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# TWO-GRID FINITE-ELEMENT SCHEMES FOR THE STEADY NAVIER–STOKES PROBLEM IN POLYHEDRA

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*Presented by J.P. Dias*

**Abstract:** We discretize a steady Navier–Stokes system on a three-dimensional polyhedron by finite-elements schemes defined on two grids. In the first step, the fully nonlinear problem is solved on a coarse grid, with mesh-size $H$. In the second step, the problem is linearized by substituting into the nonlinear term, the velocity $\mathbf{u}_H$ computed at step one, and the linearized problem is solved on a fine grid with mesh-size $h$. This approach is motivated by the fact that the contribution of $\mathbf{u}_H$ to the error analysis is measured in the $L^3$ norm, and thus, for the lowest-degree elements on a Lipschitz polyhedron, is of the order of $H^{3/2}$. Hence, an error of the order of $h$ can be recovered at the second step, provided $h = H^{3/2}$. When the domain is convex, a similar result can be obtained with $h = H^2$. Both results are valid in two dimensions.

## 0 – Introduction

Let us consider the following equation in a bounded domain $\Omega$ of $\mathbb{R}^3$:

$$-\Delta\, u + u^3 \;=\; f \quad \text{in } \;\Omega\;, \tag{0.1}$$

$$u \;=\; 0 \quad \text{on } \;\partial\Omega\;. \tag{0.2}$$

For any $f$ in $L^2(\Omega)$, it admits a unique solution in the Sobolev space $H_0^1(\Omega)$ (i.e. the space of functions $\varphi$ such that $\varphi$ and $\frac{\partial\varphi}{\partial x_i}$ belong to $L^2(\Omega)$ for $i = 1, 2, 3$, and $\varphi = 0$ on $\partial\Omega$). Its variational formulation writes:

$$\forall\, v \in H_0^1(\Omega)\,, \qquad a(u,v) + (u^3, v) = (f, v)\;, \tag{0.3}$$

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where

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, d\mathbf{x}\,, \qquad (g,h) = \int_\Omega g\, h \, d\mathbf{x}\ .$$

Assuming that $\Omega$ has a polyhedral boundary, we introduce *two* finite-element subspaces of $H_0^1(\Omega)$, denoted by $V_H$ and $V_h$, where $H$ (resp. $h$) stands for a coarse (resp. fine) regular triangulation of $\overline{\Omega}$ (the notion of regular triangulation is defined in (1.12)). We want to explain the method presented in this paper on the above simplified academic example.

We define the approximation $u_H$ on the coarse grid by: $u_H$ is the unique solution in $V_H$ of

$$\forall\, v_H \in V_H\,, \quad a(u_H, v_H) + (u_H^3, v_H) = (f, v_H)\ . \tag{0.4}$$

We proceed now in a rather formal fashion, in order to describe the key ingredient of the method to follow (in a much more interesting and less trivial situation!). It can be proven that analogues of "linear estimates" hold true for (0.4), i.e., if $u$ belongs to $H^2(\Omega)$, then we have for constants $c$ depending on the norm of $f$ in $L^2(\Omega)$,

$$\|u - u_H\|_{H^1(\Omega)} \le c\, H\,, \quad \|u - u_H\|_{L^2(\Omega)} \le c\, H^2\ , \tag{0.5}$$

the second estimate being valid on a convex polyhedron. Problem (0.4) is *non-linear* on the *coarse* grid. We define now $u_h$ on the *fine* grid as the solution of: $u_h \in V_h$,

$$\forall\, v_h \in V_h\,, \quad a(u_h, v_h) + (u_H^2 u_h, v_h) = (f, v_h)\ . \tag{0.6}$$

If $h \ll H$, the dimension of (0.6) is *much larger* than that of (0.4) but (0.6) is *linear*.

We now estimate $\|u - u_h\|_{H^1(\Omega)}$, assuming $u \in H^2(\Omega)$ and $\Omega$ convex, so that both inequalities in (0.5) hold: it follows from (0.3) and (0.6) that

$$\forall\, v_h \in V_h\,, \quad a(u - u_h,\, v_h) + (u^2 u - u_H^2 u_h,\, v_h) = 0\ ,$$

or equivalently, for all $v_h$ and $w_h$ in $V_h$:

$$\begin{aligned} a(w_h - u_h,\, v_h) + \Big(u_H^2(w_h - u_h),\, v_h\Big) &= \\ = a(w_h - u,\, v_h) + \Big((u_H^2 - u^2)\, u,\, v_h\Big) &+ \Big(u_H^2(w_h - u),\, v_h\Big)\ . \end{aligned} \tag{0.7}$$

By choosing $v_h = w_h - u_h$ in (0.7), we obtain:

$$\begin{aligned} \Big|a(w_h - u_h,\, w_h - u_h)\Big| \le \Big|a(u - w_h,\, w_h - u_h)\Big| &+ \Big|\Big((u^2 - u_H^2)\, u,\, w_h - u_h\Big)\Big| \\ &+ \Big|\Big(u_H^2(u - w_h),\, w_h - u_h\Big)\Big|\ . \end{aligned} \tag{0.8}$$

Using Sobolev's imbedding theorem: $H^1(\Omega) \subset L^6(\Omega)$ and denoting by $C$ various constants independent of $h$ and $H$, we observe that

$$
\begin{aligned}
&\Big|\Big((u^2 - u_H^2)\, u,\, w_h - u_h\Big)\Big| \;\le \\
(0.9) \qquad &\le\; \|u - u_H\|_{L^2(\Omega)}\, \|u + u_H\|_{L^6(\Omega)}\, \|u\|_{L^6(\Omega)}\, \|w_h - u_h\|_{L^6(\Omega)} \\
&\le\; C\, H^2\, \|w_h - u_h\|_{H^1(\Omega)} \;,
\end{aligned}
$$

$$
(0.10) \qquad \Big|\Big(u_H^2(u - w_h),\, w_h - u_h\Big)\Big| \;\le\; C\, \|u - w_h\|_{H^1(\Omega)}\, \|w_h - u_h\|_{H^1(\Omega)} \;.
$$

It follows from (0.8)–(0.10) that

$$
(0.11) \qquad \|w_h - u_h\|_{H^1(\Omega)} \;\le\; C\Big(\|u - w_h\|_{H^1(\Omega)} + H^2\Big) \;.
$$

Therefore

$$
\begin{aligned}
\|u - u_h\|_{H^1(\Omega)} \;&\le\; \|u - w_h\|_{H^1(\Omega)} + \|w_h - u_h\|_{H^1(\Omega)} \\
&\le\; C\Big(\|u - w_h\|_{H^1(\Omega)} + H^2\Big) \;.
\end{aligned}
$$

Hence

$$
\|u - u_h\|_{H^1(\Omega)} \;\le\; C\Big(\inf_{w_h \in V_h} \|u - w_h\|_{H^1(\Omega)} + H^2\Big) \;\le\; C(h + H^2) \;.
$$

Therefore, if we choose

$$
h = H^2 \;,
$$

we have

$$
\|u - u_h\|_{H^1(\Omega)} \;\le\; C\, h \;,
$$

i.e. *the same estimate* (when $h \to 0$) *as we would obtain by solving the nonlinear problem on the fine grid*. Here we obtain the "good estimate" by solving the *nonlinear problem on the coarse grid and a linear problem on the fine grid*.

