PORTUGALIAE MATHEMATICA
Vol. 58 Fasc. 1 – 2001
Nova Série

ON THE LACK OF NULL-CONTROLLABILITY OF THE HEAT EQUATION ON THE HALF SPACE

S. Micu * and E. Zuazua •

Abstract: We study the null-controllability property of the linear heat equation on the half-space with a L^2 Dirichlet boundary control. We rewrite the system on the similarity variables that are a common tool when analyzing asymptotic problems. By separation of variables the multi-dimensional control problem is reduced to an infinite family of one-dimensional controlled systems. Next, the results for this type of systems proved in [18] are used in order to show that, roughly speaking, controllable data have Fourier coefficients that grow exponentially for large frequencies. This result is in contrast with the existing ones for bounded domains that guarantee that every initial datum belonging to a Sobolev space of negative order may be driven to zero in an arbitrarily small time.

1 – Introduction. Problem formulation

Let Ω be a smooth domain of $\mathbb{R}^n$ with $n \geq 1$. Given $T > 0$ and an open non-empty subset of the boundary of Ω we consider the linear heat equation:

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \; Q, \\ u = v \, 1_{\Sigma_0} & \text{on } \; \Sigma, \\ u(x,0) = u_0(x) & \text{in } \; \Omega \,, \end{cases} \tag{1.1}$$

where $Q = \Omega\times(0,T)$, $\Sigma = \partial\Omega\times(0,T)$ and $\Sigma_0 = \Gamma_0\times(0,T)$, Γ_0 being an open,

Received: February 2, 2000; *Revised*: May 11, 2000.
AMS Subject Classification: 35B37, 35K05.
Keywords: Heat equation; Similarity variables; Control; Moments.
* Supported by grants PB96-0663 of the DGES (Spain) and 4/2001 of CNCSIS (Romania).
• Supported by grant PB96-0663 of the DGES (Spain).

non-empty subset of $\partial\Omega$ and where 1_{Σ_0} denotes the characteristic function of the subset Σ_0 of Σ.

In (1.1), $v \in L^2(\Sigma)$ is a *boundary control* that acts on the system through the subset Σ_0 of the boundary and $u = u(x,t)$ is the *state*.

System (1.1) is said to be *null-controllable* at time T if for any $u_0 \in L^2(\Omega)$ there exists a control $v \in L^2(\Sigma_0)$ such that the solution of (1.1) satisfies

$$u(x,T) = 0 \quad \text{in } \ \Omega \ . \tag{1.2}$$

When Ω is a *bounded* domain of class C^2 it is well-known that system (1.1) is null-controllable for any $T > 0$. We refer to D.L. Russell [20] for some particular examples treated by means of moment problems and Fourier series and to A. Fursikov and O.Yu. Imanuvilov [9] and G. Lebeau and L. Robbiano [17] for the general result covering any bounded smooth domain Ω and open, non-empty subset Γ_0 of $\partial\Omega$. Both the approaches of [9] and [17] are based on the use of Carleman inequalities.

None of the approaches mentioned above apply when Ω is an unbounded domain.

This paper is devoted to analyze the particular case in which Ω is a half-space:

$$\Omega \, = \, \mathbb{R}^n_+ \, = \, \Big\{ x = (x', x_n) : \, x' \in \mathbb{R}^{n-1}, \; x_n > 0 \Big\} \tag{1.3}$$

and

$$\Gamma_0 \, = \, \partial\Omega \, = \, \mathbb{R}^{n-1} \, = \, \Big\{ (x', 0) : \, x' \in \mathbb{R}^{n-1} \Big\} \ . \tag{1.4}$$

As we shall see, the situation is completely different to the one we have described above and, roughly speaking, one may say that there is no initial data in any negative Sobolev space that may be driven to zero in finite time. Therefore, in some sense, the situation is the opposite one to the one we encounter in the case of bounded domains.

There is a weaker notion of controllability. It is the so called approximate controllability property. System (1.1) is said to be *approximately controllable* in time T if for any $u_0 \in L^2(\Omega)$ the set of reachable states, $R(T; u_0) = \{u(T) :$ u solution of (1.1) with $v \in L^2(\Sigma_0)\}$, is dense in $L^2(\Omega)$.

With the aid of classical backward uniqueness results for the heat equation (see, for instance, J.L. Lions and E. Malgrange [16] and J.M. Ghidaglia [10]), it can be seen that null-controllability implies approximate controllability. Moreover one can easily prove the approximate controllability directly both in the case of bounded and unbounded domains.

Thus, taking into account that approximate controllability holds, it is natural to analyze whether null-controllability holds as well.

As we mentioned above, we start analyzing the case where Ω is a half-space and $\Gamma_0 = \partial\Omega$. Thus, we consider the system:

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \ Q = \mathbb{R}^n_+ \times (0,T), \\ u = v & \text{on } \ \Sigma = \mathbb{R}^{n-1} \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \ \mathbb{R}^n_+ \ . \end{cases} \tag{1.5}$$

It is easy to see that we may not expect the null controllability result of the case where Ω is bounded to be true in this case. Indeed, the null-controllability of (1.5) with initial data in $L^2\left(\mathbb{R}^n_+\right)$ and boundary control in $L^2(\Sigma)$ is equivalent to an *observability inequality* for the adjoint system

$$\begin{cases} \varphi_t + \Delta\varphi = 0 & \text{on } \ Q, \\ \varphi = 0 & \text{on } \ \Sigma \ . \end{cases} \tag{1.6}$$

More precisely, it is equivalent to the existence of a positive constant $C > 0$ such that

$$\|\varphi(0)\|^2_{L^2(\mathbb{R}^n_+)} \ \leq \ C \int_\Sigma \left| \frac{\partial \varphi}{\partial x_n} \right|^2 dx'\, dt \tag{1.7}$$

holds for every smooth solution of (1.6).

When Ω is bounded, Carleman inequalities provide the estimate (1.7) and, consequently, null-controllability holds (see for instance [9], [7]).

In the case of a half-space it is easy to see that (1.7) may not hold. Indeed, let $\varphi^0 \in \mathcal{D}\left(\mathbb{R}^n_+\right)$ and $\varphi^0_k(x) = \varphi^0(x', x_n - k)$ with $k > 0$ large enough. Let φ_k be the solution of (1.6) with initial datum φ^0_k at time $t = T$, i.e. $\varphi_k(T) = \varphi^0_k$ in $\mathbb{R}^n_+$. It is easy to see that

$$\|\varphi_k(0)\|^2_{L^2(\mathbb{R}^n_+)} \Big/ \int_\Sigma \left| \frac{\partial \varphi_k}{\partial x_n} \right|^2 d\Sigma \ \longrightarrow \ \infty, \qquad \text{as } \ k \to \infty \ . \tag{1.8}$$

Indeed, from the representation formula

$$\varphi_k(x,t) \ = \ \frac{1}{\left(2\sqrt{\pi(T-t)}\right)^n} \int_{\mathbb{R}^n} e^{-\frac{||x-\xi||^2}{4(T-t)}} \, \tilde{\varphi}^0_k(\xi) \ d\xi$$

where $\tilde{\varphi}^0_k$ is an odd extension of φ^0_k from $\mathbb{R}^n_+$ to $\mathbb{R}^n$, we obtain that $(\varphi_k)_x(x', 0, t)$ decays exponentially when $k \to \infty$. Consequently $\int_\Sigma |(\varphi_k)_x|^2 \, d\Sigma$ tends to zero when k goes to infinity whereas $\|\varphi_k(0)\|^2_{L^2(\mathbb{R}^n_+)} = \|\varphi(0)\|^2_{L^2(\mathbb{R}^n_+)}$ remains constant. Hence (1.8) follows.

We refer to L. Rosier [19] for a similar negative result in the context of exact controllability of linear, constant coefficient pde in one space dimension.

However, this translation invariance argument does not allow to exclude weighted versions of the inequality (1.7), and therefore, the null-controllability of suitable initial data u_0 may not be excluded.

This paper is devoted to analyze the class of initial data u_0 such that the solution of (1.5) may be driven to zero in time T by means of a L^2-control v.

