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ULTRAFILTER SPACES AND COMPACTIFICATIONS

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Dedicated to Professor Á. Császár on the occasion of his seventy fifth anniversary

Abstract: In this paper we shall describe a method for generating compactifications of topological spaces. We also show that standard compactifications, such as the Čech–Stone compactification and the T_0-stable compactification, can be obtained from the compactification discussed here by rendering it T_2 and T_0, respectively.

1 – Introduction

The present paper is a contribution to the study of compactifications of topological and bitopological spaces.

Compact Hausdorff spaces have been studied extensively, as well as the associated Čech–Stone compactification. The importance and usefulness of compactness properties is Topology and Functional Analysis is universally recognised. There are other types of compact spaces and compactifications that have also been studied. For instance, H. Wallman considered T_1-compactifications of spaces [12]. The Wallman compactification, as opposed to the Čech–Stone compactification, fails to be functorial [3]. More recently, there has been a growing interest in T_0-compactifications — see, for example, the work of M. Smyth [11], J. Lawson [7], H. Simmons [10], and also [9].

In all these papers, references may be found to further work. Albeit in a different direction, H. Herrlich ([4]) has argued that T_0 compact spaces deserve

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to be studied comprehensively, and has initiated a proposed hierarchy of differing degrees of compactness for these spaces.

In this paper we shall describe a method for generating compactifications of spaces. The resulting construction is functorial. We shall also show that standard compactifications are obtained from the one described in the paper by rendering it suitably separated. For instance, if it is made T_2, the resulting compactification is the Čech–Stone compactification; if is made T_0, the resulting compactification is the T_0 stable compactification; if it is made T_1, the degree of separation of the compactification is not clear: it may happen that all such compactifications are necessarily T_2, or, as we conjecture, there may be some that are strictly T_1 and not T_2, in which case, the construction would provide an interesting method for generating T_1 compactifications which is different from the Wallman one.

Throughout this paper, we shall use the notation $f\colon (X,T)\to(X',T')$ to indicate a function from X to X' which is continuous with respect to the topologies T and T' in the usual sense. We shall also use the notation $f^{\leftarrow}$ to denote the inverse image map associated with f.

2 – The ultrafilter space

Let (X,T) be a topological space. Denote by $\mathcal{U}(X)$ the set of *all ultrafilters* on X. For $A\subseteq X$, denote by A^* the set of all p in $\mathcal{U}(X)$ such that $A\in p$. It is well known that $\phi^*=\phi$, $X^*=\mathcal{U}(X)$, $(A\cup B)^*=A^*\cup B^*$ and $(A\cap B)^*=A^*\cap B^*$. Thus, the sets G^*, where $G\in T$, form a base for a topology on $\mathcal{U}(X)$ which will be denoted by $\mathcal{U}(T)$.

When $f\colon (X,T)\to(X',T')$ is a continuous function and $p\in\mathcal{U}(X)$, $f^{\#}(p)=\{A\subseteq X' \mid f^{\leftarrow}[A]\in p\}$ is indeed an ultrafilter on X' which we denote by $\mathcal{U}(f)(p)$. It is readily verified that $\mathcal{U}(f\circ g)=\mathcal{U}(f)\circ\mathcal{U}(g)$, and that $\mathcal{U}(f)\colon(\mathcal{U}(X),\mathcal{U}(T))\to(\mathcal{U}(X'),\mathcal{U}(T'))$ is continuous. There is also the natural embedding map $\eta_X : (X,T)\to(\mathcal{U}(X),\mathcal{U}(T))$, continuous, given by $\eta_X(x)=\{A\subseteq X \mid x\in A\}$, the principal ultrafilter containing $\{x\}$.

Before we can state the crucial topological properties of $(\mathcal{U}(X),\mathcal{U}(T))$, we recall ([2]) that a compact space (X,T) is *supersober* if for every ultrafilter p on X, the adherence of p is the closure of a point; the point necessarily being unique if the space is a T_0 space. Observe that a compact Hausdorff space is supersober, that a compact T_1 supersober space is necessarily a Hausdorff space, and that a compact T_0 space need not be supersober.