There is another possibility, which is simply to apply Newton's method. The fully non-linear equation on the fine grid has the variational formulation:

$$
(0.12) \qquad \forall\, v_h \in V_h\,, \qquad a(u_h, v_h) + (u_h^3, v_h) = (f, v_h) \;.
$$

By linearizing (0.12) around $u_H$, i.e. approximating $u_h^3$ by:

$$
u_H^3 \;+\; 3\, u_H^2(u_h - u_H) \;,
$$

we replace (0.12) by its linearized version:

$$
(0.13) \qquad \forall\, v_h \in V_h\,, \qquad a(u_h, v_h) + 3\,(u_H^2 u_h, v_h) \;=\; 2\,(u_H^3, v_h) + (f, v_h) \;.
$$

The system (0.4), (0.13) satisfies similar error estimates as (0.4), (0.6). But it is (slightly) more complex without gain in the a priori estimates, at least in theory. The reader will find another reference to Newton's method in Remark 0.2 below.

We now proceed with a much more significant situation. From now on, we assume that $\Omega$ is a Lipschitz domain of $\mathbb{R}^3$ with a polyhedral boundary $\partial\Omega$. Consider the steady Navier–Stokes equations:

$$-\nu\,\Delta\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla p = \mathbf{f} \quad \text{in } \Omega\,, \tag{0.14}$$

with the incompressiblity condition:

$$\operatorname{div}\mathbf{u}=0 \quad \text{in } \Omega\,, \tag{0.15}$$

and the homogeneous Dirichlet boundary condition:

$$\mathbf{u}=\mathbf{0} \quad \text{on } \partial\Omega\,. \tag{0.16}$$

Let $\mathcal{T}_H$ be a coarse triangulation of $\overline{\Omega}$ and let $X_H$ and $M_H$ be suitable finite-element spaces for discretizing the velocity $\mathbf{u}$ and pressure $p$. Similarly, let $\mathcal{T}_h$ be a fine triangulation, with corresponding finite-element spaces $X_h$ and $M_h$. The two-grid algorithm for discretizing (0.14)–(0.16) is:

• **Step One** (nonlinear problem on coarse grid): Find $(\mathbf{u}_H, p_H) \in X_H \times M_H$, solution of

$$\forall\,\mathbf{v}_H \in X_H,\quad \nu(\nabla\mathbf{u}_H,\nabla\mathbf{v}_H)+((\mathbf{u}_H\cdot\nabla)\mathbf{u}_H,\mathbf{v}_H)-(p_H,\operatorname{div}\mathbf{v}_H)=\langle\mathbf{f},\mathbf{v}_H\rangle\,, \tag{0.17}$$

$$\forall\,q_H \in M_H,\quad (q_H,\operatorname{div}\mathbf{u}_H)=0\,. \tag{0.18}$$

• **Step Two** (linearized problem on fine grid): Find $(\mathbf{u}_h, p_h) \in X_h \times M_h$, solution of

$$\forall\,\mathbf{v}_h \in X_h,\quad \nu(\nabla\mathbf{u}_h,\nabla\mathbf{v}_h)+((\mathbf{u}_H\cdot\nabla)\mathbf{u}_h,\mathbf{v}_h)-(p_h,\operatorname{div}\mathbf{v}_h)=\langle\mathbf{f},\mathbf{v}_h\rangle\,, \tag{0.19}$$

$$\forall\,q_h \in M_h,\quad (q_h,\operatorname{div}\mathbf{u}_h)=0\,. \tag{0.20}$$

**Remark 0.1.** Note that in general (0.18) does not imply that $\operatorname{div}\mathbf{u}_H = 0$. Therefore

$$\Big((\mathbf{u}_H\cdot\nabla)\mathbf{u}_H,\,\mathbf{u}_H\Big) \neq 0\,,$$

and proving existence of solutions of (0.17), (0.18) is not so obvious. For the reader's convenience we give a proof of existence in the Appendix. ▫

**Remark 0.2.** If one uses Newton's linearization around $\mathbf{u}_H$ in the full nonlinear equation, one obtains, instead of (0.19):

$$\begin{aligned} &\forall\, \mathbf{v}_h \in X_h\,, \\ &\nu(\nabla \mathbf{u}_h, \nabla \mathbf{v}_h) + \Big((\mathbf{u}_h \cdot \nabla)\mathbf{u}_H, \mathbf{v}_h\Big) + \Big((\mathbf{u}_H \cdot \nabla)\mathbf{u}_h, \mathbf{v}_h\Big) - (p_h, \operatorname{div} \mathbf{v}_h) \;= \\ &\qquad\qquad = \;\langle \mathbf{f}, \mathbf{v}_h\rangle + \Big((\mathbf{u}_H \cdot \nabla)\mathbf{u}_H, \mathbf{v}_h\Big)\;, \end{aligned} \tag{0.21}$$

and of course this is completed by (0.20). But in constrast to the system (0.17), (0.18), (0.19), (0.20) that converges without restriction on the data (we refer to the convergence proof in the appendix), it seems difficult to establish convergence if we replace (0.19) by (0.21), because we have no suitable estimate for $((\mathbf{u}_h \cdot \nabla)\mathbf{u}_H, \mathbf{v}_h)$. This is not surprizing, since in general, the proof of convergence of Newton's method uses the fact that the exact solution is isolated. In this respect, (0.19) is simpler and more robust, and hence preferable.

Newton's method is applied in a somewhat similar framework for optimal control problems by Niemistö in [27]. □

From the computational point of view, the fine grid is usually obtained by refining the coarse grid, because computing integrals over pieces of tetrahedra (or even triangles) is highly time-consuming. Thus we usually have $X_H \subset X_h$ and $M_H \subset M_h$. The coupled system (0.19),(0.20) is decoupled by a gradient algorithm (such as the Uzawa's algorithm) with $p_H$ as starting value. As in the example above, we shall see further on that the error analysis of (0.19) involves $\mathbf{u}_H$ only in the factor $\|\mathbf{u} - \mathbf{u}_H\|_{L^3(\Omega)}$. But a duality argument shows that, by using finite elements of the lowest degree, if the solution $(\mathbf{u}, p)$ belongs to $H^2(\Omega)^3 \times H^1(\Omega)$, we can obtain without extra assumption on $\Omega$ (cf. Theorem 1.1 hereafter)

$$\|\mathbf{u} - \mathbf{u}_H\|_{L^3(\Omega)} \;\leq\; C_1 H^{3/2}\;. \tag{0.22}$$

Thus, if $h$ and $H$ are related by

$$h = H^{3/2}\;, \tag{0.23}$$

the error of step two is:

$$\|\nabla(\mathbf{u} - \mathbf{u}_h)\|_{L^2(\Omega)} + \|p - p_h\|_{L^2(\Omega)} \;\leq\; C_2\, h\;.$$

For instance, if $H = 2^{-4}$, then $h = 2^{-6}$. As a consequence, on one hand the size of the nonlinear problem in Step One can be substantially reduced, and on the other hand by starting with $p_H$, the algorithm for computing $p_h$ in Step Two converges faster.