In the case of bounded domains, using Fourier series expansion, *the control problem may be reduced to a moment problem*. However, since we are working on $\mathbb{R}^n_+$, we can not use Fourier series. Nevertheless, it was observed by M. Escobedo and O. Kavian in [4] that, on suitable similarity variables and at the appropriate scale, solutions of the heat equation on conical domains may be indeed developed in Fourier series on a weighted L^2-space.

We adapt this idea to our problem. Using similarity variables and weighted Sobolev spaces and developing solutions in Fourier series we reduce the control problem to a sequence of one-dimensional controlled systems. The null-controllability properties of this type of systems were studied in [18] where it was proved that no initial datum u_0 belonging to any Sobolev space of negative order may be driven to zero in finite time. Moreover, this negative result was complemented by showing that there exist initial data with exponentially growing Fourier coefficients for which null-controllability holds in finite time with L^2-controls.

We use the results of the one-dimensional case to show that, roughly speaking, the controllable initial data of the multi-dimensional problem have exponentially growing Fourier coefficients.

The paper is organized in the following way. In Section 2 we introduce the similarity variables. In Section 3, devoted to the $1-d$ control problem, we recall some of the results proved in [18]. In Section 4, we discuss the multi-dimensional case and we give the main results. Finally, in Section 5, we briefly comment the case of general conical domains and discuss some other extensions of the results of this paper and open problems.

2 – Similarity variables and weighted Sobolev spaces

In this section we recall some basic facts about the similarity variables and weighted Sobolev spaces for the heat equation. We refer to [4] and [14] for further developments and details.

2.1. The similarity variables

We consider the solutions $u = u(x,t)$ of

$$\begin{cases} u_t - \Delta u = 0 & \text{in } Q, \\ u = v & \text{on } \Sigma, \\ u(x,0) = u_0(x) & \text{in } \Omega , \end{cases} \tag{2.1}$$

where

$$\begin{cases} \Omega = \mathbb{R}^n_+, \\ Q = \mathbb{R}^n_+ \times (0,T) , \qquad \Sigma = \mathbb{R}^{n-1} \times (0,T) = \partial\Omega \times (0,T) \ . \end{cases} \tag{2.2}$$

We now introduce the new space-time variables

$$y \ = \ x \Big/ \sqrt{t+1} \ ; \qquad s \ = \ \log(t+1) \ . \tag{2.3}$$

Then, given $u = u(x,t)$ solution of (2.1) we introduce

$$w(y,s) \ = \ e^{sn/2}\, u(e^{s/2}y,\, e^s - 1) \ . \tag{2.4}$$

It follows that u solves (2.1) if and only if w satisfies

$$\begin{cases} w_s - \Delta w - \dfrac{y \cdot \nabla w}{2} - \dfrac{n}{2}\, w \ = \ 0 & \text{in } \widetilde{Q}, \\ w = \widetilde{v} & \text{on } \widetilde{\Sigma}, \\ w(y,0) = u_0(y) & \text{in } \Omega , \end{cases} \tag{2.5}$$

where

$$\widetilde{v}(y',s) \ = \ e^{sn/2}\, v(e^{s/2}y',\, e^s - 1) \ . \tag{2.6}$$

Here and in the sequel we use the notation $y' = (y_1, ..., y_{n-1})$ so that $y = (y', y_n)$. On the other hand,

$$\begin{aligned} &\widetilde{Q} = \Omega \times (0,S) = \mathbb{R}^{n-1} \times (0,S); \\ &\widetilde{\Sigma} = \partial\Omega \times (0,S) = \mathbb{R}^{n-1} \times (0,S); \end{aligned} \qquad S = \log(T+1) \ . \tag{2.7}$$

Obviously, analyzing the null controllability of system (2.1) in time T is equivalent to studying the null controllability of system (2.5) in time $S = \log(T+1)$.

Therefore, in the sequel, we shall analyze system (2.5).

The elliptic operator involved in (2.5) may also be written as

$$Lw := -\Delta w - \frac{y \cdot \nabla w}{2} \ = \ -\frac{1}{K(y)}\, \operatorname{div}(K(y)\, \nabla w) \tag{2.8}$$

where $K = K(y)$ is the Gaussian weight

$$K(y) = \exp\Big(|y|^2/4\Big) . \tag{2.9}$$

We first analyze this operator in the whole space.

2.2. Weighted spaces and spectral analysis in the whole space

We introduce the weighted L^2-space: $L^2(K) = \{f\colon \mathbb{R}^n \to \mathbb{R}\colon \sqrt{K} f \in L^2(\mathbb{R}^n)\}$ endowed with the natural norm $\|f\|_{L^2(K)} = \left(\int_{\mathbb{R}^n} |f|^2 \, K(y) \, dy\right)^{1/2}$. Obviously, it is a Hilbert space.

We then define the unbounded operator L on $L^2(K)$ by setting

$$Lw = -\Delta w - \frac{y \cdot \nabla w}{2} ,$$

as above, and $D(L) = \{w \in L^2(K)\colon \; Lw \in L^2(K)\}$.

Integrating by parts it is easy to see that

$$\int_{\mathbb{R}^n} (Lw) \, w \, K \; dy = \int_{\mathbb{R}^n} |\nabla w|^2 \, K \; dy .$$

Therefore it is natural to introduce the weighted H^1-space:

$$H^1(K) = \Big\{ f \in L^2(K)\colon \; \partial f/\partial x_i \in L^2(K), \; i = 1, ..., n \Big\}$$

endowed with the norm

$$\|f\|_{H^1(K)} = \left[\int_{\mathbb{R}^n} \Big(|f|^2 + |\nabla f|^2 \Big) \, K \; dy \right]^{1/2} .$$

In a similar way, for any $s \in \mathbb{N}$ we may introduce the space

$$H^s(K) = \Big\{ f \in L^2(K)\colon \; D^\alpha f \in L^2(K), \; \forall \, \alpha\colon \; |\alpha| \le s \Big\} .$$

The following properties were proved in [4] and [14]:

$$\int f^2 \, |y|^2 \, K \; dy \; \le \; 16 \int |\nabla f|^2 \, K \; dy , \quad \forall \, f \in H^1(K) , \tag{2.10}$$

(2.11) the embedding $H^1(K) \hookrightarrow L^2(K)$ is compact ,

(2.12) $L\colon \; H^1(K) \to (H^1(K))'$ is an isomorphism ,

(2.13) $D(L) = H^2(K)$,

(2.14) $L^{-1}\colon \; L^2(K) \to L^2(K)$ is self-adjoint and compact ,

(2.15) the eigenvalues of L are $\lambda_j = \dfrac{n + j - 1}{2}, \; j \ge 1$, and the corresponding eigenspaces

$$\mathrm{Ker}(L - \lambda_j I) \;=\; \mathrm{span}\Big\{D^\alpha \varphi_1 \colon\; |\alpha| = j-1\Big\}\;, \tag{2.16}$$

where φ_1 is the eigenfunction associated to the first eigenvalue λ_1, which is simple, and is explicitly given by

$$\varphi_1(y) \;=\; K^{-1}(y) \;=\; \exp(-|y|^2/4)\;. \tag{2.17}$$

In one space dimension all the eigenvalues are simple. In dimensions $n \geq 2$ the multiplicity μ_j of all the eigenvalues λ_j with $j \geq 2$ is greater than one, $\mu_j = \binom{n+j-2}{n-1}$. We can then choose the eigenfunctions $\{\varphi_{j,\ell}\}_{\substack{j\geq 1\\ 1\leq \ell\leq \mu_j}}$ such that they constitute an orthonormal basis of $L^2(K)$. Note that $\varphi_{j,\ell}$ denotes an eigenfunction associated to the eigenvalue λ_j and $\ell = 1, ..., \mu_j$, μ_j being the multiplicity of λ_j.

Using this spectral decomposition the solutions of the heat equation in similarity variables in the whole space can be easily developed in Fourier series. Namely, if w solves

$$\begin{cases} w_s + Lw - \dfrac{n}{2}\, w \;=\; 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ w(y,0) = u_0(y) & \text{in } \mathbb{R}^n\;, \end{cases} \tag{2.18}$$

with

$$u_0(y) \;=\; \sum_{j=1}^{\infty} \sum_{\ell=1}^{\mu_j} a_{j,\ell}\, \varphi_{j,\ell}$$

then,

$$w(y,s) \;=\; \sum_{j=1}^{\infty} e^{-\lambda_j s} \left[\sum_{\ell=1}^{\mu_\ell} a_{j,\ell}\; \varphi_{j,\ell}(y) \right]\;.$$

2.3. Spectral analysis on the half-space

This similarity transformation may be used in any conical domain of $\mathbb{R}^n$. Indeed, under the condition that Ω is a cone (i.e. $\lambda\Omega = \Omega$, $\forall\,\lambda > 0$), the equation in the similarity variables is still posed in the domain Ω for any $s > 0$. The operator L with Dirichlet boundary conditions on a cone has basically the same properties as above, except for the fact that the spectrum is not the same (see [14]).

