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ZONOIDS AND CONDITIONALLY POSITIVE DEFINITE FUNCTIONS

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Abstract: It is well-known in convex geometry that a centrally symmetric convex body $Z \subset \mathbb{R}^d$ is a zonoid (a limit of sums of line segments), if and only if its support function h has the property that -h is conditionally positive definite. Here, we give a new proof of this result using extreme point methods.

1 – Introduction

In $\mathbb{R}^d$, let $\langle \cdot, \cdot \rangle$ denote the scalar product, $\|\cdot\|$ the Euclidean norm and

$$\Omega^{d-1} = \left\{ x \in \mathbb{R}^d \mid \ \|x\| = 1 \right\}$$

the unit sphere. $C(\mathbb{R}^d)$ is the space of all continuous functions $f \colon \mathbb{R}^d \to \mathbb{R}$, and $C_c(\mathbb{R}^d)$ the subspace of all continuous functions with compact support.

A *zonoid* is a convex body in Euclidean space $\mathbb{R}^d$ which can be approximated, in the sense of the Hausdorff metric, by finite vector sums of line segments. A zonoid has a centre of symmetry, and this may be assumed to be the origin of $\mathbb{R}^d$.

A locally integrable function $f : \mathbb{R}^d \to \mathbb{R}$ is said to be *positive definite* if, for all functions $g \in C_c(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(u-v)\, g(u)\, g(v)\; du\, dv \;\geq\; 0 \;. \tag{1}$$

If f satisfies (1) at least for functions g with the additional property

$$\int_{\mathbb{R}^d} g(u)\; du \;=\; 0 \;,$$

then f is called *conditionally positive definite*. We deduce that the function

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$f \in C(\mathbb{R}^d)$ is positive definite if and only if, for all natural numbers k, all real numbers w_i and all $x_i \in \mathbb{R}^d$,

$$\sum_{i,j=1}^{k} f(x_i - x_j)\, w_i\, w_j \;\geq\; 0 \; . \tag{2}$$

A continuous f is conditionally positive definite if and only if f satisfies (2) at least for all real numbers w_i with the additional property $\sum_{i=1}^{k} w_i = 0$.

It is well known (see Bolker [2]) that a convex body Z is a zonoid if and only if its support function h is of the form

$$h(u) \;=\; \int_{\Omega^{d-1}} |\langle x, u\rangle|\, d\mu(x) \tag{3}$$

with some even measure μ on Ω^{d-1}. Here by a *measure* on Ω^{d-1} we understand a nonnegative, σ-additive, real valued function on the Borel subsets of Ω^{d-1}. The measure μ is uniquely determined by Z.

The following classical result describes the connection between support functions of zonoids and conditionally positive definite functions. Note that in $\mathbb{R}^d$ the support function of a convex body, which is centrally symmetric to the origin, is a seminorm.

Theorem. *For a seminorm h on $\mathbb{R}^d$, the following conditions are equivalent:*

(**a**) *h is the support function of a zonoid, i.e.,*

$$h \;=\; \int_{\Omega^{d-1}} |\langle x, \cdot\rangle|\, d\mu(x) \;,$$

with an even measure μ on Ω^{d-1}.

(**b**) *$-h$ is conditionally positive definite.*

The implication from (a) to (b) is relatively straight forward. Suppose h has the integral representation (3). Kelly [8] has shown that, if $|\cdot|$ denotes the absolute value function, then $-|\cdot|$ is conditionally positive definite. This implies that $-|\langle x, \cdot\rangle|$ is a conditionally positive definite function on $\mathbb{R}^d$, for each $x \in \mathbb{R}^d$. Clearly this extends to the integral representation (3) because the measure μ is nonnegative. The difficult part of the theorem concerns the reverse implication from (b) to (a). For a proof of this result, most authors in convex geometry (see e.g. Choquet [3, Vol. III, pp. 55–59], Witsenhausen [12], Goodey–Weil [5]) refer to Lévy's [10] general results on stable probability distributions. Lévy's classical approach uses the fact that (b) implies e^{-h} to be positive definite. By Bochner's theorem, e^{-h} is the Fourier transform of a probability measure which can then be used to establish (a). In connection with embedding results for finite dimensional Banach spaces, the Lévy representation (3) of seminorms has been generalized and studied further in recent years (see Koldobsky [9], for a survey and further references).