We shall perform the error analysis under the assumption that problem (0.14)–(0.16) has a nonsingular solution $(\mathbf{u}, p) \in H^2(\Omega)^3 \times H^1(\Omega)$ (see (2.1),(2.2) for the definition of a nonsingular solution), and use the approach of Brezzi, Rappaz & Raviart [8]. This avoids imposing restrictive sufficient conditions on the data $\mathbf{f}$ and $\nu$ that guarantee uniqueness. But of course, the assumption of nonsingularity requires some knowledge of the solution. Interestingly, apart from this regularity hypothesis on the solution $(\mathbf{u}, p)$, we shall need no assumption on the domain for proving (0.23): all the numerical analysis will be done on a Lipschitz polyhedron and regular triangulations. The crucial point is the duality argument of Proposition 1.5 that relies on Theorem 1.1, which is a sharp regularity result due to Costabel & Dauge [11], generalizing the work of Fabes, Kenig & Verchotta [13] for the three-dimensional Stokes problem on a Lipschitz domain.

**Remark 0.3.** Increasing the exponent of $H$ in (0.23) does not appear possible without some condition on the angles of the domain; in this respect, (0.23) seems optimal. In contrast, in a convex polyhedron, we shall prove in Section 4 that we can take

$$h = H^2 \ . \square \tag{0.24}$$

**Remark 0.4.** A simpler argument than that of Theorem 1.1 yields a slightly deteriorated version of (0.23): we obtain $h = H^{3/2-\varepsilon}$ for arbitrary $\varepsilon > 0$. □

**Remark 0.5.** In two dimensions, the analogue of Theorem 1.1 is given by formula (3.12), that is due to Grisvard [16]. □

The technique used in this article was introduced by Layton in [20] and by Layton & Lenferink in [21] and [22] for solving the steady Navier–Stokes problem in two and three dimensions, in the velocity-pressure formulation. The stream-function formulation was studied by Layton & Ye in [23] and by Fairag in [14]. General semilinear elliptic equations were studied by Xu in [32] and [33] and semilinear elliptic equations with semilinear constraints were studied by Niemistö in [27]. All these references treated only the problem on convex polyhedra or convex polygons. In the present article, we extend their results on the Navier–Stokes equations to arbitrary Lipschitz polyhedra (or polygons in two dimensions) and confirm their results on convex domains.

After this introduction, this article is organized as follows. In Section 1, we study a general conforming discretization of the Stokes problem and prove an $L^3$-error estimate for the velocity. The same estimate is derived in Section 2 for the corresponding discretization of the Navier–Stokes problem. The two-grid algorithm is studied in Section 3 and the particular case of a convex polyhedron is studied in Section 4. Finally, in the Appendix, we establish the convergence of the two-grid algorithm, for all data.

In the sequel, we shall use the following notation and results, stated in a domain of $\mathbb{R}^3$ whose boundary is Lipschitz-continuous (cf. [16]), referred to as a Lipschitz-continuous domain. Let $(k_1, k_2, k_3)$ denote a triple of non-negative integers, set $|k| = k_1 + k_2 + k_3$ and define the partial derivative $\partial^k$ by

$$\partial^k v \;=\; \frac{\partial^{|k|} v}{\partial x_1^{k_1}\, \partial x_2^{k_2}\, \partial x_3^{k_3}} \;.$$

Then, for any non-negative integer $m$ and number $r \geq 1$, recall the classical Sobolev space (cf. Adams [1] or Nečas [26])

$$W^{m,r}(\Omega) \;=\; \Big\{ v \in L^r(\Omega)\,;\; \partial^k v \in L^r(\Omega) \;\; \forall\, |k| \leq m \Big\} \;,$$

equipped with the seminorm

$$|v|_{W^{m,r}(\Omega)} = \left[ \sum_{|k|=m} \int_\Omega |\partial^k v|^r \, d\mathbf{x} \right]^{1/r} \;,$$

and norm (for which it is a Banach space)

$$\|v\|_{W^{m,r}(\Omega)} = \left[ \sum_{0 \leq |k| \leq m} |v|^r_{W^{m,r}(\Omega)} \right]^{1/r} \;,$$

with the usual extension when $r = \infty$. The reader can refer to Lions & Magenes [25] and [16] for extensions of this definition to non-integral values of $m$. When $r = 2$, this space is the Hilbert space $H^m(\Omega)$. In particular, the scalar product of $L^2(\Omega)$ is denoted by $(\cdot,\cdot)$. The definitions of these spaces are extended straightforwardly to vectors, with the same notation, but with the following modification for the norms in the non-Hilbert case. Let $\mathbf{u} = (u_1, u_2, u_3)$; then we set

$$\|\mathbf{u}\|_{L^r(\Omega)} = \left[ \int_\Omega \|\mathbf{u}(\mathbf{x})\|^r \, d\mathbf{x} \right]^{1/r} \;,$$

where $\|\cdot\|$ denotes the Euclidean vector norm.

For vanishing boundary values, we define

$$H_0^1(\Omega) = \Big\{ v \in H^1(\Omega)\,;\; v|_{\partial\Omega} = 0 \Big\} \;,$$

and its dual space, $H^{-1}(\Omega)$. We shall often use Sobolev's imbeddings: in three dimensions, for any real number $1 \le p \le 6$, there exists a constant $S_p$ such that

$$\forall\, v \in H_0^1(\Omega)\,, \quad \|v\|_{L^p(\Omega)} \le S_p\, |v|_{H^1(\Omega)}\ . \tag{0.25}$$

In two dimensions, (0.25) is valid for any finite $p$. When $p = 2$, in any dimension, (0.25) reduces to Poincaré's inequality and $S_2$ is Poincaré's constant. Owing to Poincaré's inequality, we use the seminorm $|\cdot|_{H^1(\Omega)}$ on $H_0^1(\Omega)$ to define the dual norm:

$$\|f\|_{H^{-1}(\Omega)} = \sup_{v \in H_0^1(\Omega)} \frac{\langle f, v\rangle}{|v|_{H^1(\Omega)}}\ ,$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

We shall also use the standard spaces for Navier–Stokes equations

$$\begin{aligned}
V &= \Big\{\mathbf{v} \in H_0^1(\Omega)^3\,;\ \operatorname{div}\mathbf{v} = 0 \ \text{ in } \Omega\Big\}\ ,\\
V^\perp &= \Big\{\mathbf{v} \in H_0^1(\Omega)^3\,;\ \forall\, \mathbf{w} \in V,\ (\nabla\mathbf{v}, \nabla\mathbf{w}) = 0\Big\}\ ,\\
L_0^2(\Omega) &= \Big\{q \in L^2(\Omega)\,;\ \int_\Omega q\, d\mathbf{x} = 0\Big\}\ .
\end{aligned}$$