However, when $\Omega = \mathbb{R}^n_+$, the spectrum may be easily computed explicitly. In fact, with $\varphi_{j,\ell}$ as above, it is easy to see that $\varphi_{j,\ell}$ is an eigenfunction of

L satisfying the Dirichlet homogeneous boundary condition on $\partial\Omega = \mathbb{R}^{n-1}$, i.e. on $y_n = 0$, provided $\varphi_{j,\ell}$ is odd with respect to y_n. Taking into account that $\varphi_{j,\ell} = c_{j,\ell}\, D^\alpha K^{-1}$ for a suitable $\alpha = (\alpha_1, ..., \alpha_n)$ with $|\alpha| = j-1$, we see that this holds if and only if α_n is odd.

In one space dimension $(n=1)$ we deduce that φ_j is an eigenfunction of L in $\Omega = (0,\infty)$ with Dirichlet boundary conditions on the left extreme $y = 0$, i.e. such that $\varphi_j(0) = 0$, if and only if j is even. Consequently, in this case the eigenfunctions are

$$\phi_m(y) = \varphi_{2m}(y)\,, \qquad m \geq 1\ , \tag{2.19}$$

and the corresponding eigenvalues

$$\omega_m = \lambda_{2m} = \frac{2m}{2} = m\,, \qquad m \geq 1\ . \tag{2.20}$$

When $n \geq 2$, and $\Omega = \mathbb{R}^n_+$ we have to exclude only the eigenvalue λ_1. Therefore

$$\omega_m = \lambda_{m+1} = \frac{n+m}{2}\,, \qquad m \geq 1\ , \tag{2.21}$$

and the corresponding eigenfunctions are then multiples of

$$D^\alpha(K^{-1}), \qquad \text{with}\ \ |\alpha| = m, \ \ \alpha_n = \text{odd}\ . \tag{2.22}$$

Obviously, in this case, the multiplicity $\widehat{\mu}_m$ of each eigenvalue is strictly less that μ_m. We shall denote by $\{\phi_{m,\ell}\}_{\substack{m\geq 1\\ \ell=1,\cdots,\widehat{\mu}_m}}$ an orthonormal basis of $L^2\left(\mathbb{R}^n_+; K\right)$ constituted by the eigenfunctions of L vanishing on $y_n = 0$.

Here and in the sequel, by $L^2\left(\mathbb{R}^n_+; K\right)$ we denote the weighted L^2-space:

$$L^2(\mathbb{R}^n_+; K) \ = \ \left\{ f\colon\ \mathbb{R}^n_+ \to \mathbb{R}\colon\ \sqrt{K} f \in L^2\left(\mathbb{R}^n_+\right) \right\}$$

endowed with the canonical norm. We will also use the weighted Sobolev spaces:

$$H^1(\mathbb{R}^n_+; K) \ = \ \left\{ f \in L^2(\mathbb{R}^n_+; K)\colon\ \nabla f \in (L^2(\mathbb{R}^n_+; K))^n \right\}$$

and

$$H^1_0(\mathbb{R}^n_+; K) \ = \ \left\{ f \in H^1(\mathbb{R}^n_+; K)\colon\ f = 0\ \text{ in }\ y_n = 0 \right\}\ ,$$

endowed with the canonical norms. Finally, by $H^{-1}\left(\mathbb{R}^n_+; K\right)$ we denote the dual of $H^1_0\left(\mathbb{R}^n_+; K\right)$.

3 – The $1-d$ control problem

In this section we recall some of the main results for the $1-d$ control problem in the half-line proved in [18]. We consider the control problem

$$\begin{cases} u_t - u_{xx} = 0, & x > 0, \quad t > 0, \\ u(0,t) = v(t), & t > 0, \\ u(x,0) = u_0(x), & x > 0 \ . \end{cases} \tag{3.1}$$

Here, $u = u(x,t)$ is the state and $v = v(t)$ is the control.

Given $T > 0$ we are interested on the structure of the space of initial data that may be driven to zero in time T by means of a control $v \in L^2(0,T)$. In other words, we want to describe the space of data u_0 for which there exists $v \in L^2(0,T)$ such that the solution of (3.1) satisfies

$$u(x,T) = 0\,, \qquad \forall\, x > 0\ . \tag{3.2}$$

We define w by means of the similarity transformation (2.4). Then, u solves (3.1) and satisfies (3.2) if and only if w solves

$$\begin{cases} w_s - w_{yy} - \dfrac{y\, w_y}{2} - \dfrac{1}{2} w \ = \ 0, & y > 0, \ 0 < s < S, \\ w(0,s) = \widetilde{v}(s), & 0 < s < S, \\ w(y,0) = u_0(y), & y > 0\ , \end{cases} \tag{3.3}$$

and satisfies

$$w(y,S) = 0\,, \qquad \forall\, y > 0\ , \tag{3.4}$$

with

$$S \ = \ \log(T+1)\ , \tag{3.5}$$

and $\widetilde{v}$ as in (2.6).

Obviously, $v \in L^2(0,T)$ if and only if $\widetilde{v} \in L^2(0,S)$. Therefore, the problem is reduced to analyze the structure of the space of initial data u_0 for which there exists $\widetilde{v} \in L^2(0,S)$ such that the solution of (3.3) satisfies (3.4).

Let us rewrite the initial datum u_0 as

$$u_0(y) \ = \ \sum_{m\ge 1} a_m\, \phi_m\ , \tag{3.6}$$

where, recall, $\{\phi_m\}_{m\ge1}$ is an orthonormal basis of $L^2(\mathbb{R}_+;K)$ constituted of eigenfunctions of the operator L in $H^1_0(\mathbb{R}_+;K)$ and eigenvalues $\omega_m = m$.

The following controllability result was proved in [18]:

Theorem 3.1. *There is no non-trivial initial datum u_0 which is null-controllable in finite time and with Fourier coefficients $\{a_m\}$ satisfying*

$$|a_m| \le C_\delta\, e^{\delta m}\,, \quad \forall\, m \ge 1\,, \tag{3.7}$$

for all $\delta > 0$.

Moreover, for any $\delta > 0$ and $S > \delta$ there exist non-trivial initial data u_0 that are null-controllable in time S and such that

$$C_1\, \frac{e^{\delta m}}{m^{3/4}} \le |a_m| \le C_2\, \frac{e^{\delta m}}{m^{3/4}}\,, \quad \forall\, m \ge 1\,, \tag{3.8}$$

for suitable positive constants C_1, $C_2 > 0$. In this case a control $f \in L^2(0,S)$ with $\mathrm{supp}(f) \subseteq [0,\delta]$ can be found.

Theorem 3.1 indicates, roughly speaking, that null-controllable initial data have exponentially growing Fourier coefficients. Actually, the Fourier coefficients need to be exponentially large for any $0 \le s < S$ along the controlled trajectory.

Observe that $u_0 \in H^{-\alpha}(\mathbb{R}_+;K)$ with $\alpha > 0$ if $\sum_{m\ge1} |a_m|^2\, m^{-\alpha} < \infty$. Consequently the null-controllable initial data that Theorem 3.1 provides satisfying (3.8) do not belong to any Sobolev space of negative order $H^{-\alpha}(\mathbb{R}_+;K)$.

On the other hand, if $u_0 \in H^{-\alpha}(\mathbb{R}_+;K)$ for some $\alpha > 0$, we have

$$|a_m| = |\langle u_0,\, \phi_m\rangle| \le \|u_0\|_{H^{-\alpha}(\mathbb{R}_+;K)}\, \|\phi_m\|_{H^{\alpha}(\mathbb{R}_+;K)}\,.$$

Taking into account that $\|\phi_m\|_{H^{\alpha}(\mathbb{R}_+;K)}$ grows polynomially as $m \to \infty$ we deduce that (3.7) holds and therefore u_0 is not null-controllable, except when $u_0 \equiv 0$.