Theorem 1. *$(\mathcal{U}(X),\mathcal{U}(T))$ is compact, locally compact and supersober.*

Proof: To show that $(\mathcal{U}(X),\mathcal{U}(T))$ is compact and locally compact, it suffices to show that every basic open set of the form G^*, where $G \in T$, is compact. Consider a family $\mathcal{F} = \{F_i^* \mid i \in I\}$ of basic closed subsets of $\mathcal{U}(X)$ such that $\{F_i^* \cap G^* \mid i \in I\}$ has the finite intersection property. By Zorn's lemma, there is $p \in \mathcal{U}(X)$ such that $(F_i \cap G) \in p$ for all i in I. Then $p \in \bigcap\{F_i^* \mid i \in I\} \cap G^*$, showing that G^* is compact.

To prove that $(\mathcal{U}(X),\mathcal{U}(T))$ is supersober, consider an ultrafilter $\mathcal{U}$ on $\mathcal{U}(X)$. Then $p = \{A \subseteq X \mid A^* \in \mathcal{U}\}$ is an ultrafilter on X. We show that the adherence of $\mathcal{U}$ in $(\mathcal{U}(X),\mathcal{U}(T))$ is the closure of $\{p\}$. Firstly, p is in the adherence of $\mathcal{U}$, otherwise there is an open set G such that $p \in G^*$, $G^* \notin \mathcal{U}$, so that $(X-G)^* \in \mathcal{U}$, hence $(X-G) \in p$, which contradicts $p \in G^*$. Secondly, if q is an adherence point of $\mathcal{U}$ we observe that q must be in the $\mathcal{U}(T)$-closure of p, otherwise there is an open set H such that $q \in H^*$ and $p \notin H^*$, so that $X - H \in p$, hence $(X-H)^* \in \mathcal{U}$, which is impossible since $H^* \in \mathcal{U}$ as q is in the adherence of $\mathcal{U}$. The proof is complete. ∎

The embedding of X in $\mathcal{U}(X)$ is expressed concisely in terms of the patch topology ([7], [2]). We recall the relevant definitions. For a topological space (X,T), the *co-compact topology* T_K has sets of the form $X-K$ as subbasic open sets, where K is a saturated compact set, i.e. K is compact and is the intersection of all open sets containing K. The *patch topology* is $T \vee T_K$.

Proposition 2. *$\eta_X\colon (X,T) \to (\mathcal{U}(X),\mathcal{U}(T))$ maps (X,T) homeomorphically onto a patch-dense subspace of $(\mathcal{U}(X),\mathcal{U}(T))$.*

Proof: Let $H \in T$, K a saturated compact subset of $(\mathcal{U}(X),\mathcal{U}(T))$, and $p \in H^*-K$. Since $p \notin K$, there is an open set which contains K but not p. By compactness of K, there are open sets G_i, $1 \le i \le n$, such that $K \subseteq \bigcup_{i=1}^n G_i^* \subseteq \mathcal{U}(X) - \{p\}$. Let $G = \bigcup_{i=1}^n G_i$. Then $K \subseteq G^*$ and $p \notin G^*$. Hence $X - G \in p$, so that $H \cap (X-G) \neq \phi$ since $H \in p$. Choose x in $H - G$. Then $\eta_X(x) \in H^* - G^* \subseteq H^* - K$, as required. ∎

In what follows, a compact, locally compact, supersober space will be called *stably compact* ([5], [7]) for ease of reference. We examine the behaviour of stably compact spaces under the ultrafilter space construction.

Proposition 3. *Let (X,T) be stably compact. There is a retraction, not necessarily unique, $r_X\colon (\mathcal{U}(X),\mathcal{U}(T)) \to (X,T)$, such that $r_X \circ \eta_X = 1_X$.*

Proof: Given $p \in \mathcal{U}(X)$, there is x such that the adherence of p is $cl_T x$. Note that if $p = \eta_X(x_0)$, we have that the adherence of p is $cl_T x_0$. For $p \in \eta_X(X)$, define $r_X(p) = x$, where $p = \eta_X(x)$. For $p \in \mathcal{U}(X) - \eta_X(X)$, let $r_X(p) = x$, where x is *any* element of X whose closure is precisely the adherence of p. It is clear that $r_X \circ \eta_X = 1_X$. To prove continuity of r_X at p , with $r_X(p) = x$, consider an open set H containing x. Let G be an open neighborhood of x and K a compact set which contains G and is contained in H. If $p \notin G^*$, then $p \in (X - G)^*$, so that $X - G \in p$. But then x is not an adherence point of p since $x \in G$ and $G \cap (X - G) = \phi$, contradicting our definition of $x = r_X(p)$. Thus $p \in G^*$. We show that, when $q \in G^*$, then $r_X(q) \in H$. Because $q \in K^*$, we have $K \in q$. Since K is compact we have $A = K \cap (\text{adherence}(q)) \neq \phi$. Let $x_1 \in A$. Then $x_1 \in H \cap cl_T(r_X(q))$, so that $r_X(q) \in H$, as required. ∎