## 1 – Conforming discretization of the Stokes problem

In this section, we derive by duality an approximation result for the Stokes problem. Recall the homogeneous, normalized Stokes problem (with viscosity $\nu = 1$), called Problem T: For $\mathbf{f}$ given in $H^{-1}(\Omega)^3$, find $\mathbf{u}$ in $H_0^1(\Omega)^3$ and $p$ in $L_0^2(\Omega)$, solution of

$$-\Delta\,\mathbf{u} + \nabla p = \mathbf{f} \qquad \text{in } \ \Omega\ , \tag{1.1}$$

$$\operatorname{div}\mathbf{u} = 0 \qquad \text{in } \ \Omega\ . \tag{1.2}$$

It is well-known (cf. Girault & Raviart [15]) that Problem T has the two equivalent variational formulations:

**1.** Find $(\mathbf{u}, p) \in H_0^1(\Omega)^3 \times L_0^2(\Omega)$, such that

$$\forall\, \mathbf{v} \in H_0^1(\Omega)^3\,, \quad (\nabla\mathbf{u}, \nabla\mathbf{v}) - (p, \operatorname{div}\mathbf{v}) = \langle\mathbf{f}, \mathbf{v}\rangle\ , \tag{1.3}$$

$$\forall\, q \in L_0^2(\Omega)\,, \qquad (q, \operatorname{div}\mathbf{u}) = 0\ . \tag{1.4}$$

**2.** Find $\mathbf{u} \in V$ such that

$$\forall \mathbf{v} \in V\,, \quad (\nabla \mathbf{u}, \nabla \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle\ . \tag{1.5}$$

Problem T has a unique solution $(\mathbf{u}, p)$ that depends continuously on $\mathbf{f}$:

$$|\mathbf{u}|_{H^1(\Omega)} \le \|\mathbf{f}\|_{H^{-1}(\Omega)}\,, \quad \|p\|_{L^2(\Omega)} \le \frac{1}{\beta}\, \|\mathbf{f}\|_{H^{-1}(\Omega)}\ , \tag{1.6}$$

where $\frac{1}{\beta} > 0$ is the constant of the divergence isomorphism of $V^\perp$ onto $L^2_0(\Omega)$:

$$\forall \mathbf{v} \in V^\perp\,, \quad |\mathbf{v}|_{H^1(\Omega)} \le \frac{1}{\beta}\, \|\mathrm{div}\, \mathbf{v}\|_{L^2(\Omega)}\ . \tag{1.7}$$

This is equivalent to the inf-sup condition (cf. Babuška [3], Brezzi [6], [15] or Brenner & Scott [5]):

$$\forall q \in L^2_0(\Omega)\,, \quad \sup_{\mathbf{v} \in H^1_0(\Omega)^3} \frac{1}{|\mathbf{v}|_{H^1(\Omega)}} \int_\Omega q\, \mathrm{div}\, \mathbf{v}\, d\mathbf{x} \ \ge\ \beta\, \|q\|_{L^2(\Omega)}\ . \tag{1.8}$$

Thus, setting $u = (\mathbf{u}, p)$, Problem T defines a linear, continuous mapping:

$$T\colon\, \mathbf{f} \mapsto u \ =\ T(\mathbf{f})\,, \quad T \in \mathcal{L}(H^{-1}(\Omega)^3; H^1_0(\Omega)^3 \times L^2_0(\Omega))\ . \tag{1.9}$$

The forthcoming analysis is based on the following crucial regularity result. It extends a similar result of [13] where $\mathbf{f} = \mathbf{0}$, but $\mathbf{u} = \mathbf{g}$ on $\partial\Omega$ with $\mathbf{g} \in H^1(\partial\Omega)^3$.

**Theorem 1.1.** *Let $\Omega$ be a Lipschitz polyhedron. Then*

$$T\ \in\ \mathcal{L}(L^{3/2}(\Omega)^3; H^{3/2}(\Omega)^3 \times H^{1/2}(\Omega))\ . \tag{1.10}$$

**Proof:** This proof is due to Costabel & Dauge [11]. Let $\mathbf{f}$ be given in $L^{3/2}(\Omega)^3$ and let $u = (\mathbf{u}, p) = T(\mathbf{f})$.

**1)** Let $\tilde{\mathbf{f}}$ be the extension of $\mathbf{f}$ by zero outside $\Omega$ and let $w = (\mathbf{w}, q)$ be the restriction to $\Omega$ of $(\mathbf{U} \star \tilde{\mathbf{f}}, \mathbf{q} \star \tilde{\mathbf{f}})$ where $(\mathbf{U}, \mathbf{q})$ is the elementary solution of the Stokes problem in $\mathbb{R}^3$:

$$-\Delta\, \mathbf{U}_i + \nabla q_i = \delta\, \mathbf{e}_i, \quad \mathrm{div}\, \mathbf{U}_i = 0\,, \quad i = 1, 2, 3\ .$$

Since $\tilde{\mathbf{f}}$ belongs to $L^{3/2}(\mathbb{R}^3)^3$ and has compact support, then $(\mathbf{w}, q)$ belongs to $W^{2,3/2}(\Omega)^3 \times W^{1,3/2}(\Omega)$ with continuous dependence on $\mathbf{f}$. Furthermore, in $\mathbb{R}^3$,

$$W^{2,3/2}(\Omega) \subset H^{3/2}(\Omega) \quad \text{(note that the exponent 3/2 is sharp)}\ .$$

But $v = (\mathbf{v}, r) = w - u$ satisfies

$$-\Delta \mathbf{v} + \nabla r = \mathbf{0}, \quad \operatorname{div} \mathbf{v} = 0 \quad \text{in } \Omega ,$$
$$\mathbf{v}|_{\partial\Omega} = \mathbf{w}|_{\partial\Omega} .$$

Therefore, owing to [13], we shall have

$$v \in H^{3/2}(\Omega)^3 \times H^{1/2}(\Omega) ,$$

which together with the above imbedding implies (1.10), provided we show that

$$\mathbf{w}|_{\partial\Omega} \in H^1(\partial\Omega)^3 .$$

Hence the proof reduces to establishing that, on a Lipschitz polyhedron $\Omega$, the trace on $\partial\Omega$ of a function of $W^{2,3/2}(\Omega)$ belongs to $H^1(\partial\Omega)$, and by the above imbedding, it suffices to consider the trace on $\partial\Omega$ of a function of $H^{3/2}(\Omega)$.

**2**) Now, the fact that $\Omega$ is a Lipschitz polyhedron implies that the space of traces of $H^{3/2}(\Omega)$ on $\partial\Omega$ is precisely $H^1(\partial\Omega)$. Indeed, on a Lipschitz polyhedron $\Omega$, the local charts defining the boundary $\partial\Omega$ are piecewise affine. The trace of a function of $H^{3/2}(\Omega)$ on each face $F$ of $\partial\Omega$ belongs to $H^1(F)$ and the trace on each edge $A$ shared by two adjacent faces $F_1$ and $F_2$ coincide. Therefore, after composing the trace with the local chart that "flattens" $\partial\Omega$ in a neighborhood of an edge or a corner, we obtain a function defined on a polygonal subdivision of the plane, that is $H^1$ in each polygon and is continuous across the edges of adjacent polygons. Such a function belongs globally to $H^1$.