Theorem 3.1 refers to the null-controllability of system (3.3) in the similarity variables. However, due to the equivalence of the null-controllability of system (3.1) and (3.3) the same holds for (3.1):

Corollary 3.1. *There is no non trivial initial datum u_0 which is null-controllable in finite time for system (3.1) by means of L^2 boundary controls and such that*

$$|a_m| \le C_\delta\, e^{\delta m}\,, \quad \forall m \ge 1\,, \tag{3.9}$$

for all $\delta > 0$.

Moreover, for any $\delta > 0$ and $T > e^{\delta} - 1$ there exist non-trivial initial data u_0 for system (3.1) that are null-controllable in time T with L^2 controls supported in $[0,\delta]$ and such that its Fourier coefficients $\{a_m\}$ satisfy (3.8).

Remark 3.1. According to Corollary 3.1 we deduce, in particular, that the following initial data are not null-controllable in any time T for system (3.1):

- $u_0(x) = x^k \exp(-x^2/2)$, $\forall\, k \geq 0$;
- $u_0 \in \mathcal{D}(\mathbb{R}_+)$.

As we mentioned in the introduction, this result is in contrast with the existing ones for bounded intervals that guarantee that any initial datum in any Sobolev space of negative order is null-controllable in an arbitrarily short time.

The examples we have mentioned above show that the lack of null-controllability on the half-line is not due to the lack of regularity or to the lack of decay at infinity of the initial data. In fact, there is no initial datum $u_0 \in \mathcal{D}(\mathbb{R}_+)$, except for the trivial one, that is null-controllable in any time! □

4 – The multi-dimensional case: The half space

In this section we discuss the case where $\Omega = \mathbb{R}^n_+$, $n \geq 2$, and the control acts on the whole boundary $\partial\Omega$. As we shall see, the situation is similar to the one encountered in one space dimension. Namely:

a) Initial data with Fourier coefficients that grow less than any exponential are not null-controllable in any time;

b) There are initial data with exponentially growing Fourier coefficients that are null-controllable.

4.1. Reduction to the one-dimensional case

We consider the system in the similarity variables

$$\begin{cases} w_s - \Delta w - \dfrac{y \cdot \nabla w}{2} - \dfrac{n}{2}\, w \,=\, 0 & \text{in } \ \Omega \times (0,S), \\ w = v & \text{on } \ \partial\Omega \times (0,S), \\ w(y,0) = u_0(y) & \text{in } \ \Omega\,, \end{cases} \tag{4.1}$$

where

$$\Omega = \mathbb{R}^n_+\,, \qquad \partial\Omega = \mathbb{R}^{n-1}\ . \tag{4.2}$$

The space variable $y = (y_1, ..., y_n)$ will be split as $y = (y', y_n)$ with $y' = (y_1, ..., y_{n-1})$.

According to the analysis of Section 2.3 the eigenvalues of the corresponding elliptic operator $Lw = -\Delta w - \frac{y \cdot \nabla w}{2}$ are:

$$\omega_m = \frac{n+m}{2}, \quad \forall m \geq 1 . \tag{4.3}$$

The corresponding eigenfunctions are of the form

$$\phi \,=\, D^\alpha [K^{-1}] \,=\, D^\alpha [e^{-|y|^2/4}] \tag{4.4}$$

with $\alpha = (\alpha_1, ..., \alpha_n)$ such that

$$\begin{cases} |\alpha| \,=\, \alpha_1 + \cdots + \alpha_n \,=\, m, \\ \alpha_n \,=\, \text{odd} \,. \end{cases} \tag{4.5}$$

We denote by $\widehat{\mu}_m$ the multiplicity of the eigenvalues ω_m, which coincides with the number of multi-indexes α satisfying (4.5).

The eigenfunctions may be chosen to constitute an orthonormal basis of $L^2(\mathbb{R}^n_+; K)$. We shall denote them as $\{\phi_{m,\ell}\}_{\substack{m \geq 1 \\ 1 \leq \ell \leq \widehat{\mu}_m}}$. Then,

$$\phi_{m,\ell} \,=\, C_{m,\ell}\, D^\alpha [e^{-|y|^2/4}] \tag{4.6}$$

for some α as in (4.5).

We shall re-index the elements of the orthonormal basis $\{\phi_{m,\ell}\}_{\substack{m \geq 1 \\ 1 \leq \ell \leq \widehat{\mu}_m}}$ in a different way. Let us define the set $\mathcal{E} = \{(\alpha', 2j-1) :\, j \in \mathbb{N}^*,\ \alpha' \in \mathbb{N}^{n-1}\}$. For each $(\alpha', 2j-1) \in \mathcal{E}$ there is an unique eigenfunction

$$\phi_{\alpha',j} \,=\, C_{\alpha',j}\, D^{(\alpha',2j-1)} [e^{-|y|^2/4}]$$

which belongs to the orthonormal basis $\{\phi_{m,\ell}\}_{\substack{m \geq 1 \\ 1 \leq \ell \leq \widehat{\mu}_m}}$. In fact, all the elements of the orthonormal basis can be obtained in this way. Hence, $\{\phi_{\alpha',j}\}_{(\alpha',2j-1) \in \mathcal{E}}$ forms an orthonormal basis of $L^2(\mathbb{R}^n_+, K)$.

To each eigenfunction $\phi_{\alpha',j}$ corresponds an eigenvalue $\omega_m = \frac{n+m}{2}$ where $m = |\alpha'| + 2j - 1$.

Any initial datum u_0 can be developed in Fourier series as

$$u_0(y) \,=\, \sum_{\alpha' \in \mathbb{N}^{n-1}} \left[\sum_{j \geq 1} a_{\alpha',j}\, \phi_{\alpha',j} \right]. \tag{4.7}$$

To simplify things we suppose that the control $v = v(y', s)$ lies in $L^2(0, S; L^2(\partial\Omega; K))$, i.e.

$$\|v\|^2_{L^2(0,S;L^2(\partial\Omega;K))} \,=\, \int_0^S \int_{\mathbb{R}^{n-1}} v^2(y', s)\, K(y')\, dy'\, ds \,<\, \infty \,. \tag{4.8}$$

Here and in the sequel by $K(\cdot)$ we denote the $n-1$-dimensional gaussian weight

$$K(y') = \exp\Big(|y'|^2/4\Big) . \tag{4.9}$$

Note that K denotes as well the gaussian weight in n variables.

The control $v \in L^2(0,S;L^2(\mathbb{R}^{n-1};K))$ may also be developed in Fourier series as

$$v(y',s) = \sum_{\alpha' \in \mathbb{N}^{n-1}} v_{\alpha'}(s)\, \psi_{\alpha'}(y') \tag{4.10}$$

where $\{\psi_{\alpha'}\}_{\alpha' \in \mathbb{N}^{n-1}}$ is the orthonormal basis of $L^2(\mathbb{R}^{n-1};K)$ constituted by the eigenfunctions of the operator $-\Delta' - \frac{y'\cdot\nabla'}{2}$ in $\mathbb{R}^{n-1}$. Here and in the sequel Δ' and ∇' stand for the Laplacian and gradient operators in the $n-1$-variables y'.

The eigenfunctions $\psi_{\alpha'}$ are also explicit:

$$\psi_{\alpha'}(y') = d_{\alpha'}\, D^{\alpha'}\Big[e^{-|y'|^2/4}\Big], \quad \text{with } \alpha' \in \mathbb{N}^{n-1} . \tag{4.11}$$

To each eigenfunction $\psi_{\alpha'}$ corresponds an eigenvalue $\nu_j = \frac{n-2+j}{2}$ where $j = |\alpha'|+1$ (see Section 2.2).