The map r_X is also continuous with respect to the co-compact topologies induced by $\mathcal{U}(T)$ and T. In the proof we shall make use of the following remark suggested by the anonymous reference to whom we express our appreciation.

Remark. If $r: (\mathcal{U}(X), \mathcal{U}(T)) \to (X, T)$ is any retraction satisfying $r \circ \eta_X = 1_X$, then $r[F^*] = F$ for every T-closed set F. In fact, $x \in F$ implies $\eta_X(x) \in F^*$, so $x = r(\eta_X(x)) \in r[F^*]$. Conversely, assume $p \in F^*$. If $r(p) \notin F$, then p has an open neighbourhood G^* with $G \in T$, $G \in p$ such that $r[G^*] \subseteq X - F$. However, $F \in p$ and $G \in p$ imply the existence of $x \in G \cap F$. Then $\eta_X(x) \in G^*$, so $x = r(\eta_X(x)) \in [G^*] \subseteq X - F$, a contradiction. □

Proposition 4. *Let (X, T) be stably compact and $r_X : (\mathcal{U}(X), \mathcal{U}(T)) \to (X, T)$ a retraction map with $r_X \circ \eta_X = 1_X$. If K is compact, then so is ${r_X}^{\leftarrow}[K]$.*

Proof: Consider a filter base of basic closed sets $\{F_i^* \mid F_i \text{ closed}, F_i \subseteq X\}$ such that $F_i^* \cap {r_X}^{\leftarrow}[K] \neq \phi$. Then $r_X(F_i^*) \cap K \neq \phi$ for all i, i.e. $F_i \cap K \neq \phi$ for all i in I. By compactness of K, there exists $x \in (\bigcap_{i \in I} F_i) \cap K$, so that $\eta_X(x) \in \bigcap_{i \in I} F_i^* \cap r_X^{\leftarrow}[K]$. The proof is complete. ∎

The converse of Proposition 3 is also true. We prove a more general result.

Proposition 5. *Let $e: (X, T) \to (X', T')$ and $r: (X', T') \to (X, T)$ be such that $r \circ e = 1_X$. If (X', T') is stably compact, then so is (X, T).*

Proof: It is clear that (X, T) is compact. To prove local compactness, consider a neighborhood V of x. Then $r^{\leftarrow}[V]$ is a neighborhood of $e(x)$. Hence there is an open set W and a compact set K such that $e(x) \in W \subseteq K \subseteq r^{\leftarrow}[V]$.

Now observe that $x \in e^{\leftarrow}[W] \subseteq r[K] \subseteq V$. Finally, to prove that (X,T) is supersober, let p be an ultrafilter on X. Then $e^{\#}(p) = \{A \subseteq X' : | : e^{\leftarrow}[A] \in p\}$ is an ultrafilter $\mathcal{F}$, say, on X'. By assumption, there is p_0 in X' such that $cl_{T'} p_0$ is the adherence of $\mathcal{F}$. We show that the adherence of p is $cl_T\, r(p_0)$: if x is in the adherence of p, then $e(x)$ is in the adherence of $e^{\#}(p)$, hence in $cl_{T'} p_0$, so that $x = r(e(x)) \in r(cl_{T'}\, p_0) \subseteq cl_T\, r(p_0)$; conversely $r(p_0)$ is in the adherence of p, since, given V open, $r(p_0) \in V$, we have $p_0 \in r^{\leftarrow}[V]$, so that $r^{\leftarrow}[V] \in e^{\#}(p)$ since p_0 is in the adherence of $e^{\#}(p)$, hence $V = e^{\leftarrow}[r^{\leftarrow}[V]] \in p$. The proof is complete. ∎

Because compact Hausdorff spaces are stably compact, we have the following useful retraction property which follows from Proposition 3 and the fact that continuous mappings into Hausdorff spaces that coincide on dense subspaces are necessarily equal.