Note that this result is not valid on an arbitrary Lipschitz domain. We refer to Jerrison & Kenig [18], Proposition 3.2, for a counter-example on a plane domain with a $\mathcal{C}^1$ boundary. ∎

**Remark 1.2.** When $\Omega$ is a smooth bounded domain (which is not the case here!), Theorem 1.1 follows from a regularity result of Cattabriga [9]:

$$T \in \mathcal{L}(L^2(\Omega)^3; H^2(\Omega)^3 \times H^1(\Omega)) . \tag{1.11}$$

By using interpolation theory (cf. [25]), it follows from (1.9) and (1.11) that

$$T \in \mathcal{L}(H^{-1/2}(\Omega)^3; H^{3/2}(\Omega)^3 \times H^{1/2}(\Omega)) .$$

As a consequence of the fractional imbedding theorem, we have $H^{1/2}(\Omega) \subset L^3(\Omega)$. This amounts to $L^{3/2}(\Omega) \subset H^{-1/2}(\Omega)$ and we recover precisely (1.10). But this proof is not valid when $\Omega$ is a Lipschitz polyhedron. □

**Remark 1.3.** The proof of Theorem 1.1 is not trivial, because it is a limiting case. It extends the result of Dauge in [12] stating that if $\mathbf{f}$ belongs to $L^{3/2}(\Omega)^3$ then $(\mathbf{u}, p)$ belongs to $H^{3/2-\varepsilon}(\Omega)^3 \times H^{1/2-\varepsilon}(\Omega)$ for arbitrary $\varepsilon > 0$. Thus it gives estimates independent of $\varepsilon$. □

Now, we consider a general conforming finite-element discretization of Problem T. For simplicity, we restrict the discussion to conforming methods, but all the subsequent analysis should be easily extended to non-conforming methods. Let $h > 0$ be a discretization parameter, that will tend to zero, and for each $h$, let $\mathcal{T}_h$ be a family of regular triangulations of $\overline{\Omega}$, consisting of tetrahedra with diameters bounded by $h$. As usual, any pair of tetrahedra of $\mathcal{T}_h$ are either disjoint or share a whole face, a whole edge or a vertex. For any tetrahedron $K$, we denote by $h_K$ the diameter of $K$ and by $\rho_K$ the diameter of its inscribed sphere. By regular we mean (cf. Ciarlet [10]): there exists a constant $\sigma > 0$, independent of $h$ such that

$$\forall\, K \in \mathcal{T}_h\,, \quad \frac{h_K}{\rho_K} = \sigma_K \le \sigma\,. \tag{1.12}$$

Next, let $X_h \subset H_0^1(\Omega)^3$ and $M_h \subset L_0^2(\Omega)$ be two finite-element spaces satisfying a uniform inf-sup condition: there exists a constant $\beta^* > 0$, independent of $h$, such that:

$$\forall\, q_h \in M_h\,, \quad \sup_{\mathbf{v}_h \in X_h} \frac{1}{|\mathbf{v}_h|_{H^1(\Omega)}} \int_\Omega q_h \operatorname{div} \mathbf{v}_h \, d\mathbf{x} \;\ge\; \beta^* \, \|q_h\|_{L^2(\Omega)}\,. \tag{1.13}$$

In addition, we suppose that $X_h$ and $M_h$ are at least of order one, i.e. there exist operators $\Pi_h \in \mathcal{L}(H_0^1(\Omega)^3; X_h)$ and $r_h \in \mathcal{L}(L_0^2(\Omega); M_h)$ such that, for $m = 0, 1$ and $k = 1, 2$,

$$\forall\, \mathbf{v} \in \left(H^k(\Omega) \cap H_0^1(\Omega)\right)^3, \quad |\Pi_h(\mathbf{v}) - \mathbf{v}|_{H^m(\Omega)} \;\le\; C\, h^{k-m}\, |\mathbf{v}|_{H^k(\Omega)}\,, \tag{1.14}$$

$$\forall\, q \in H^k(\Omega) \cap L_0^2(\Omega), \quad \|r_h(q) - q\|_{L^2(\Omega)} \;\le\; C\, h^k\, |q|_{H^k(\Omega)}\,, \quad k = 0, 1\,. \tag{1.15}$$

By interpolating (1.14) between $k = 1$ and 2 and (1.15) between $k = 0$ and 1 (cf. [25]), we extend (1.14) to any real number $s \in [1, 2]$ and (1.15) to $s \in [0, 1]$:

$$\begin{aligned} &\forall\, \mathbf{v} \in \left(H^s(\Omega) \cap H_0^1(\Omega)\right)^3, \\ &\qquad |\Pi_h(\mathbf{v}) - \mathbf{v}|_{H^m(\Omega)} \;\le\; C\, h^{s-m}\, |\mathbf{v}|_{H^s(\Omega)}\,, \quad m = 0, 1\,, \end{aligned} \tag{1.16}$$

$$\forall\, q \in H^s(\Omega) \cap L_0^2(\Omega)\,, \quad \|r_h(q) - q\|_{L^2(\Omega)} \;\le\; C\, h^s\, |q|_{H^s(\Omega)}\,. \tag{1.17}$$

We define the discrete analogue of $V$:

$$V_h \;=\; \left\{ \mathbf{v}_h \in X_h\,;\; \forall\, q_h \in M_h,\; \int_\Omega q_h \operatorname{div} \mathbf{v}_h \, d\mathbf{x} = 0 \right\}. \tag{1.18}$$

Usually, $V_h$ is not a subspace of $V$, unless the finite elements in $M_h$ have sufficiently high degree (cf. Scott & Vogelius [29]), which is seldom the case in practice. However as a consequence of the inf-sup condition (1.13), there exists an operator $P_h \in \mathcal{L}(H_0^1(\Omega)^3; X_h)$ such that (cf. [15]):

$$\forall \mathbf{v} \in H_0^1(\Omega)^3, \ \ \forall q_h \in M_h , \qquad \int_\Omega q_h \operatorname{div}\Big(P_h(\mathbf{v}) - \mathbf{v}\Big)\, d\mathbf{x} \,=\, 0 \,, \tag{1.19}$$

and

$$\forall \mathbf{v} \in H_0^1(\Omega)^3 , \qquad |P_h(\mathbf{v}) - \mathbf{v}|_{H^1(\Omega)} \,\le\, \left(1 + \frac{\sqrt{3}}{\beta^*}\right) |\Pi_h(\mathbf{v}) - \mathbf{v}|_{H^1(\Omega)} \,, \tag{1.20}$$

so that in particular, $P_h \in \mathcal{L}(V; V_h)$. Constructions of $P_h$ are sketched in the proofs of Lemmas 3.3, 4.5 and 4.6. Then we discretize the Stokes Problem T by:

Find a pair $(\mathbf{u}_h, p_h) \in X_h \times M_h$ solution of:

$$\forall \mathbf{v}_h \in X_h\,, \quad (\nabla \mathbf{u}_h, \nabla \mathbf{v}_h) - (p_h, \operatorname{div} \mathbf{v}_h) = \langle \mathbf{f}, \mathbf{v}_h \rangle \,, \tag{1.21}$$

$$\forall q \in M_h\,, \qquad (q_h, \operatorname{div} \mathbf{u}_h) = 0 \,. \tag{1.22}$$

Owing to (1.13), this problem has a unique solution that depends continuously on $\mathbf{f}$:

$$|\mathbf{u}_h|_{H^1(\Omega)} \le \|\mathbf{f}\|_{H^{-1}(\Omega)}\,, \qquad \|p_h\|_{L^2(\Omega)} \le \frac{1}{\beta^*}\, \|\mathbf{f}\|_{H^{-1}(\Omega)} \,. \tag{1.23}$$

Moreover, it is equivalent to:

Find $\mathbf{u}_h \in V_h$ such that

$$\forall \mathbf{v}_h \in V_h\,, \qquad (\nabla \mathbf{u}_h, \nabla \mathbf{v}_h) = \langle \mathbf{f}, \mathbf{v}_h \rangle \,. \tag{1.24}$$

Similarly, setting $u_h = (\mathbf{u}_h, p_h)$, this discrete problem defines a linear, continuous mapping:

$$T_h \colon \mathbf{f} \mapsto u_h \,=\, T_h(\mathbf{f})\,, \qquad T_h \in \mathcal{L}(H^{-1}(\Omega)^3; X_h \times M_h) \,. \tag{1.25}$$

The next propositions give an upper bound for the error $u - u_h$ (cf. for instance [15]).

**Proposition 1.4.** *If the inf-sup condition (1.13) holds, we have*

$$|\mathbf{u} - \mathbf{u}_h|_{H^1(\Omega)} \,\le\, 2\left(1 + \frac{\sqrt{3}}{\beta^*}\right) |\mathbf{u} - \Pi_h(\mathbf{u})|_{H^1(\Omega)} + \sqrt{3}\, \|p - r_h(p)\|_{L^2(\Omega)} \,, \tag{1.26}$$

$$\|p - p_h\|_{L^2(\Omega)} \,\le\, \left(1 + \frac{\sqrt{3}}{\beta^*}\right) \left(\|p - r_h(p)\|_{L^2(\Omega)} + \frac{1}{\beta^*}\, |\mathbf{u} - \Pi_h(\mathbf{u})|_{H^1(\Omega)}\right) . \tag{1.27}$$

**Proposition 1.5.** *Let $\Omega$ be a Lipschitz polyhedron and assume the triangulation and finite-element spaces satisfy (1.12), (1.13), (1.16) and (1.17). Let $\mathbf{u}$ be the solution of (1.5) and $\mathbf{u}_h$ the solution of (1.24). Then there exists a constant $C$ independent of $h$ such that*

$$\|\mathbf{u}-\mathbf{u}_h\|_{L^3(\Omega)} \le C\,h^{1/2}\Big(|\mathbf{u}-\mathbf{u}_h|_{H^1(\Omega)} + \inf_{s_h\in M_h}\|p-s_h\|_{L^2(\Omega)}\Big)\,. \tag{1.28}$$

**Proof:** The proof is an easy variant of a duality result that can be found in [15], Chapter II, Section 1, p. 119. By duality, we have

$$\|\mathbf{u}-\mathbf{u}_h\|_{L^3(\Omega)} = \sup_{\mathbf{w}\in L^{3/2}(\Omega)^3}\frac{(\mathbf{u}-\mathbf{u}_h,\mathbf{w})}{\|\mathbf{w}\|_{L^{3/2}(\Omega)}}\,. \tag{1.29}$$

Let $v=(\mathbf{v},q)=T(\mathbf{w})$; then

$$(\mathbf{u}-\mathbf{u}_h,\mathbf{w}) = \Big(\nabla(\mathbf{u}-\mathbf{u}_h),\nabla\mathbf{v}\Big) - \Big(\mathrm{div}(\mathbf{u}-\mathbf{u}_h),q\Big)\,.$$

Therefore, we have for any $\mathbf{v}_h\in V_h$ and $q_h\in M_h$:

$$(\mathbf{u}-\mathbf{u}_h,\mathbf{w}) = \Big(\nabla(\mathbf{u}-\mathbf{u}_h),\nabla(\mathbf{v}-\mathbf{v}_h)\Big)+\Big(\nabla(\mathbf{u}-\mathbf{u}_h),\nabla\mathbf{v}_h\Big)-\Big(\mathrm{div}(\mathbf{u}-\mathbf{u}_h),q-q_h\Big)\,.$$

But (1.5) and (1.24) imply that

$$\forall\,\mathbf{v}_h\in V_h,\ \ \forall\,s_h\in M_h\,,\qquad \Big(\nabla(\mathbf{u}-\mathbf{u}_h),\nabla\mathbf{v}_h\Big) = \Big(p-s_h,\ \mathrm{div}(\mathbf{v}_h-\mathbf{v})\Big)\,.$$

Hence

$$\begin{aligned}(\mathbf{u}-\mathbf{u}_h,\mathbf{w}) &= \Big(\nabla(\mathbf{u}-\mathbf{u}_h),\nabla(\mathbf{v}-\mathbf{v}_h)\Big)\\ &\quad+\Big(p-s_h,\mathrm{div}(\mathbf{v}_h-\mathbf{v})\Big)-\Big(\mathrm{div}(\mathbf{u}-\mathbf{u}_h),q-q_h\Big)\,.\end{aligned} \tag{1.30}$$

Now, Theorem 1.1 implies that $(\mathbf{v},q)\in H^{3/2}(\Omega)^3\times H^{1/2}(\Omega)$ with continuous dependence on $\mathbf{w}$. Therefore, (1.20) and (1.16) with $s=3/2$, and (1.17) with $s=1/2$ yield:

$$|\mathbf{v}-P_h(\mathbf{v})|_{H^1(\Omega)} \le C\,h^{1/2}\,|\mathbf{v}|_{H^{3/2}(\Omega)} \le C\,h^{1/2}\,\|\mathbf{w}\|_{L^{3/2}(\Omega)}\,,$$

$$\|q-r_h(q)\|_{L^2(\Omega)} \le C\,h^{1/2}\,|q|_{H^{1/2}(\Omega)} \le C\,h^{1/2}\,\|\mathbf{w}\|_{L^{3/2}(\Omega)}\,,$$

and (1.28) follows by substituting these two inequalities into (1.30). ∎

The first part of the following corollary follows immediately by substituting (1.6) and (1.23) into (1.28); its second part follows by substituting (1.14) with $k=2$ and (1.15) with $k=1$ into (1.26) and the result into (1.28).