Let us remark that

$$\begin{aligned} \phi_{\alpha',j} &= C_{\alpha',j}\, D^{\alpha'}\Big[e^{-|y'|^2/4}\Big]\, \frac{\partial^{2j-1}}{\partial y_n^{2j-1}}\Big[e^{-|y_n|^2/4}\Big] \\ &= \frac{C_{\alpha',j}}{d_{\alpha'}}\, \psi_{\alpha'}(y')\, \frac{\partial^{2j-1}}{\partial y_n^{2j-1}}\Big[e^{-|y_n|^2/4}\Big] = \frac{C_{\alpha',j}}{C_j d_{\alpha'}}\, \psi_{\alpha'}(y')\, \phi_j(y_n) . \end{aligned}$$

Taking into account that ϕ_j and $\psi_{\alpha'}$ are both normalized in $L^2(\mathbb{R}^n;K)$ and $L^2(\mathbb{R}^{n-1};K)$ respectively, it follows that $\frac{C_{\alpha',j}}{C_j d_{\alpha'}} = 1$. Hence

$$\phi_{\alpha',j} = \psi_{\alpha'}(y')\, \phi_j(y_n) . \tag{4.12}$$

Now, we reduce the controllability problem (4.1) to a family of one-dimensional control problems. Let us decompose the initial datum u_0 as in (4.7) and the control v as in (4.10).

Then, the solution w of problem (4.1) can be written as

$$w(y,s) = \sum_{\alpha' \in N^{n-1}} \Bigg[\sum_{j\geq 1} w_{\alpha',j}(s)\ \phi_{\alpha',j}(y)\Bigg] = \sum_{\alpha' \in N^{n-1}} z_{\alpha'}(y_n,s)\ \psi_{\alpha'}(y')$$

where $z_{\alpha'}(y_n,s)$ is the solution of the one-dimensional heat equation

$$\begin{cases} z_s - \dfrac{\partial^2 z}{\partial y_n^2} - \dfrac{y_n}{2}\dfrac{\partial z}{\partial y_n} + \frac{|\alpha'|-1}{2} z = 0, & y_n > 0, \quad 0 < s < S, \\ z(0,s) = v_{\alpha'}(s), & 0 < s < S, \\ z(y_n,0) = z_0(y_n), & y_n > 0 , \end{cases} \tag{4.13}$$

with z_0 such that

$$\sum_{j\geq 1} a_{\alpha',j}\,\phi_{\alpha',j} \;=\; \sum_{j\geq 1} a_{\alpha',j}\,\phi_j(y_n)\,\psi_{\alpha'}(y') \;=\; z_0(y_n)\,\psi_{\alpha'}(y')\;. \tag{4.14}$$

Obviously, (4.13) is a $1-d$ control problem similar to the one we have considered in Section 3. In fact, the equation arising in (4.13) can be easily transformed into (3.3) multiplying z by a suitable time-dependent exponential. This explains the analogy between the results of Sections 3 and 4. We have

Lemma 4.1. *Let $\alpha'\in\mathbb{N}^{n-1}$ and let us consider an initial datum for (4.1) of the form $u_0(y)=\sum_{j\geq 1} a_{\alpha',j}\,\phi_{\alpha',j}$. Let also z_0 be such that $u_0(y)=z_0(y_n)\,\psi_{\alpha'}(y')$. Then u_0 is $L^2(0,S;L^2(\mathbb{R}^{n-1},K))$ null-controllable if and only if the initial datum z_0 of (4.13) is $L^2(0,S)$ null-controllable.*

Proof: Let us suppose first that the initial datum z_0 of (4.13) is $L^2(0,S)$ null-controllable. Hence, there exists $v_{\alpha'}\in L^2(0,S)$ such that $z(y_n,S)=0$.

If we define now $v(y',s)=v_{\alpha'}(s)\,\psi_{\alpha'}(y')\in L^2(0,S;L^2(\mathbb{R}^{n-1},K))$ we obtain that the corresponding solution of (4.1) is $w(y,s)=z(y_n,s)\,\psi_{\alpha'}(y')$. Therefore, the initial datum u_0 of (4.1) is $L^2(0,S;L^2(\mathbb{R}^{n-1},K))$ null-controllable.

Let us now suppose that the initial datum u_0 of (4.1) is $L^2(0,S;L^2(\mathbb{R}^{n-1},K))$ null-controllable. Hence, there exists $v\in L^2(0,S;L^2(\mathbb{R}^{n-1},K))$ such that the corresponding solution of (4.1), w, satisfies $w(y,s)=0$.

Let us now introduce the adjoint system

$$\begin{cases} \xi_s+\Delta\xi+\dfrac{y\cdot\nabla\xi}{2}+\dfrac{n}{2}\xi \;=\; 0, & y\in\mathbb{R}^n_+,\quad 0<s<S,\\ \xi(y',0,s)=0, & y'\in\mathbb{R}^{n-1},\ 0<s<S,\\ \xi(y,T)=\xi_0(y), & y\in\mathbb{R}^n_+\ . \end{cases} \tag{4.15}$$

Multiplying in (4.1) by ξK and integrating by parts we deduce that

$$\int_{\mathbb{R}^n_+} w\,\xi\,K\;dy\,\Big|_0^S-\int_0^S\!\!\int_{\mathbb{R}^{n-1}} v(y',s)\,\frac{\partial\xi}{\partial y_n}(y',0,s)\,K(y')\;dy'\,ds \;=\; 0\;. \tag{4.16}$$

Therefore,

$$w(y,S)=0 \quad \text{in}\ \ \mathbb{R}^n_+ \tag{4.17}$$

if and only if

$$\int_0^S\!\!\int_{\mathbb{R}^{n-1}} v(y',s)\,\frac{\partial\xi}{\partial y}(y',0,s)\,K(y')\;dy'\,ds \;=\; -\int_{\mathbb{R}^n_+} u_0(y)\,\xi(y,0)\,K(y)\;dy\;, \tag{4.18}$$

for all $\xi_0\in L^2(\mathbb{R}^n_+;K)$.

Let us now consider $\xi_0 = \psi_{\alpha'}(y')\,\zeta_0(y_n)$ where $\zeta_0(y_n) \in L^2(\mathbb{R}_+, K)$. The corresponding solution of the adjoint system (4.15) is $\xi(y,s) = \psi_{\alpha'}(y')\,\zeta(y_n, s)$ where ζ is the solution of the adjoint one-dimensional system

$$\begin{cases} \zeta_s + \dfrac{\partial^2 \zeta}{\partial y_n^2} + \dfrac{y_n}{2}\,\dfrac{\partial \zeta}{\partial y_n} - \frac{|\alpha'|-1}{2}\,\zeta \;=\; 0, & y_n > 0, \quad 0 < s < S, \\ \zeta(0,s) = 0, & 0 < s < S, \\ \zeta(y_n, 0) = \zeta_0(y_n), & y_n > 0\ . \end{cases} \tag{4.19}$$

From (4.18) we obtain that

$$\int_0^S \int_{\mathbb{R}^{n-1}} v(y',s)\,\psi_{\alpha'}(y')\,\frac{\partial \zeta}{\partial y_n}(0,s)\,K(y')\;dy'\,ds \;= \\ = \;-\int_{\mathbb{R}^n_+} u_0(y)\,\psi_{\alpha'}(y')\,\zeta(y_n,0)\,K(y)\;dy$$

for all $\zeta_0 \in L^2(\mathbb{R}_+, K)$.

By taking into account that

$$\int_{\mathbb{R}^n_+} u_0(y)\,\psi_{\alpha'}(y')\,\zeta(y_n,0)\,K(y)\;dy \;=\; \int_{\mathbb{R}_+} z_0(y_n)\,\zeta(y_n,0)\,K(y_n)\;dy_n$$

we obtain that, for all $\zeta_0 \in L^2(R_+, K)$,

$$\int_0^S v_{\alpha'}(s)\,\frac{\partial \zeta}{\partial y_n}(0,s)\;ds \;=\; -\int_{\mathbb{R}_+} z_0(y_n)\,\zeta(y_n,0)\,K(y_n)\;dy_n\ , \tag{4.20}$$

where $v_{\alpha'}(s) = \int_{\mathbb{R}^{n-1}} v(y',s)\,\psi_{\alpha'}(y')\,K(y')\;dy'$.

Since $v_{\alpha'} \in L^2(0,S)$, from (4.20) it follows that the initial datum z_0 of (4.13) is $L^2(0,S)$ null-controllable and the proof finishes. ∎

Remark 4.1. The previous Lemma allows us to reduce the multi-dimensional problem to a family of one-dimensional problems depending on α'.

Roughly speaking, if all the corresponding $1-d$ problems are null-controllable then the multi-dimensional problem is null-controllable. Note however that, in this situation, whether the control lies in $L^2(0,S;L^2(\mathbb{R}^{n-1},K))$ depends heavily on how the size of the control of the $1-d$ problem (4.13) depends on α'.