Proposition 6. *If (X,T) is a compact Hausdorff space, then there is a unique retraction $r_X : (\mathcal{U}(X),\mathcal{U}(T)) \to (X,T)$ such that $r_X \circ \eta_X = 1_X$.* ∎

The retraction property together with functoriality of $\mathcal{U}$ allow us to establish concretely and effortlessly the weak reflectivity of the category of stably compact spaces in <u>Top</u>.

Proposition 7. *Let $f\colon (X,T) \to (X',T')$, where (X',T') is stably compact. Then there exists $F\colon (\mathcal{U}(X),\mathcal{U}(T)) \to (X',T')$ such that $F \circ \eta_X = f$.*

Proof: We have $\mathcal{U}(f)\colon (\mathcal{U}(X),\mathcal{U}(T)) \to (\mathcal{U}(X'),\mathcal{U}(T'))$.
Let $r_{X'}\colon (\mathcal{U}(X'),\mathcal{U}(T')) \to (X',T')$ be such that $r_{X'} \circ \eta_{X'} = 1_{X'}$. Then $F = r_{X'} \circ \mathcal{U}(f)$ is such that $F \circ \eta_X = r_X \circ \mathcal{U}(f) \circ \eta_X = r_{X'} \circ \eta_{X'} \circ f = 1_{X'} \circ f = f$, as required. ∎

We conclude by mentioning specific examples of ultrafilter spaces.

Examples 8.

1. If X is finite, then $(X,T) = (\mathcal{U}(X),\mathcal{U}(T))$. The converse also holds, assuming the axiom of choice.
2. If (X,T) is a discrete space, then $(\mathcal{U}(X),\mathcal{U}(T))$ is the Čech–Stone compactification $\beta(X,T)$. Conversely, if $(\mathcal{U}(X),\mathcal{U}(T))$ is Hausdorff, even T_1, then (X,T) is discrete: If X is a finite space, then $X = \mathcal{U}(X)$ and there is nothing to prove. If X were infinite and not discrete, then there would exist a non-empty subset of X, say A, and x_0 in X which is in $\overline{A}$ but

not in A. Observe that $(V - \{x_0\}) \cap A \neq \phi$ for all V which are neighbourhoods of x_0. Thus, there exists an ultrafilter on X, q, which contains all such sets as well as $X - \{x_0\}$. Let p denote the principal ultrafilter consisting of all subsets of X which contain x_0. Clearly, $p \neq q$. We now show that $p \in cl_{\mathcal{U}(T)} q$, so that $(\mathcal{U}(X), \mathcal{U}(T))$ would fail to be a T_1-space. Let $p \in G^*$, where $G \in T$, then G is an open set containing x_0. Because (X,T) is topologically embedded in $(\mathcal{U}(X), \mathcal{U}(T))$ which is compact Hausdorff, hence completely regular, it follows that (X,T) is regular. Thus, there is $H \in T$ such that $x_0 \in H \subseteq \overline{H} \subseteq G$. We show that $\overline{H} \in q$. If not, then $X - \overline{H} \in q$, hence $(X - \overline{H}) \cap (H - \{x_0\}) \cap A \neq \phi$ which is impossible. Thus $\overline{H} \in q$, so that $G \in q$, hence $q \in G^*$. Thus $p \in cl_{\mathcal{U}(T)} q$, as required.

3. Let (w^+, D^+) denote the one point compactification of $w = \{1, 2, ...\}$ with the discrete topology D. $(\mathcal{U}(w^+), \mathcal{U}(D^+))$ is a T_0 space, but not T_1. □

3 – Separated compactifications

3.1. T_0 compactifications

We shall first examine the simplest case: that of the T_0 reflection. Let $e_{0X}: (X,T) \to (X_0, T_0)$ denote the reflection map. It is well known that e_{0X} is an open and initial map, and there is a continuous section $s_{0X}: (X_0, T_0) \to (X,T)$, i.e. $e_{0X} \circ s_{0X} = 1_{X_0}$.

As expected, the T_0-reflection of a stably compact space is a T_0 stably compact space.