**Corollary 1.6.** *Under the assumptions of Proposition 1.5, we have*

$$\|\mathbf{u} - \mathbf{u}_h\|_{L^3(\Omega)} \leq C\, h^{1/2}\, \|\mathbf{f}\|_{H^{-1}(\Omega)} \; . \tag{1.31}$$

*If in addition,* $(\mathbf{u}, p) \in H^2(\Omega)^3 \times H^1(\Omega)$ *then*

$$\|\mathbf{u} - \mathbf{u}_h\|_{L^3(\Omega)} \leq C\, h^{3/2} \Big( |\mathbf{u}|_{H^2(\Omega)} + |p|_{H^1(\Omega)} \Big) \; . \tag{1.32}$$

## 2 – General discretization of the Navier–Stokes problem

The results in this section are valid for the general discretization of Section 1; they will be applied in the next section to the discretization of the Navier–Stokes problem on a coarse mesh.

Let us write the exact problem (0.14)–(0.16) as a perturbation of the Stokes problem:

$$u = \left( \mathbf{u}, \frac{p}{\nu} \right), \qquad F(u) = 0 \; ,$$

where

$$F(u) = u + T\, G(u) \, , \qquad G(u) = \frac{1}{\nu} \Big( (\mathbf{u} \cdot \nabla)\, \mathbf{u} - \mathbf{f} \Big) \; .$$

Note that $G(u)$ does not depend on the second argument of $u$; hence to simplify, we denote it by $G(\mathbf{u})$. Following [8] we define a nonsingular solution $(\mathbf{u}, p)$ of (0.14)–(0.16) by the two conditions:

$$F(u) = 0 \; , \tag{2.1}$$

(2.2) $\quad F'(u)$ is an isomorphism of $X \times M$ where $X = H_0^1(\Omega)^3, \;\; M = L_0^2(\Omega)$ .

Here $F'(u) = I + T\, G'(\mathbf{u})$ and

$$G'(\mathbf{u}) \cdot \mathbf{v} \; = \; \frac{1}{\nu} \Big( (\mathbf{u} \cdot \nabla)\, \mathbf{v} + (\mathbf{v} \cdot \nabla)\, \mathbf{u} \Big) \; .$$

Let $T_h$ be the Stokes discretization operator defined in (1.25). We retain the hypotheses on the triangulation and finite-element spaces of Section 1 and we discretize $F$ by the mapping

$$F_h(u) \; = \; u + T_h\, G(\mathbf{u}) \; . \tag{2.3}$$

Note that $F_h$ maps $X \times M$ into itself and $X_h \times M_h$ also into itself. Then we discretize (2.1) by:

Find $u_h = (\mathbf{u}_h, p_h/\nu)$ in $X_h \times M_h$ solution of

$$F_h(u_h) = 0 \; . \tag{2.4}$$

When expanded, (2.4) gives the nonlinear system of equations:

$$\forall \mathbf{v}_h \in X_h\,, \quad \nu(\nabla \mathbf{u}_h, \nabla \mathbf{v}_h) + \Big((\mathbf{u}_h \cdot \nabla)\mathbf{u}_h, \mathbf{v}_h\Big) - (p_h, \operatorname{div} \mathbf{v}_h) \;=\; \langle \mathbf{f}, \mathbf{v}_h \rangle\ , \tag{2.5}$$

$$\forall q_h \in M_h\,, \quad (q_h, \operatorname{div} \mathbf{u}_h) = 0\ . \tag{2.6}$$

**Remark 2.1.** In all practical situations, efficient algorithms are obtained by this direct discretization of the nonlinear term (cf. for instance Gunzburger [17], Pironneau [28]). This form is preferred because in the time-dependent problem, which is what ultimately one has to solve, we have

$$\frac{d\,\mathbf{u}}{d\,t} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\,\mathbf{u}\ .$$

But in theory, the discrete nonlinear term is written in an antisymmetric form (cf. for instance [31]):

$$\frac{1}{2}\bigg(\Big((\mathbf{u}_h \cdot \nabla)\mathbf{u}_h, \mathbf{v}_h\Big) - \Big((\mathbf{u}_h \cdot \nabla)\mathbf{v}_h, \mathbf{u}_h\Big)\bigg)\ ,$$

so that it vanishes when $\mathbf{v}_h = \mathbf{u}_h$. In fact, this is not necessary; for the reader's convenience, we prove in the Appendix existence of solutions of (2.5), (2.6) for $h$ small enough and their strong convergence in $X \times M$, without restriction on the data $\mathbf{f}$ and $\nu$. □

The error analysis of (2.5), (2.6) cannot be done without additional assumptions. This can be easily seen by taking the difference between (0.14) and (2.5). Either we impose on the size of the data the same restriction that guarantees uniqueness (cf. [24], [31], or [15]), or we allow for multiple solutions and we assume that the particular solution we want to discretize is nonsingular, as in [15]. We shall adopt here the second option, because it is less restrictive. Thus, from now on, we assume that $u$ is a nonsingular solution of (0.14)–(0.16). Then, according to [15], there exists $\eta_0 > 0$ such that for all $h \le \eta_0$, (2.4) has a nonsingular solution $u_h$ that is unique in a neighborhood $\mathcal{O}$ of $u$, and the radius of $\mathcal{O}$ can be bounded below by a constant independent of $h$. Moreover, there exists a constant $K$, also independent of $h$, such that

$$\|u - u_h\|_{X \times M} \;\le\; K\, \|(T - T_h)\, G(\mathbf{u})\|_{X \times M}\ . \tag{2.7}$$

**Remark 2.2.** The fact that $u = -T\,G(\mathbf{u})$ means that $(T - T_h)G(\mathbf{u})$ is the discretization error of a Stokes problem whose solution is $u$. This is used in estimating the discretization error of the Navier–Stokes problem. □

**Remark 2.3.** Since $u_h$ is a nonsingular solution of (2.4), Newton's method is well-adapted to its numerical computation. The algorithm is (compare with (0.21)):

$$\forall\, \mathbf{v}_h \in X_h\ ,$$

$$\nu(\nabla \mathbf{u}_h^{n+1}, \nabla \mathbf{v}_h) + \Big((\mathbf{u}_h^n \cdot \nabla)\mathbf{u}_h^{n+1}, \mathbf{v}_h\Big) + \Big((\mathbf{u}_h^{n+1} \cdot \nabla)\mathbf{u}_h^n, \mathbf{v}_h\Big) - (p_h^{n+1}, \operatorname{div} \mathbf{v}_h) = \tag{2.8}$$

$$= \langle \mathbf{f}, \mathbf{v}_h \rangle + \Big((\mathbf{u}_h^n \cdot \nabla)\mathbf{u}_h^n, \mathbf{v}_h\Big)\ ,$$

$$\forall\, q_h \in M_h\ , \qquad (q_h, \operatorname{div} \mathbf{u}_h^{n+1}) = 0\ . \tag{2.9}$$

The theory in [15] can be easily extended to prove that, for all $h$ sufficiently small, the algorithm (2.8), (2.9) defines a unique sequence $(\mathbf{u}_h^n, p_h^n)$ provided the starting value $(\mathbf{u}_h^0, p_h^0)$ belongs to a neighborhood of $u$, whose radius can be bounded below independently of $h$. The sequence $(\mathbf{u}_h^n, p_h^n)$ converges to $(\mathbf{u}_h, p_h)$ in $X_h \times M_h$ and the convergence is quadratic. □

The next theorem extends the $L^3$ estimate of Proposition 1.5 to the solution of the Navier–Stokes equation.