On the other hand, if there exists at least one $1-d$ problem which is not null-controllable then the multi-dimensional problem is not null-controllable. □

4.2. Main results

As an immediate corollary of the results of Section 3, the following holds:

Theorem 4.1. *Assume that the initial datum u_0 with Fourier coefficients $\{a_{\alpha',j}\}_{(\alpha',j)\in\mathcal{E}}$ is null-controllable for system* (4.1) *in time S.*

Assume also that, for some α', the corresponding Fourier coefficients $\{a_{\alpha',j}\}_j$ satisfy

$$|a_{\alpha',j}| \leq C_\delta\, e^{\delta m}, \qquad \text{as } \; m\to\infty \tag{4.21}$$

for all $\delta>0$. Then, necessarily,

$$a_{\alpha',j}=0\,, \qquad \forall\, j\geq 1\ . \tag{4.22}$$

Proof: Since $u_0=\sum_{\alpha'\in\mathbb{N}^{n-1}}\sum_{j\geq 1} a_{\alpha',j}\,\phi_{\alpha',j}$ is null-controllable it follows that, for each $\alpha'\in\mathbb{N}^{n-1}$, $w_0=\sum_{j\geq 1}a_{\alpha',j}\,\phi_{\alpha',j}$ is null-controllable. This is a direct consequence of the orthogonality of the traces of the normal derivatives of the eigenfunctions $\left.\frac{\partial\phi_{\alpha',j}(y)}{\partial\nu}\right|_{\partial\Omega}$ in $L^2(\partial\Omega;K|_{\partial\Omega})$. If the control corresponding to u_0 is $\sum_{\alpha'\in\mathbb{N}^{n-1}}v_{\alpha'}(s)\,\psi_{\alpha'}(y')$ then the control corresponding to w_0 is $v_{\alpha'}(s)\,\psi_{\alpha'}(y')$. Let z_0 be such that $w_0=z_0\,\psi_{\alpha'}$. From Lemma 4.1 it follows that the solution of

$$\begin{cases} z_s-\dfrac{\partial^2 z}{\partial y_n^2}-\dfrac{y_n}{2}\,\dfrac{\partial z}{\partial y_n}+\dfrac{|\alpha'|-1}{2}\,z\;=\;0, & y_n>0,\;\; 0<s<S,\\ z(0,s)=v_{\alpha'}(s), & 0<s<S,\\ z(y_n,0)=z_0(y_n), & y_n>0\ , \end{cases} \tag{4.23}$$

is null-controllable.

Let us now define $\zeta(y_n,s)=z(y_n,s)\,e^{-\frac{|\alpha'|}{2}s}$. Then ζ satisfies

$$\begin{cases} \zeta_s-\dfrac{\partial^2\zeta}{\partial y_n^2}-\dfrac{y_n}{2}\,\dfrac{\partial\zeta}{\partial y_n}-\frac12\,\zeta\;=\;0, & y_n>0,\;\; 0<s<S,\\ \zeta(0,s)=v_{\alpha'}(s), & 0<s<S,\\ \zeta(y_n,0)=z_0(y_n), & y_n>0\ . \end{cases} \tag{4.24}$$

We can now apply Theorem 3.1 to deduce that $a_{\alpha',j}=0$, for all $j\geq 1$ and the proof finishes. ∎

As a positive counterpart to this result we now show that there are initial data with coefficients that grow exponentially and that are null-controllable:

Theorem 4.2. *Let the sequence of coefficients $\{a_{\alpha',j}\}$ be such that*

$$a_{\alpha',j} = 0\,, \quad \forall\, \alpha' \neq \gamma\,, \tag{4.25}$$

where $\gamma = (\gamma_1, ..., \gamma_{n-1})$ is a given multi-index and such that

$$a_{\gamma,j} = F(j)\, e^{jS/2}\,, \quad \forall\, j \geq 1\,, \tag{4.26}$$

where

$$F(z) \;=\; \frac{\sin(i\,\delta\, z)}{i\,\delta\, z} \tag{4.27}$$

with $\delta > 0$.

Then:

(**a**) *The corresponding initial datum u_0 is null-controllable by means of a control $v \in L^2(0,S;L^2(\mathbb{R}^{n-1};K))$ in time $S = \delta$.*

(**b**) *There exist positive constants $C_1, C_2 > 0$ such that*

$$C_1\, \frac{e^{2\delta j}}{j^{3/4}} \;\leq\; |a_{\gamma,j}| \;\leq\; C_2\, \frac{e^{2\delta j}}{j^{3/4}}\,, \quad \forall\, j\,. \tag{4.28}$$

Proof: From (4.25) it follows that the corresponding initial datum u_0 has the form:

$$u_0 \;=\; \sum_{j\geq 1} a_{\gamma,j}\, \phi_{\gamma,j} \;=\; \left[\sum_{j\geq 1} a_{\gamma,j}\, \phi_j\right] \psi_\gamma\,.$$

Hence, the initial datum u_0 is null-controllable if and only if the following one-dimensional problem is null-controllable

$$\begin{cases} z_s - \dfrac{\partial^2 z}{\partial y_n^2} - \dfrac{y_n}{2}\,\dfrac{\partial z}{\partial y_n} + \dfrac{|\gamma| - 1}{2}\, z \;=\; 0, & y_n > 0, \quad 0 < s < S, \\ z(0,s) = v_{\alpha'}(s), & 0 < s < S, \\ z(y_n, 0) = z_0(y_n), & y_n > 0\,, \end{cases} \tag{4.29}$$

where $z_0 = \sum_{j\geq 1} a_{\gamma,j}\, \phi_j$.

Now we only have to apply Theorem 3.1 to this one-dimensional problem and the proof finishes. ∎

Undoing the similarity transformation, i.e. going back to the original space-time variables (x,t) and to the state u, as a consequence of Theorem 4.1 and 4.2 we can immediately deduce the corresponding results for the original system

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}^n_+ \times (0,T), \\ u = v & \text{on } \mathbb{R}^{n-1} \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^n_+\,. \end{cases} \tag{4.30}$$

Recall that the control $v = v(x', t)$ of system (4.30) and the control $\widetilde{v} = \widetilde{v}(y', s)$ for system (4.1) are related by the transformation

$$\widetilde{v}(y', s) = e^{sn/2}\, v\Big(e^{s/2}y',\, e^s - 1\Big)\,. \tag{4.31}$$

Up to now we have been dealing with controls $\widetilde{v} \in L^2(0, S; L^2(\mathbb{R}^{n-1}; K))$. Consequently:

$$\begin{aligned}
&\int_0^S \int_{\mathbb{R}^{n-1}} |\widetilde{v}(y', s)|^2 \exp\Big(\frac{|y'|^2}{4}\Big)\, dy'\, ds = \\
&\qquad = \int_0^S e^{sn} \int_{\mathbb{R}^{n-1}} \Big|v\Big(e^{s/2}y',\, e^s - 1\Big)\Big|^2 \exp\Big(\frac{|y'|^2}{4}\Big)\, dy'\, ds \\
&\qquad = \int_0^T (t+1)^{n/2} \int_{\mathbb{R}^{n-1}} v^2(x', t) \exp\Big(\frac{|x'|^2}{4\,(t+1)}\Big)\, dx'\, dt \;<\; \infty\,,
\end{aligned}$$

or, in other words,

$$\int_0^T \int_{\mathbb{R}^{n-1}} v^2(x', t) \exp\Big(\frac{|x'|^2}{4\,(t+1)}\Big)\, dx'\, dt \;<\; \infty\,. \tag{4.32}$$

The following holds:

Corollary 4.1. *Assume that the initial datum u_0 with Fourier coefficients $\{a_{\alpha',j}\}$ in $L^2(\mathbb{R}^n_+; K)$ is null controllable for system (4.30) in time $T > 0$ with control $v = v(x', t)$ satisfying (4.32).*

Then, if for some α'

$$|a_{\alpha',j}| \le C_\delta\, e^{\delta j}\,, \qquad \forall j\,, \tag{4.33}$$

for all $\delta > 0$, necessarily

$$a_{\alpha',j} = 0\,, \qquad \forall j\,. \tag{4.34}$$

Moreover there exist initial data with Fourier coefficients as in (a)–(b) of Theorem 4.2 that are null-controllable in time $T = e^\delta - 1$ with a control $v = v(x', t)$ satisfying (4.32) for system (4.30).