Bochner's theorem is a typical example of an integral representation which can be obtained directly by extreme point methods. Choquet [3, Vol. II, pp. 245–262] gives a proof of that kind and emphasizes the role of extreme point methods in connection with integral representations (see also Edwards [4]). Other examples include a proof of a Lévy–Khintchin formula by Johansen [6]. It is therefore a natural question whether the implication from (b) to (a) can also be obtained by a suitable application of the Krein–Milman theorem (or Choquet's extension of that result). It is the aim of this paper to provide a proof of that kind.

Our attempt is motivated by the fact that the set H of all support functions of zonoids is a closed convex cone with compact base in $C(\mathbb{R}^d)$. The extreme rays of this cone consist of the functions

$$|\langle\cdot, x\rangle|\,, \qquad x \in \mathbb{R}^d\,, \tag{4}$$

and thus (3) gives the extremal representation of the functions in H. Hence, for the implication from (b) to (a), one has to prove that the seminorms h which fulfill (b) form a closed convex cone D as well and that the extreme rays of D consist exactly of the functions (4). Here, the difficult part is to show that only the functions (4) are elements of an extreme ray of D. To overcome this problem, we will have to remove the homogeneity property and thus consider a larger cone C, for which it is possible to determine the extreme rays. We will then embed C in $L^\infty(\mathbb{R}^d)$ and use the weak-$*$ topology to obtain a compact base. Applying Choquet's theorem we get an integral representation for the functions in C. From this, we finally deduce that a seminorm h in $\mathbb{R}^d$, which fulfills (b), can be written in the form (3).

2 – Extreme points of a class of conditionally positive definite functions

Let C denote the cone of all conditionally positive definite even functions $f \in C(\mathbb{R}^d)$ satisfying $f(0) = 0$. A base B of C is given by the set of all $f \in C$ with

$$\int_I f(u)\,du = -1\,, \qquad I = [-1,1]^d\,.$$

From (2) we deduce that the functions in C are nowhere positive.

To get some information about the set $\operatorname{ext} B$ of extreme points of the convex set B, we need two lemmata. For these lemmata we can generalize a proof for the one-dimensional case from Johansen [6] without effort.

For $f\colon \mathbb{R}^d \to \mathbb{R}$, $w \in \mathbb{R}^d$ and $\varepsilon \in \mathbb{R}$, we use the notation $f_{w,\varepsilon}$ for the function

$$f_{w,\varepsilon}(u) := f(u) + \varepsilon\Big(f(u+w) + f(u-w) - 2\,f(w)\Big)\,, \qquad u \in \mathbb{R}^d \ .$$

Lemma 1. *Let $w \in \mathbb{R}^d$, $\varepsilon \in [-\frac{1}{2}, \frac{1}{2}]$, and $f\colon \mathbb{R}^d \to \mathbb{R}$ be a locally integrable function. If f is conditionally positive definite, then $f_{w,\varepsilon}$ is also conditionally positive definite. For $f \in C$, we have $f_{w,\varepsilon} \in C$.*

Proof: Let $g \in C_c(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} g(u)\,du = 0$, then

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} 2\,\varepsilon\, f(w)\, g(u)\, g(v)\; du\, dv \;=\; 0\ .$$

So, it is sufficient to show that

$$f(u) + \varepsilon\Big(f(u+w) + f(u-w)\Big)$$

is conditionally positive definite. For $x \in \mathbb{R}$ we define $h\colon \mathbb{R}^d \to \mathbb{R}$ by

$$h(u) \;=\; g(u) + x\, g(u+w)\,, \qquad u \in \mathbb{R}^d\ ,$$

and we have

$$\int_{\mathbb{R}^d} h(u)\, du \;=\; \int_{\mathbb{R}^d} g(u)\, du \;+\; x\int_{\mathbb{R}^d} g(u+w)\, du \;=\; 0\ . \tag{5}$$

Because f is conditionally positive definite, we get

$$\begin{aligned}
0 &\le \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(u-v)\, h(u)\, h(v)\; du\, dv \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\Big(f(u-v)\, g(u)\, g(v) + x\, f(u-v)\, g(u)\, g(v+w) \\
&\qquad + x\, f(u-v)\, g(u+w)\, g(v) + x^2\, f(u-v)\, g(u+w)\, g(v+w)\Big)\; du\, dv \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\Big(f(u-v)\, g(u)\, g(v) + x\, f(u-v+w)\, g(u)\, g(v) \\
&\qquad + x\, f(u-v-w)\, g(u)\, g(v) + x^2\, f(u-v)\, g(u)\, g(v)\Big)\; du\, dv\ .
\end{aligned}$$