Proposition 9. *If (X,T) is a stably compact, then so is (X_0, T_0).*

Proof: The mappings e_{0X}, s_{0X} show that (X_0, T_0) is a retract of (X,T). The result now follows from Proposition 5. ■

Proposition 10. *Let $f: (X,T) \to (X', T')$ be a continuous map from the T_0 space (X,T) to the T_0 stably compact space (X', T'). There is a continuous map F from the stably compact space $((\mathcal{U}(X))_0, (\mathcal{U}(T))_0)$ to (X', T') such that $F \circ e_{0\,\mathcal{U}(X)} \circ \eta_X = f$.*

Proof: We have $\mathcal{U}(f): (\mathcal{U}(X), \mathcal{U}(T)) \to (\mathcal{U}(X'), \mathcal{U}(T'))$, hence $r_{X'} \circ \mathcal{U}(f): (\mathcal{U}(X), \mathcal{U}(T)) \to (X', T')$. $F = (r_{X'} \circ \mathcal{U}(f))_0 = (r_{X'})_0 \circ (\mathcal{U}(f))_0$ is the required map. ■

In view of Proposition 2 and Proposition 10, we conclude from [7] that $((\mathcal{U}(X))_0, (\mathcal{U}(T))_0)$ is indeed the stable compactification $\beta_0(X,T)$ of (X,T) ([9], [10], [11]).

3.2. T_2 compactifications

Let $e_{2X}\colon (X,T) \to (X_2, T_2)$ denote the T_2 reflection map. The compactification $((\mathcal{U}(X))_2, (\mathcal{U}(T))_2)$ of (X,T) is indeed the Čech–Stone compactification of (X,T) as shown by the following proposition, and the fact that $e_{2X}(\eta_X(X))$ is dense in $((\mathcal{U}(X))_2, (\mathcal{U}(T))_2)$.

Proposition 11. *Let f be a continuous map from (X,T) to the compact T_2 space (X',T'). Then there is a continuous map F from the compact T_2 space $((\mathcal{U}(X))_2, (\mathcal{U}(T))_2)$ to (X',T') such that $F \circ e_{2\mathcal{U}(X)} \circ \eta_X = f$.* ∎

The proof is analogous to that of Proposition 10 and will be omitted.

3.3. T_1 compactifications

Let $e_{1X}\colon (X,T) \to (X_1, T_1)$ denote the T_1 reflection map. The T_1 compactification assigning $((\mathcal{U}(X))_1, (\mathcal{U}(T))_1)$ to (X,T) is clearly functorial and pointed with $e_{1\mathcal{U}(X)} \circ \eta_X\colon (X,T) \to ((\mathcal{U}(X))_1, (\mathcal{U}(T))_1)$. It is thus not the same as the Wallman compactification of (X,T). We denote $((\mathcal{U}(X))_1, (\mathcal{U}(T))_1)$ by $\beta_1(X,T)$, and denote $e_{1\mathcal{U}(X)} \circ \eta_X$ by η_{1X}.

Examples.

1. Consider $w = \{1, 2, ...\}$ with topology T with finite subsets as basic closed sets. Then $((\mathcal{U}(w))_1, (\mathcal{U}(T))_1)$ is a singleton set with its unique topology, hence, so is $\beta_1(w,T)$.

2. Let $X = [0,1]$ and let T denote the usual topology on $[0,1]$. Then $\beta_1(X,T)$ is (X,T).

3. Let (w^+, D^+) denote the one-point compactification of (w,D). Then $\beta_1(w^+, D^+)$ is (w^+, D^+). □

Problem. It is not clear whether or not $\beta_1(X,T)$ is always a T_2 space, and it would be interesting to ascertain this, especially in the light of Propositions 12 and 13. □

Independently of the answer to the problem, $\beta_1 X$ has the following extension property in common with the Wallman compactification WX.

Proposition 12. *Let $f\colon (X,T) \to (X',T')$ be a continuous map from the T_1 space (X,T) to the compact T_2 space (X',T'). There is a continuous map $F\colon \beta_1(X,T) \to (X',T')$ such that $F \circ \eta_{1X} = f$.*

Proof: We have, as in the proof of Proposition 10, $\mathcal{U}(f)\colon (\mathcal{U}(X),\mathcal{U}(T)) \to (\mathcal{U}(X'),\mathcal{U}(T'))$. Then $r_{X'} : (\mathcal{U}(X'),\mathcal{U}(T')) \to (X',T')$ gives $r_{X'} \circ \mathcal{U}(f) : (\mathcal{U}(X),\mathcal{U}(T)) \to (X',T')$. Hence $(r_{X'} \circ \mathcal{U}(f))_1 \colon \beta_1(X,T) \to (X',T')$, as required. ∎

It is quite natural to enquire as to the characterization of the spaces (X',T') for which, whenever $f\colon (X,T) \to (X',T')$ is given, there is $F\colon \beta_1(X,T) \to (X',T')$ such that $F \circ \eta_{1X} = f$. We shall refer to such spaces as β_1-*injective*. The previous proposition shows that compact T_2 spaces are β_1-injective. The following asserts that the converse is true.