Remark 4.2. Very often the control is restricted to be with support on a given subset of the boundary. Let $\Gamma_0 \subset \mathbb{R}^{n-1}$ be a bounded open subset of the boundary of $\mathbb{R}^n_+$. Let us denote by 1_{Γ_0} the characteristic function of this set. Consider the control problem

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}^n_+ \times (0, T), \\ u = v\, 1_{\Gamma_0} & \text{on } \mathbb{R}^{n-1} \times (0, T), \\ u(x, 0) = u_0(x) & \text{in } \mathbb{R}^n_+\,. \end{cases} \tag{4.35}$$

The negative result of Corollary 4.1 applies in this case as well. Indeed, taking into account that $v1_{\Gamma_0}$ satisfies (4.32) for any $v \in L^2(\Gamma_0 \times (0,T))$ one deduces that if the controllable initial datum u_0 for system (4.35) is such that (4.33) holds with control $v \in L^2(\Gamma_0 \times (0,T))$, then, necessarily, (4.34) holds as well. □

5 – Further comments and open problem

5.1. General conical domains

Assume that Ω is a conical domains, i.e. such that $\tau\,\Omega = \Omega$ for any $\tau > 0$. Consider the heat equation with control

$$\left\{\begin{array}{ll} u_t - \Delta u = 0 & \text{in } \ \Omega \times (0,T), \\ u = v & \text{on } \ \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \ \Omega\ . \end{array}\right. \tag{5.1}$$

Using the similarity transformation of Section 2, system may be transformed into

$$\left\{\begin{array}{ll} w_s - \Delta w - \dfrac{y \cdot \nabla w}{2} - \dfrac{n}{2}\, w \, = \, 0 & \text{in } \ \Omega \times (0,S), \\ w = \widetilde{v} & \text{on } \ \partial\Omega \times (0,S) \\ w(y,0) = w_0(y) & \text{in } \ \Omega\ . \end{array}\right. \tag{5.2}$$

The operator $L = -\Delta - \frac{y \cdot \nabla}{2}$ defines an isomorphism $L\colon H_0^1(\Omega;K) \to H^{-1}(\Omega;K)$ with

$$\begin{aligned} L^2(\Omega;K) \ &= \ \left\{ f\colon\ \Omega \to \mathbb{R}\colon\ \int_\Omega f^2 K < \infty \right\} \\ H_0^1(\Omega;K) \ &= \ \left\{ f \in L^2(\Omega;K)\colon\ \nabla f \in (L^2(\Omega;K))^n,\ f = 0 \ \text{ on } \partial\Omega \right\}\,, \end{aligned}$$

$H^{-1}(\Omega;K)$ being the dual of $H_0^1(\Omega;K)$.

One can also obtain a spectral decomposition of L in $L^2(\Omega;K)$. The eigenfunctions of L may be written in separated spherical coordinates and they constitute a two-parameter family $\phi_{m,\ell}(y) = \rho_{m,\ell}(r)\,\theta_\ell(\sigma)$ where $r = |y|$ is the radious and $\sigma \in S^{n-1} \cap \Omega$.

The angular components θ_ℓ may be determined as the eigenfunctions of the corresponding Laplace–Beltrami operator in $S^{n-1} \cap \Omega$ with Dirichlet boundary conditions on the boundary and the radial components $\rho_{m,\ell}(r)$ are determined as the solutions of a Bessel-like equation in $\mathbb{R}^+$ with the integrability condition

$$\int_0^\infty r^{n-1}\,\rho^2(r)\,e^{r^2/4}\,dr \;<\; \infty\;.$$

We denote by $\lambda_{m,\ell}$ the associated eigenvalues.

With this Fourier decomposition in mind the null-control problem may be reduced to a moment problem:

$$\int_0^S\int_{\partial\Omega} \widetilde{v}(y',s)\,e^{(\lambda_{m,\ell}-\frac{n}{2})s}\;\frac{\partial\phi_{m,\ell}(y)}{\partial\nu}\,K(y)\;d\sigma\,ds \;=\; a_{m,\ell}\;, \tag{5.3}$$

$\{a_{m,\ell}\}$ being the Fourier coefficients of the initial datum to be controlled on the orthonormal basis $\{\phi_{m,\ell}\}$ of $L^2(\Omega;K)$ and where $\partial\cdot/\partial\nu$ denotes the derivative in the direction of the unit outward normal.

Of course, the moment problem consists on finding $\widetilde{v}(y',s)$ such that (5.3) holds for all m,ℓ simultaneously. However, in this case we may not use the orthogonality of the traces of the normal derivatives of the eigenfunctions $\frac{\partial\phi_{m,\ell}(y)}{\partial\nu}\Big|_{\partial\Omega}$ in $L^2(\partial\Omega;K|_{\partial\Omega})$. This fact was essential in Section 4 when analyzing the problem in $\Omega=\mathbb{R}^n_+$.

However, one expects negative results like those in Theorem 3.1 and 4.1 to be true in this more general case.

The analysis of Section 4 may be easily extended to the case where, for instance

$$\Omega=\Big\{y\in\mathbb{R}^n:\; y_j\ge 0,\;\forall j=1,...,n\Big\}\;,$$

or, more generally, to the case where the cone Ω is such that, by a finite number of reflections one may cover $\mathbb{R}^n$. This allows indeed to compute the spectrum of L in $H_0^1(\Omega;K)$ which turns out to be a suitable subset of the whole spectrum of the operator L in $L^2(\mathbb{R}^n;K)$.

However, the analysis of the general case remains to be done.

We refer to [14] and [3] for the analysis of the large time behaviour of solutions of semilinear heat equations in conical domains.

We also recall that the approximate controllability problem for the semilinear heat equation was studied in [22] in the frame of the weighted Sobolev spaces above. The methods in [22] which are based on unique continuation properties do apply in any conical domain. But, as we mentioned in the introduction, the null-control property is stronger than the approximate controllability one and requires further analysis.

5.2. General domains

The problem of null-controllability of the heat equation arises in fact in any domain Ω of $\mathbb{R}^n$. As we have described in the introduction, when Ω is bounded and of class C^2 the null-controllability is well known. In the context of unbounded domains, in this work we have shown the lack of null-controllability (for "nice" initial data) when, for instance, $\Omega = \mathbb{R}^n_+$. As we have described in the previous section, the approach based on the use of the similarity variables may also be used in general conical domains. But, due to the lack of orthogonality of the traces of the normal derivatives of the eigenfunctions, the corresponding moment problem is more complex and remains to be solved.

When Ω is a general unbounded domain, the similarity transformation does not seem to be of any help since the domain one gets after transformation depends on time.

Therefore, a completely different approach seems to be needed when Ω is not conical. However, one may still expect a bad behaviour of the null-control problem. Indeed, assume for instance that Ω contains $\mathbb{R}^n_+$. If one is able to control to zero in Ω an initial datum u_0 by means of a boundary control acting on $\partial\Omega\times(0,T)$, then, by restriction, one is able to control the initial datum $u_0|_{\mathbb{R}^n_+}$ with the control being the restriction of the solution in the larger domain $\Omega\times(0,T)$ to $\mathbb{R}^{n-1}\times(0,T)$. A careful development of this argument and of the result it may lead to remains to be done.

The approximate control problem for the semilinear heat equation in general unbounded domains was addressed in [23]. There an approximation method was developed. The domain Ω was approximated by bounded domains (essentially by $\Omega\cap B_R$, B_R being the ball of radius R) and the approximate control in the unbounded domain Ω was obtained as limit of the approximate control on the approximating bounded domain $\Omega\cap B_R$.

However, this approach does not apply in the context of the null-control problem.

The approach described in [13] and [15] is also worth mentioning. In this articles is proved that, for any $T > 0$, the heat equation has a fundamental solution which is C^∞ away from the origin and with support in the strip $0 \leq t \leq T$. This allows to build a solution u of the heat equation

$$u_t - \Delta u = 0 \quad \text{in } \mathbb{R}^n\times(0,T) \tag{5.4}$$

which is continuous in $\mathbb{R}^n\times[0,T]$, and that matches the initial and final conditions

$$u(x,0) = u_0 \quad \text{in } \mathbb{R}^n \ , \tag{5.5}$$

$$u(x,T)=0 \quad \text{in } \mathbb{R}^n \ , \tag{5.6}$$

for any continuous function u_0.