This means that, for $x \in \mathbb{R}$, the function

$$u \;\mapsto\; (1+x^2)\, f(u) + x\,\Big(f(u+w) + f(u-w)\Big)\;, \tag{6}$$

is conditionally positive definite. The range of values of the function $x \mapsto x/(1+x^2)$ is the interval $[-\frac{1}{2}, \frac{1}{2}]$, and this completes the proof. Finally we remark that $x = -1$ implies (5) for every $g \in C_c(\mathbb{R}^d)$. Hence, for $x = -1$, the function (6) is positive definite. ∎

Lemma 2. *If $f \in C$, $n \in \mathbb{N}$, $u \in \mathbb{R}^d$, then $f(n\,u) \ge n^2 f(u)$.*

Proof: Lemma 1 shows that $f \in C$ implies $f_{u,-1/2} \in C$. So we have $f_{u,-1/2}(v) \le 0$, $v \in \mathbb{R}^d$, and for $v = k\,u$, $k \in \mathbb{N}$, we get

$$f\Big((k+1)\,u\Big) - f(k\,u) \;\ge\; f(k\,u) - f\Big((k-1)\,u\Big) + 2\,f(u)\;.$$

Summing over k we get for all $n \in \mathbb{N}$

$$\sum_{m=1}^{n}\sum_{k=1}^{m}\Big(f\Big((k+1)\,u\Big) - f(k\,u)\Big) \;\ge\; \sum_{m=1}^{n}\sum_{k=1}^{m}\Big(f(k\,u) - f\Big((k-1)\,u\Big) + 2\,f(u)\Big)\;,$$

which gives

$$\sum_{m=1}^{n}\Big(f\Big((m+1)\,u\Big) - f(u)\Big) \;\ge\; \sum_{m=1}^{n}\Big(f(mu) + 2\,m\,f(u)\Big)\;,$$

and we arrive at

$$f\Big((n+1)\,u\Big) \;\ge\; (n+1)^2\, f(u)\;. \;∎$$

Proposition 1. *$f \in \operatorname{ext} B$ implies either that there is a positive semidefinite $(d\times d)$-matrix $\mathcal{A}$ such that*

$$f(u) \;=\; -u\;\mathcal{A}\;u^\top\;, \tag{7}$$

or that there exists a $v \in \mathbb{R}^d\backslash\{0\}$ and a constant $c(v)$ such that

$$f(u) = c(v)\,(\cos\langle u, v\rangle - 1)\;. \tag{8}$$

Proof: Let $f \in \operatorname{ext} B$ and

$$g(w) := -\int_I \Big(f(u+w) + f(u-w) - 2\,f(w)\Big)\,du\,, \qquad w \in \mathbb{R}^d\;.$$

The functions $f_{w,\pm 1/2}$ are nonpositive, and so we have $\|g\|_\infty \leq 2$. For fixed $w \in \mathbb{R}^d$ we get the convex combination

$$f \;=\; \frac{1+g(w)/3}{2} \cdot \frac{f_{w,1/3}}{1+g(w)/3} + \frac{1-g(w)/3}{2} \cdot \frac{f_{w,-1/3}}{1-g(w)/3}$$

with $\frac{f_{w,1/3}}{1+g(w)/3}, \frac{f_{w,-1/3}}{1-g(w)/3} \in B$.

From $f \in \operatorname{ext} B$ we deduce, for all $w \in \mathbb{R}^d$,

$$f = \frac{f_{w,1/3}}{1+g(w)/3} = \frac{f_{w,-1/3}}{1-g(w)/3} \;,$$

and we get

$$g(w)\, f(u) \;=\; f(u+w) + f(u-w) - 2\, f(w) \;. \tag{9}$$

Interchanging w and u and subtracting the equations gives us

$$g(w)\, f(u) - g(u)\, f(w) \;=\; 2\, f(u) - 2\, f(w) \;,$$

which means

$$(g(w)-2)\, f(u) \;=\; (g(u)-2)\, f(w) \;.$$

Case 1: $g \equiv 2$.

Then, for all $w, u \in \mathbb{R}^d$, (9) yields

$$f(u+w) + f(u-w) - 2\, f(w) - 2\, f(u) \;=\; 0 \;.$$

The nonpositive solutions of this functional equation are precisely the functions

$$f(u) \;=\; -u\; \mathcal{A}\; u^\top \;,$$

with $\mathcal{A}$ a positive semidefinite $(d\times d)$-matrix (see Aczél and Dhombres [1, p. 166]).

Case 2: There exists a $w_0 \in \mathbb{R}^d$ such that $g(w_0) \neq 2$.