Proposition 13. *β_1-injective spaces are compact T_2.*

Proof: If (X,T) is a β_1-injective space, then $1_X : (X,T) \to (X,T)$ determines $F\colon \beta_1(X,T) \to (X,T)$ such that $F \circ \eta_{1X} = 1_X$. Thus (X,T) is a retract of $\beta_1(X,T)$, hence (X,T) is a retract of $(\mathcal{U}(X),\mathcal{U}(T))$. Hence (X,T) is a T_1 stably compact space. It follows that ultrafilters on X have unique cluster points, so that (X,T) is T_2. ∎

It is clear that the above proposition would be more interesting if, indeed, as we suspect, $\beta_1(X,T)$ is not a T_2 space for some (X,T).

3.4. Other separated compactifications

The method described in the previous section will yield separated compactifications for other reflective subcategories.

One such example is provided by the T_D spaces. They constitute a reflective subcategory of <u>*Top*</u>. Let the reflector be R_D, and write β_D for $R_D \circ \mathcal{U}$.

An interesting problem associated with β_D is the characterization of the β_D-injective spaces, especially in view of the fact that the β_0-injectives are the T_0-stably compact spaces, and the β_1-injectives are the compact Hausdorff spaces.

4 – Compactifications of bitopological spaces

Given (X, P, Q), there is the associated $(\mathcal{U}(X), \mathcal{U}(P), \mathcal{U}(Q))$, which we shall designate by the *ultrafilter bispace*.

It is clear that $\mathcal{U}(P) \vee \mathcal{U}(Q) = \mathcal{U}(P \vee Q)$. Thus $\mathcal{U}(P) \vee \mathcal{U}(Q)$ is a stably compact topology on $\mathcal{U}(X)$.

We shall now consider the separated compactifications of bispaces, which will illustrate similarities and differences when compared with the topological context described in section 3.

4.1. T_0-stable compactifications of bispaces

The minimal requirement with regards to the separation of points of (X, P, Q) is that points are separated by sets that are either P-open or Q-open, i.e. $P \vee Q$ is T_0. Denote by $\underline{BTop}$ the category of bitopological spaces and maps $f : (X, P, Q) \to (X', P', Q')$. The full subcategory consisting of bispaces (X, P, Q) for which $P \vee Q$ is T_0 is a reflective subcategory. Denote the reflector by S_0, and let $\beta_0 = S_0 \circ \mathcal{U}$, with $\beta_0(X, P, Q) = (\overline{X}, \overline{P}, \overline{Q})$. Let $e_X : (X, P, Q) \to \beta_0(X, P, Q)$ denote the natural embedding map. The following universal property holds:

Proposition 14. *Let $f : (X, P, Q) \to (X', P', Q')$, where $(X', P' \vee Q')$ is a T_0 stably compact space. Then there exists $F : \beta_0(X, P, Q) \to (X', P', Q')$ such that $F \circ e_X = f$.* ∎

We thus have an analogue for bispaces of the T_0 stable compactification of topological spaces. It should be noted that there are other natural analogues, for instance, the full subcategory of $\underline{BTop}$ whose objects (X, P, Q) are such that both P and Q are T_0 topologies is reflective. Let the associated reflector be DS_0. Then $D\beta_0 = DS_0 \circ \mathcal{U}$ is a special compactification of (X, P, Q) which is different from β_0. As far as we are aware, a study of this compactification has not appeared in the literature.

4.2. Hausdorff compactifications of bispaces

In the bitopological context there are several natural notions of the "Hausdorff" separation property. We shall mention only three to stress the difference between the topological and the bitopological situations.