This may also be interpreted as a null-controllability result in a general domain Ω. Indeed, by setting $v=u|_{\partial\Omega\times(0,T)}$, we deduce that $u|_{\Omega\times(0,T)}$, the restriction of u solution of (5.4)–(5.6) to $\Omega\times(0,T)$ satisfies

$$\begin{cases} u_t-\Delta u=0 & \text{in } \Omega\times(0,T), \\ u=v & \text{on } \partial\Omega\times(0,T), \\ u(x,0)=u_0(x) & \text{in } \Omega, \\ u(x,T)=0 & \text{in } \Omega \ . \end{cases} \tag{5.7}$$

This argument applies when Ω is unbounded as well. In particular, when $n=1$ and $\Omega=\mathbb{R}^+$, this shows that for any $u_0\in C(\mathbb{R}^+)$ there exists $v\in C[0,T]$ and a solution u of

$$\begin{cases} u_t-u_{xx}=0, & 0<x, \qquad 0<t<T, \\ u(0,t)=v(t), & 0<t<T, \\ u(x,0)=u_0(x), & 0<x \ , \end{cases} \tag{5.8}$$

such that

$$u(x,T)=0 \quad \text{in } \mathbb{R}^+ \ . \tag{5.9}$$

Note however that the solutions of (5.8)–(5.9) that the approach of [13] and [15] provides do not feet in the context of our negative result since they grow too fast as $|x|\to\infty$ and therefore, these are not solutions in the sense of transposition.

It is also worth comparing this result with the positive one of Section 3. In Theorem 3.1 we prove that initial data with exponentially growing Fourier coefficients are null-controllable by means of $L^2(0,T)$-controls. Moreover, the trajectory we obtain has necessarily exponentially growing Fourier coefficients during the whole time interval. This is in agreement with the result of [13] and [15] in which the trajectory has also a fast growth rate as $|x|\to\infty$.

5.3. More general equations

The same problems arise in the context of more general parabolic equations including variable coefficients, semilinear terms, ...

We refer to the works [6] and [8] for the analysis of the null-control problem of the semilinear heat equation in bounded domains and to [12] for the case of linear heat equations with variable coefficients.

The approach we have adopted in this work does not seem to extend to these more general problems even in $\mathbb{R}^n_+$ except for very particular cases. Indeed, Fourier series decompositions of the solutions apply for equations of the form

$$w_s - \Delta w - \frac{y \cdot \nabla w}{2} - \frac{n}{2}\, w + a(y)\, w + b(s)\, w \;=\; 0 \;. \tag{5.10}$$

Obviously, in order to get an equation of the form (5.10) in the similarity variables, the original equation

$$u_t - \Delta u + c(x,t)\, u \;=\; 0 \tag{5.11}$$

needs to have a variable coefficient $c(x,t)$ of a very special structure.

The extension of the results of this paper in, say $\mathbb{R}^n_+$, to the general case of coefficients $c \in L^\infty(\mathbb{R}^n_+ \times (0,T))$ remains to be done.

5.4. Necessary and sufficient conditions for null-controllability

The result of this paper show that, when $\Omega = \mathbb{R}^n_+$,

- ⋆ Initial data with Fourier coefficients growing slower than any exponential may not be controlled in finite time by means of controls in a suitable weighted L^2 space of the boundary;
- ⋆ Some initial data with exponentially growing Fourier coefficient are controllable.

It would be desirable to obtain a more explicit characterization of the Fourier coefficients of the null-controllable initial data.

REFERENCES

[1] Avdonin, S.A. and Ivanov, S.A. – *Families of Exponentials. The Method of Moments in Controllability Problems for Distributed Parameter Systems*, Cambridge University Press, 1995.

[2] Boas, R.P. – *Entire Functions*, Academic Press, New York, 1954.

[3] Bandle, C. and Levine, H.A. – On the existence and nonexistence of global solutions of reaction-diffusion equations in sectorial domains, *Trans. Amer. Math. Soc.,* 316(2) (1989), 595–622.

[4] Escobedo, M. and Kavian, O. – Variational problems related to self-similar solutions of the heat equation, *Nonlinear Analysis TMA,* 11(10) (1987), 1103–1133.

[5] Fattorini, H. and Russell, D.L. – Uniform bounds on biorthogonal functions for real exponentials with an application to the control theory of parabolic equations, *Quarterly J. Appl. Math.,* 32 (1974), 45–69.

[6] Fernández-Cara, E. – Null controllability of the semilinear heat equation, *ESAIM:COCV,* 2 (1997), 87–107, (http://www.emath.fr/cocv/).

[7] Fernández-Cara, E. and Zuazua, E. – The cost of approximate controllability for heat equations: The linear case, *Advances Diff. Eqs.,* 5 (2000), 465–514.

[8] Fernández-Cara, E. and Zuazua, E. – Null controllability of weakly blowing-up semilinear heat equations, *Ann. IHP, Analyse non-linéaire,* 17(5) (2000), 583–616.

[9] Fursikov, A. and Imanuvilov, O.Yu. – *Controllability of evolution equations*, Lecture Notes Series #34, Research Institute of Mathematics, Global Analysis Research Center, Seoul National University, 1996.

[10] Ghidaglia, J.M. – Some backward uniqueness results, *Nonlinear Anal. TMA,* 10 (1986), 777–790.

[11] Güichal, E.N. – A lower bound on the norm of the control operator for the heat equation, *J. Math. Anal. Appl.,* 110 (1985), 519–527.

[12] Imanuvilov, O.Yu. and Yamamoto, M. – *On Carleman inequalities for parabolic equations in Sobolev spaces of negative order and exact controllability for semilinear parabolic equations*, Preprint nº 98-46, University of Tokyo, Graduate School of Mathematics, Japan, 1998.

[13] Jones, Jr., B.F. – A fundamental solution of the heat equation which is supported in a strip, *J. Math. Anal. Appl.,* 60 (1977), 314–324.

[14] Kavian, O. – Remarks on the large time behavior of a nonlinear diffusion equation, *Ann. Inst. Henri Poincaré, Analyse non linéaire,* 4(5) (1987), 432–452.

[15] Littman, W. – Boundary control theory for hyperbolic and parabolic partial differential equations with constant coefficients, *Annali Scuola Norm. Sup. Pisa, Serie IV,* 3 (1978), 567–580.

[16] Lions, J.L. and Malgrange, E. – Sur l'unicité rétrograde dans les problèmes mixtes paraboliques, *Math. Scan.,* 8 (1960), 277–286.

[17] Lebeau, G. and Robbiano, L. – Contrôle exact de l'équation de la chaleur, *Comm. P.D.E.,* 20 (1995), 335–356.

[18] Micu, S. and Zuazua, E. – On the lack of null-controllability of the heat equation on the half-line, *Trans. AMS,* 353 (2001), 1635–1659.

[19] Rosier, L. – *Exact controllability for the linear Korteweg–de Vries equation on the half-line*, preprint, 1998.

[20] Russell, D.L. – Controllability and stabilizability theory for linear partial differential equations. Recent progress and open questions, *SIAM Rev.,* 20 (1978), 639–739.

[21] Schwartz, L. – *Etude des Sommes d'Exponentielles*, Herman, Paris, 1959.

[22] Teresa, L. de – Approximate controllability of a semilinear heat equation in $\mathbb{R}^N$, *SIAM J. Cont. Optim.,* 36(6) (1996), 855–884.

[23] Teresa, L. de and Zuazua, E. – Approximate controllability of the heat equation in unbounded domains, *Nonlinear Anal. TMA,* 37 (1999), 1059–1090.

[24] Yosida, K. – *Functional Analisis*, Springer Verlag, 1980.

[25] Young, R.M. – *An introduction to Nonharmonic Fourier Series*, Academic Press, 1980.

Sorin Micu and Enrique Zuazua
Departamento de Matemática Aplicada, Universidad Complutense
28040 Madrid – SPAIN
E-mails: `sorin@sunma4.mat.ucm.es`
`zuazua@eucmax.sim.ucm.es`