We define

$$c := \frac{f(w_0)}{g(w_0)-2} \;,$$

and have

$$f(u) = c\,(g(u)-2)\,, \qquad u \in \mathbb{R}^d \;. \tag{10}$$

Substituting (10) into (9) and using the fact that $c \neq 0$, we get

$$g(w)\, g(u) \;=\; g(u+w) + g(u-w)\,, \qquad w, u \in \mathbb{R}^d \;.$$

All nontrivial solutions of this functional equation are

$$g(u) = 2\cos\langle v, u\rangle \qquad \text{or} \qquad g(u) = 2\cosh\langle v, u\rangle$$

with $v \in \mathbb{R}^d$ (see Kannappan [7]). The function $g(u) = 2\cosh\langle v, u\rangle$ cannot occur in our situation, since we have $\|g\|_\infty \le 2$. So the only solution in Case 2 is

$$f(u) \;=\; 2\,c\,(\cos\langle v, u\rangle - 1)\,, \qquad u \in \mathbb{R}^d\;,$$

with $v \in \mathbb{R}^d$. ∎

We denote the set of all functions f with representation (7) by E_1 and use the notation $f_{\mathcal{A}}$ for f, in this case. The set of all functions f with representation (8) is denoted by E_2, and we will write f_v for f, in this case. Further, we put $E := E_1 \cup E_2$. Note that if f has representation (8), then

$$2\,c(v) \;=\; -\Big(\int_I (\cos\langle u, v\rangle - 1)\,du\Big)^{-1}\;,$$

and, as a function of v, c is continuous.

3 – Application of Choquet's theorem

The following lemma shows that we can identify B with the continuous functions in $\widehat{B}$, the set of all measurable, even, conditionally positive definite functions $f\colon \mathbb{R}^d \to \mathbb{R}$, which are essentially bounded in any neighbourhood of 0, and fulfill

$$\sup\operatorname{ess} f = 0 \qquad \text{and} \qquad \int_I f(u)\,du = -1\;.$$

Lemma 3. *For each $f \in \widehat{B}$ there is a $g \in B$ such that $f = g$ almost surely with respect to the Lebesgue measure.*

Proof: The proof of Lemma 1 shows that, for every $w \in \mathbb{R}^d$, the function

$$u \;\mapsto\; 2\,f(u) - f(u+w) - f(u-w)$$

is positive definite. By integration over w and Fubini's theorem we get that the function

$$(11) \qquad 2^d f(u) \;-\; \frac{1}{2}\int_I \Big(f(u+w) + f(u-w)\Big)\,dw \;=\; 2^d f(u) - 2\int_{I+u} f(w)\,dw$$

is also positive definite. Now we make use of the well-known fact that there is a continuous function which equals the function (11) almost surely (see, e.g., Edwards [4, pp. 719]). The continuity of

$$u \mapsto \int_{I+u} f(w)\, dw$$

shows that there is an even positive definite function $g \in C(\mathbb{R}^d)$ with $f = g$ almost surely. The positive definiteness of g implies

$$2\,g(0) - 2\,g(u) \geq 0\,, \qquad u \in \mathbb{R}^d\ ,$$

and together with $\sup g = \operatorname{sup\,ess} f = 0$ we get $g(0) = 0$. ∎

Choquet's theorem requires a compact set, therefore we embed $\widehat{B}$ in the space $L^\infty(\mathbb{R}^d)$, supplied with the weak-$*$ topology. To do so, we need the following result.

Lemma 4. *There exists a constant c_0 such that, for all $f \in B$,*

$$|f(u)| \leq c_0(1 + \|u\|^2)\,, \qquad u \in \mathbb{R}^d\ .$$

Proof: We denote the smallest integer greater than a real number x by $r(x)$, then Lemma 2 gives

$$0 \geq f(u) = f\left(r(\|u\|)\,\frac{u}{r(\|u\|)}\right) \geq r^2(\|u\|)\, f\left(\frac{u}{r(\|u\|)}\right)\,,$$

which implies

$$|f(u)| \leq (\|u\| + 1)^2 \cdot \max\{|f(u)| \mid \|u\| \leq 1\}\ .$$

For $w \in \mathbb{R}^d$, $\int_I f_{w,1/2}(u)\, du$ is nonpositive and we get

$$\begin{aligned}
0 &\geq -1 - 2^d f(w) + \frac{1}{2}\left(\int_I f(u+w)\, du + \int_I f(u-w)\, du\right)\\
&= -1 - 2^d f(w) + \frac{1}{2}\left(\int_{I+w} f(u)\, du + \int_{I+w} f(-u)\, du\right)\\
&= -1 - 2^d f(w) + 2^d \int_{\frac{1}{2}(I+w)} f(2u)\, du\\
&\geq -1 - 2^d f(w) + 2^{d+2} \int_{\frac{1}{2}(I+w)} f(u)\, du\ ,
\end{aligned}$$