Firstly, let us recall that with every bitopological space (X,P,Q) there is a natural partial order denoted by $\leq_{PQ}$, or more simply $\leq$, defined by $x \leq y \iff (x \in cl_P y \text{ and } y \in cl_Q x)$.

The following definition provides a natural analogue of the Hausdorff property.

Definition 15. (X,P,Q) is monotonically separated if, when $x \not\leq y$, there is a P-open set V and a disjoint Q-open set W such that $y \in V$, $x \in W$. □

There are also other "natural" analogues of the Hausdorff property: (X,P,Q) is $2T_2$ if $P \vee Q$ is T_2 ; (X,P,Q) is DT_2 if both P and Q are T_2.

All these properties determine reflective subcategories of $\underline{BTop}$, hence different compactifications in $\underline{BTop}$. However, these compactifications are not necessarily pairwise completely regular, in contrast with the topological situation.

4.3. Hausdorff pairwise completely regular compactifications of bispaces

We shall say that a bispace (X,P,Q) is a pairwise Tychonoff 2-compact space when (X,P,Q) is a pairwise completely regular space [6], such that $P \vee Q$ is compact T_0 (equivalently, $P \vee Q$ is compact and T_2). There is a bitopological analogue of the Čech–Stone compactification introduced and studied in ([8], see also [9]) which has appeared in the literature in many different contexts. It is characterized as follows:

Proposition 16 ([8]). *For every bispace (X,P,Q) there is a pairwise Tychonoff 2-compact space $(\overline{X},\overline{P},\overline{Q})$, denoted by $\beta_2(X,P,Q)$ and a map $e_X : (X,P,Q) \to (\overline{X},\overline{P},\overline{Q})$ such that if $f\colon (X,P,Q) \to (X',P',Q')$ and (X',P',Q') is pairwise Tychonoff and 2-compact, then there is a unique map $F\colon (\overline{X},\overline{P},\overline{Q}) \to (X',P',Q')$ such that $F \circ e_X = f$.* ∎

We now show that $\beta_2(X,P,Q)$ can be obtained from the ultrafilter space $(\mathcal{U}(X),\mathcal{U}(P),\mathcal{U}(Q))$. The pairwise Tychonoff spaces form a reflective subcategory of $\underline{BTop}$. Let PT denote the corresponding reflector. We then have $\beta_2 = PT \circ \mathcal{U}$. This result follows from the fact that pairwise Tychonoff 2-compact spaces are retracts of their ultrafilter bispaces, as we shall prove in Proposition 17. Note that it suffices to take the pairwise regular $2T_0$-reflector, rather than the pairwise Tychonoff reflector in order to obtain β_2 from $\mathcal{U}$.

Let us recall [9] that if (X,P) is a compact, locally compact, T_0 supersober space, then there is a unique topology Q such that (X,P,Q) is a pairwise regular

2-compact, $2T_0$ bispace: Q is precisely the co-compact topology P_K. Conversely [9], when (X,P,Q) is a pairwise regular 2-compact space then (X,P) is a compact, locally compact, supersober T_0-space. As a consequence, we have the following.

Proposition 17. *Let (X,P,Q) be a pairwise regular, 2-compact, $2T_0$ space. There is a retraction map $r_X\colon (\mathcal{U}(X),\mathcal{U}(P),\mathcal{U}(Q))\to(X,P,Q)$ such that $r_X\circ\eta_X=1_X$.*

Proof: (X,P) is a stably compact T_0-space. Hence there is $r_X\colon (\mathcal{U}(X),\mathcal{U}(P))\to(X,P)$, by Proposition 3. By Proposition 4, if K is P-compact, then $r_X{}^{\leftarrow}[K]$ is $\mathcal{U}(P)$-compact. Hence $r_X\colon (\mathcal{U}(X),\mathcal{U}(P)_K)\to(X,P_K)$ is continuous. Now $P_K=Q$, by the uniqueness property quoted above, hence $r_X\colon (\mathcal{U}(X),\mathcal{U}(P),\mathcal{U}(Q))\to(X,P,Q)$ as required. ∎

In conclusion, we have established that $\beta_2(X,P,Q)$ is the pairwise-regular $2T_0$ reflection of $(\mathcal{U}(X),\mathcal{U}(P),\mathcal{U}(Q))$.

Added in Proof: It was noted above that compact Hausdorff spaces stay fixed under the universal T_1 compactification. The converse is also true.

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