathbb{R}$. As $\gamma_s$ is the point in which $\hat{G}(x) - s\,x$ attains its absolute minimum, it is easy to conclude that $\mathcal{H}(x,y) < \mathcal{H}(\gamma_s, 0)$ if $(x,y) \neq (\gamma_s, 0)$. Moreover we have $\nabla\mathcal{H}(x,y) = (-\hat{g}(x) + s, -y)$, which is different from $(0,0)$, provided that $(x,y) \neq (\gamma_s, 0)$, and hence applying Corollary 1 we conclude that global uniqueness holds for system (4). Finally, we conclude that uniqueness of solutions of equation (3) follows from the fact that $\hat{g} \equiv g$ for $x \geq 0$. In this example, we can also conclude that $(x(t), y(t)) \equiv (\gamma_s, 0)$ is the unique solution which touchs $(\gamma_s, 0)$. □

*ACKNOWLEDGEMENT* – I want to thank Fabio Zanolin and Luis Sanchez for some useful discussions about this argument.

## REFERENCES

**[1]** Agarwal, R.P. and Lakshmikantham, V. – *Uniqueness and nonuniqueness criteria for ordinary differential equations*, Series in Real Analysis, 6, World Scientific, Singapore, New Jersey, 1993.

**[2]** Fernandes, M.L.C. – *On an elementary phase-plane analysis for solving second-order BVPs,* Ricerche di Matematica, 37 (1988), 189–202.

**[3]** Piccinini, L.C. – *Remarks on a uniqueness problem for dynamical systems,* Boll. Un. Mat. Ital. A, 18 (1981), 132–137.

**[4]** Rebelo, C. – *Periodic Solutions of Nonautonomous Planar Systems via the Poincaré–Birkhoff Fixed Point Theorem*, Ph.D. Thesis, Universidade Clássica de Lisboa, Lisboa, 1996.

C. Rebelo,
Centro de Matemática e Aplicações Fundamentais,
Av. Prof. Gama Pinto 2, 1649-003 Lisboa – PORTUGAL