where we have used Lemma 2 in the last inequality. If $\|w\| < 1$, then $\frac{1}{2}(I+w) \subset I$, and we deduce

$$2^d f(w) \geq -1 - 2^{d+2} ,$$

which means

$$|f(w)| \leq 2^{-d} + 4 \leq 5 .$$

This completes the proof. ∎

Let $c_0 \geq 1$ fulfill the assertion of Lemma 4, then we define the function $g_0 : \mathbb{R}^d \to \mathbb{R}$ as

$$g_0(u) = c_0(1 + \|u\|^2) ,$$

the set

$$\widehat{B}/g_0 := \left\{ \frac{f}{g_0} \mid f \in \widehat{B} \right\} ,$$

and, analogously, the sets $E/g_0, E_1/g_0$ and E_2/g_0.

From Lemma 4 we deduce that $\widehat{B}/g_0$ is a subset of the unit sphere of $L^\infty(\mathbb{R}^d)$ which, when supplied with the weak-$*$-topology, is compact. We easily verify that $\widehat{B}/g_0$ is closed and therefore also compact.

The theorem of Choquet now shows that, for all $s \in \widehat{B}/g_0$, there is a probability measure μ on $L^\infty(\mathbb{R}^d)$ with

$$L(s) = \int_{\mathrm{ext}(\widehat{B}/g_0)} L(t)\, d\mu(t) \tag{12}$$

for all continuous linear functionals L on $L^\infty(\mathbb{R}^d)$.

4 – Transformation of the integral representation

Our next goal is to use the integral representation (12) which was obtained for functions in $\widehat{B}/g_0$, to get the representation

$$f(u) = -\int_{\Omega^{d-1}} |\langle u, x\rangle|\, d\bar{\mu}(x)$$

for continuous, conditionally positive definite, positively homogeneous, even f.

Proposition 1 shows that $t \in \mathrm{ext}(\widehat{B}/g_0)$ implies $t \in E/g_0$. More precisely, each equivalence class of $\mathrm{ext}(\widehat{B}/g_0)$ contains a continuous representative t with

$t = g/g_0$ and $g \in E$. Further the sets E_1/g_0 and E_2/g_0 are measurable. We therefore get from (12), for each function $f \in B$, a probability measure μ with

$$\int_{\mathbb{R}^d} \frac{f(u)}{g_0(u)}\, h(u)\, du \;=\; \int_{E/g_0} \int_{\mathbb{R}^d} t(u)\, h(u)\, du\, d\mu(t) \tag{13}$$

for all $h \in L^1(\mathbb{R}^d)$.

Proposition 2. *If $f \in B$ is positively homogeneous, then there is a probability measure ν on $\mathbb{R}^d$ with*

$$f(u) \;=\; \int_{\mathbb{R}^d} f_v(u)\, d\nu(v) \;.$$

Proof: Let U be an arbitrary neighborhood of zero, λ a positive number, and

$$h_\lambda(u) := \begin{cases} -g_0(u) & \text{for } u \in \dfrac{1}{\lambda}\, U, \\ 0 & \text{elsewhere} \;. \end{cases}$$

The functions in E/g_0 are nonpositive, and so

$$\int_{E_2/g_0} \int_{\mathbb{R}^d} t(u)\, h_\lambda(u)\, du\, d\mu(t) \;\geq\; 0 \;,$$

which implies

$$\int_{\frac{1}{\lambda} U} -f(u)\, du \;\geq\; \int_{E_1/g_0} \int_{\frac{1}{\lambda} U} -t(u)\, g_0(u)\, du\, d\mu(t) \;.$$

We deduce

$$\lambda^d \int_U -f(\lambda u)\, du \;\geq\; \lambda^d \int_{E_1/g_0} \int_U -t(\lambda u)\, g_0(\lambda u)\, du\, d\mu(t) \;,$$

and because f is positive homogeneous we get

$$\lambda \int_U -f(u)\, du \;\;\geq\;\; \lambda^2 \int_{E_1/g_0} \int_U -t(u)\, g_0(u)\, du\, d\mu(t) \;.$$

Since λ is an arbitrary positive number, it follows that

$$\int_{E_1/g_0} \int_U -t(u)\, g_0(u)\, du\, d\mu(t) \;=\; 0$$

and

$$\int_U f(u)\, du \;=\; \int_{E_2/g_0} \int_U t(u)\, g_0(u)\, du\, d\mu(t) \;. \tag{14}$$

Hence the measure μ in (13) is concentrated on E_2/g_0. We consider the equivalence relation $\sim$ on $\mathbb{R}^d$,

$$x \sim y \quad \text{if and only if} \quad x = \pm y \ ,$$

and supply $(\mathbb{R}^d\backslash\{0\})_{/\sim}$ with the quotient topology. Using the bijective function

$$G\colon \begin{cases} (\mathbb{R}^d\backslash\{0\})_{/\sim} \to E_2/g_0 \ , \\ \tilde{v} \to \dfrac{f_v}{g_0} \ , \end{cases}$$

we define the measure η as the image of the restriction of μ to E_2/g_0, with respect to the mapping G^{-1}. For this, we have to prove that G^{-1} is measurable.

To do so, let A be an arbitrary Borel set of $(\mathbb{R}^d\backslash\{0\})_{/\sim}$. Using

$$B_k := \left\{ \tilde{v} \in (\mathbb{R}^d\backslash\{0\})_{/\sim} \mid \ \frac{1}{k} \le \|v\| \le k \right\}, \qquad k \in \mathbb{N} \ ,$$

and $G_k\colon B_k \to G(B_k)$, with $G_k(\tilde{v}) = G(\tilde{v})$, $\tilde{v} \in B_k$, we have

$$G(A) = \bigcup_{k=1}^{\infty} G_k(A \cap B_k) \ .$$

G_k is bijective, and we prove that $G(A)$ is a Borel set by showing that G_k^{-1} is continuous. Since B_k is compact, it is sufficient to prove that G_k is continuous.

We use the fact that there is a countable set of functions $g_n \in L^1(\mathbb{R}^d)$, $n \in \mathbb{N}$, with $\|g_n\| = 1$ so that the unit sphere of $L^\infty(\mathbb{R}^d)$ supplied with the weak-$*$ topology is metrizable by

$$d(h_1, h_2) \ = \ \sum_{n=1}^{\infty} 2^{-n} \left| \int_{\mathbb{R}^d} g_n(u) \, (h_1 - h_2)(u) \, du \right|, \qquad h_1, h_2 \in L^\infty(\mathbb{R}^d) \ .$$

For each $g \in L^1(\mathbb{R}^d)$ with $\|g\| = 1$ and $v_1, v_2 \in \mathbb{R}^d$ we have

$$\begin{aligned} &\left| \int_{\mathbb{R}^d} g(u) \Big(G(v_1) - G(v_2) \Big)(u) \, du \right| \ = \\ &\qquad = \ \left| \int_{\mathbb{R}^d} \left(\frac{g}{g_0} \right)(u) \Big(c(v_1) \cos\langle u, v_1 \rangle - c(v_2) \cos\langle u, v_2 \rangle - c(v_1) + c(v_2) \Big) \, du \right| \\ &\qquad \le \ c(v_1) \int_{\mathbb{R}^d} \left| \left(\frac{g}{g_0} \right)(u) \right| \Big| \cos\langle u, v_1 \rangle - \cos\langle u, v_2 \rangle \Big| \, du \\ &\qquad\quad + \Big| c(v_1) - c(v_2) \Big| \int_{\mathbb{R}^d} \Big(|\cos\langle u, v_2 \rangle| + 1 \Big) \left| \left(\frac{g}{g_0} \right)(u) \right| du \ . \end{aligned}$$

Using $|\cos\langle u,v_1\rangle - \cos\langle u,v_2\rangle| \leq \|u\|\,\|v_1-v_2\|$ we get

$$\left|\int_{\mathbb{R}^d} g(u)\,\Big(G(v_1)-G(v_2)\Big)(u)\,du\right| \;\leq$$

$$\leq\; c(v_1)\int_{\mathbb{R}^d}\|u\|\,\|v_1-v_2\|\left|\left(\frac{g}{g_0}\right)(u)\right|du \;+\; 2\Big|c(v_1)-c(v_2)\Big|\int_{\mathbb{R}^d}\left|\left(\frac{g}{g_0}\right)(u)\right|du$$

$$\leq\; c(v_1)\,\|v_1-v_2\| + 2\Big|c(v_1)-c(v_2)\Big|\;.$$

For an arbitrary $\varepsilon > 0$ exists an $m \in \mathbb{N}$ with

$$d\Big(G(v_1),G(v_2)\Big) \;\leq\; \frac{\varepsilon}{2} + \sum_{n=1}^{m} 2^{-n}\left|\int_{\mathbb{R}^d} g_n(u)\,\Big(G(v_1)-G(v_2)\Big)(u)\,du\right|\;.$$

Furthermore, since the function $c(v)$ is continuous, there exists for fixed $v_1 \in \mathbb{R}^d\backslash\{0\}$ a positive δ such that $\|v_1-v_2\| < \delta$ implies

$$\left|\int_{\mathbb{R}^d} g(u)\,\Big(G(v_1)-G(v_2)\Big)(u)\,du\right| < \frac{\varepsilon}{m}\;.$$

Thus G is continuous at v_1 and, since v_1 was arbitrary, G is continuous.

From equation (14) we now get

$$\int_U f(u)\,du \;=\; \int_{(\mathbb{R}^d\backslash\{0\})_{/\sim}}\int_U f_v(u)\,du\,d\eta(\tilde{v})$$

for each neighborhood U of zero and arrive at

$$f(u) \;=\; \int_{(\mathbb{R}^d\backslash\{0\})_{/\sim}} f_v(u)\,d\eta(\tilde{v})\;.$$

Let π denote the canonical projection from $\mathbb{R}^d\backslash\{0\}$ to $(\mathbb{R}^d\backslash\{0\})_{/\sim}$, then we define the even measure ϑ on $\mathbb{R}^d\backslash\{0\}$ by

$$\vartheta(A) = \eta(\pi(A)) \quad \text{for all centrally symmetric Borel sets } A \text{ of } \mathbb{R}^d\backslash\{0\}\;.$$

We get

$$f(u) \;=\; \int_{\mathbb{R}^d\backslash\{0\}} f_v(u)\,d\vartheta(v)\;,$$

and complete the proof by defining the measure ν,

$$\nu(A) = \vartheta(A\backslash\{0\})$$

for all Borel sets $A \subset \mathbb{R}^d$.

The measure μ is concentrated on E_2/g_0, hence ν is a probability measure. ∎

We now transform the integral representation of Proposition 2 and show a corresponding uniqueness result.

Proposition 3. *For each positively homogeneous function in B, there exists a unique even measure η such that*

$$f(u) = \int_{\mathbb{R}^d} \frac{\cos\langle u,v\rangle - 1}{\|v\|^2}\, d\eta(v)\ .$$

Proof: We use Proposition 2 and define η as the measure, absolutely continuous with respect to ν, and with density

$$v \mapsto \begin{cases} \|v\|^2\, c(v) & \text{for } v \neq 0, \\ 0 & \text{for } v = 0\ . \end{cases}$$

The desired representation then follows directly. For the uniqueness, we first observe that

$$\begin{aligned} \int_{u+I} \cos\langle v,w\rangle\, dw &= \int_I \cos\langle v, w+u\rangle\, dw \\ &= \int_I \Big(\cos\langle v,w\rangle \cos\langle v,u\rangle - \sin\langle v,w\rangle \sin\langle v,u\rangle\Big)\, dw \\ &= \cos\langle v,u\rangle \int_I \cos\langle v,w\rangle\, dw \\ &= \cos\langle v,u\rangle \left(2^d - \frac{1}{c(v)}\right)\ , \end{aligned}$$

for $v \in \mathbb{R}^d \backslash \{0\},\ u \in \mathbb{R}^d$. It follows that

$$\begin{aligned} &\int_{\mathbb{R}^d} \cos\langle v,u\rangle\, d\nu(v) + \int_{\mathbb{R}^d} c(v) \int_{u+I} \Big(\cos\langle v,w\rangle - 1\Big)\, dw\, d\nu(v) = \\ &\quad = \int_{\mathbb{R}^d} \cos\langle v,u\rangle\, d\nu(v) + \int_{\mathbb{R}^d} c(v) \left(-2^d + \cos\langle v,u\rangle \left(2^d - \frac{1}{c(v)}\right)\right) d\nu(v) \\ &\quad = 2^d \int_{\mathbb{R}^d} c(v) \Big(\cos\langle v,u\rangle - 1\Big)\, d\nu(v) = 2^d f(u)\ . \end{aligned}$$

Since ν is an even measure, we have

$$\int_{\mathbb{R}^d} \sin\langle v,u\rangle\, d\nu(v) = 0\ ,$$

and we arrive at

$$2^d f(u) = \int_{\mathbb{R}^d} e^{i\langle v,u\rangle}\, d\nu(v) + \int_{u+I} f(w)\, dw\ . \tag{15}$$

If there is a second even measure $\bar{\nu}$ such that

$$f(u) = \int_{\mathbb{R}^d} c(v) \Big(\cos\langle u,v\rangle - 1\Big)\, d\bar{\nu}(v) \ ,$$

then equation (15) is also true for the measure $\bar{\nu}$ instead of ν. This gives $\nu = \bar{\nu}$, because the two measures have the same Fourier transform. ∎

We finally use this result to give the *Proof of the implication (b)→(a) in the Theorem.*

Let h be a seminorm on $\mathbb{R}^d$ with the property that $-h$ is conditionally positive definite.

Using $c := \int_I h(u)\, du$, we have $-h/c \in B$, and from Proposition 3 we obtain the existence of an unique even measure η such that

$$h(u) = c \int_{\mathbb{R}^d} \frac{1 - \cos\langle u,v\rangle}{\|v\|^2}\, d\eta(v) \ .$$

Matheron [11] has shown that this integral representation implies (3). For completeness we reproduce the short proof.

Since h is positively homogeneous we get for positive α

$$\begin{aligned} c\,\alpha \int_{\mathbb{R}^d} \frac{1 - \cos\langle u,v\rangle}{\|v\|^2}\, d\eta(v) &= \alpha\, h(u) = h(\alpha\, u) \\ &= c\,\alpha^2 \int_{\mathbb{R}^d} \frac{1 - \cos\langle \alpha\, u,v\rangle}{\|\alpha\, v\|^2}\, d\eta(v) \ . \end{aligned}$$

Let

$$f(u,v) := c\, \frac{1 - \cos\langle u,v\rangle}{\|v\|^2} \ ,$$

then we have for all $u \in \mathbb{R}^d$ and all positive α

$$\int_{\mathbb{R}^d} f(u,v)\, d\eta(v) = \alpha \int_{\mathbb{R}^d} f(u,\alpha\, v)\, d\eta(v) \ . \tag{16}$$

For each positive number α, we define an even measure ν_α on $\mathbb{R}^d$ by

$$\nu_\alpha(A) := \alpha\, \eta(\alpha^{-1} A)$$

for all Borel sets A of $\mathbb{R}^d$. Equation (16) gives

$$\int_{\mathbb{R}^d} f(u,v)\, d\eta(v) = \int_{\mathbb{R}^d} f(u,v)\, d\nu_\alpha(v) \ ,$$

for all $u \in \mathbb{R}^d$.

Since the measure η is unique we have $\eta = \nu_\alpha$, thus for all positive α and all Borel sets A of $\mathbb{R}^d$,

$$\eta(\alpha A) = \alpha\,\eta(A) \ .$$

Let A be an arbitrary Borel set in the unit sphere Ω^{d-1}, then

$$\widetilde{A} := \bigcup_{0\le\beta\le 1} \beta\, A \ ,$$

is a Borel set in $\mathbb{R}^d$, and we define the measure φ on Ω^{d-1} by $\varphi(A) := \eta(\widetilde{A})$. Fubini's theorem gives

$$\begin{aligned} h(u) \;&=\; c \int_{\mathbb{R}^d} \frac{1-\cos\Bigl(\|v\|\,\langle u, \frac{v}{\|v\|}\rangle\Bigr)}{\|v\|^2}\, d\eta(v) \\ &=\; c \int_{\Omega^{d-1}} \int_0^\infty \frac{1-\cos(\beta\langle u,x\rangle)}{\beta^2}\, d\beta\, d\varphi(x) \ , \end{aligned} \tag{17}$$

because $\alpha\,\varphi(A) = \eta(\alpha\widetilde{A})$, for all positive numbers α. Using the equation

$$\frac{4}{\pi} \int_0^\infty \frac{1-\cos(\gamma\beta)}{\beta^2}\, d\beta \;=\; |\gamma|\,, \qquad \gamma\in\mathbb{R} \ ,$$

it follows that

$$|\langle u,x\rangle| \;=\; \frac{4}{\pi} \int_0^\infty \frac{1-\cos(\beta\langle u,x\rangle)}{\beta^2}\, d\beta \ ,$$

and from (17) we deduce

$$h(u) \;=\; c\,\frac{4}{\pi} \int_{\Omega^{d-1}} |\langle u,x\rangle|\; d\varphi(x) \ ,$$

which completes the proof. The uniquess of φ follows from that of η. ∎

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