**Theorem 2.4.** *Suppose that the triangulation $\mathcal{T}_h$ satisfies (1.12) and the finite element spaces $X_h$ and $M_h$ satisfy (1.13), (1.16) and (1.17). Let $u$ be a nonsingular solution of (0.14)–(0.16) and let $u_h$ be the nonsingular solution of (2.4) in a neighborhood of $u$. Then there exists $\eta_1$ with $0 < \eta_1 \le \eta_0$, such that for all $h \le \eta_1$, the first argument of $u - u_h$ satisfies:*

$$\|\mathbf{u} - \mathbf{u}_h\|_{L^3(\Omega)} \le C\Big(|\mathbf{u} - \mathbf{u}_h|^2_{H^1(\Omega)} + \|(T - T_h)\, G(\mathbf{u})\|_{L^3(\Omega)}\Big)\ , \tag{2.10}$$

*with a constant $C$ independent of $h$.*

**Proof:** Let us check the assumptions of Theorem 3.5, p. 310 of [15]. First, the imbedding of $H^1(\Omega)$ into $L^3(\Omega)$ is compact.

Second, for all $\mathbf{v}$ in $H^1(\Omega)^3$, $G'(\mathbf{v})$ belongs to $\mathcal{L}(L^3(\Omega)^3; H^{-1}(\Omega)^3)$. Indeed, take $\mathbf{v}$ in $H^1(\Omega)^3$, $\mathbf{w}$ in $L^3(\Omega)^3$ and $\mathbf{z}$ in $H_0^1(\Omega)^3$. Since $H^1(\Omega)$ is continuously imbedded into $L^6(\Omega)$, the product $v_i z_j$ belongs to $W_0^{1,3/2}(\Omega)$; and therefore the duality product $\langle (\mathbf{v} \cdot \nabla)\mathbf{w}, \mathbf{z} \rangle$ is well-defined. Similarly, the product $(\mathbf{w} \cdot \nabla) v_i$ belongs to $L^{6/5}(\Omega)$ and hence the integral $((\mathbf{w} \cdot \nabla)\mathbf{v}, \mathbf{z})$ is also well-defined. Therefore, $G'(\mathbf{v}) \cdot \mathbf{w}$ belongs to $H^{-1}(\Omega)^3$ and there exists a constant $C$ such that for all $\mathbf{v} \in H^1(\Omega)^3$, $\mathbf{w} \in L^3(\Omega)^3$ and $\mathbf{z} \in H_0^1(\Omega)^3$, we have

$$|\langle G'(\mathbf{v}) \cdot \mathbf{w}, \mathbf{z} \rangle| \le C\, \|\mathbf{w}\|_{L^3(\Omega)}\, \|\mathbf{v}\|_{H^1(\Omega)}\, |\mathbf{z}|_{H^1(\Omega)}\ .$$

Third, the first part of Corollary 1.6 implies that, restricting $T$ and $T_h$ to their first argument (i.e. formulating $T$ by (1.5) and $T_h$ by (1.24)), we have

$$\|T - T_h\|_{\mathcal{L}(H^{-1}(\Omega)^3;L^3(\Omega)^3)} \le C\, h^{1/2} \,. \tag{2.11}$$

Finally, let us prove that $F'(\mathbf{u})$, restricted to its first argument, is an isomorphism of $L^3(\Omega)^3$ onto itself. We have seen that $T\,G'(\mathbf{u})$ belongs to $\mathcal{L}(L^3(\Omega)^3; H_0^1(\Omega)^3)$. Therefore, $T\,G'(\mathbf{u})$ is a compact operator of $L^3(\Omega)^3$ into itself. Hence, we can apply Fredholm's alternative: $F'(\mathbf{u}) = I + T\,G'(\mathbf{u})$ is an isomorphism of $L^3(\Omega)^3$ onto itself if the equation:

$$\mathbf{v} \in L^3(\Omega)^3\,, \qquad F'(\mathbf{u})\cdot\mathbf{v} = \mathbf{0}\,, \tag{2.12}$$

implies $\mathbf{v} = \mathbf{0}$. But from the expression of $F'(\mathbf{u})$, every solution $\mathbf{v}$ of (2.12) belongs to $H_0^1(\Omega)^3$. And by assumption, $F'(\mathbf{u})$, restricted to its first argument, is an isomorphism of $H_0^1(\Omega)^3$ onto itself. Hence $\mathbf{v} = \mathbf{0}$.

Then the conclusion of Theorem 3.5, p. 310 of [15] yields (2.10). ∎

**Remark 2.5.** The condition $h \le \eta_0$ ensures that $u_h$ is a nonsingular solution of (2.4). And the condition $h \le \eta_1$ ensures that ${F'}_h(\mathbf{u})$ is also an isomorphism of $L^3(\Omega)^3$ onto itself. But in view of the rate of convergence in (2.11), $\eta_1$ will not be much smaller than $\eta_0$. □

**Corollary 2.6.** *In addition to the hypotheses of Theorem 2.4, we suppose that $\mathbf{f} \in L^{3/2}(\Omega)^3$ and $h \le \eta_1$. Then there exists a constant $C_1(\mathbf{f})$ independent of $h$, such that*

$$\|\mathbf{u} - \mathbf{u}_h\|_{L^3(\Omega)} \le C_1(\mathbf{f})\, h\,. \tag{2.13}$$

*In addition, if $(\mathbf{u}, p) \in H^2(\Omega)^3 \times H^1(\Omega)$ (in which case, $\mathbf{f} \in L^2(\Omega)^3$), then there exists another constant $C(\mathbf{u}, p)$, independent of $h$, such that*

$$\|\mathbf{u} - \mathbf{u}_h\|_{L^3(\Omega)} \le C(\mathbf{u}, p)\, h^{3/2}\,. \tag{2.14}$$

**Proof:** Since $\mathbf{u}$ belongs to $H^1(\Omega)^3$, then $(\mathbf{u}\cdot\nabla)\mathbf{u}$ is in $L^{3/2}(\Omega)^3$ and the assumption on $\mathbf{f}$ implies that $u$ is the solution of a homogeneous Stokes problem with right-hand side in $L^{3/2}(\Omega)^3$. Thus, according to the regularity Theorem 1.1, we have $\mathbf{u} \in H^{3/2}(\Omega)^3$ and $p \in H^{1/2}(\Omega)$. Therefore, (2.7), (1.26), (1.27), (1.16) with $m = 1$ and $s = 3/2$ and (1.17) with $s = 1/2$ yield:

$$|\mathbf{u} - \mathbf{u}_h|_{H^1(\Omega)} \le C\, h^{1/2}\,.$$

Similarly, substituting this estimate into (1.28), we obtain

$$\|\mathbf{u} - \mathbf{u}_h\|_{L^3(\Omega)} \le C\,h\ ,$$

thus proving (2.13). Then the additional regularity of $(\mathbf{u}, p)$ yields (2.14). ∎

## 3 – Two-grid algorithm

Let $H > h$ be two mesh-sizes that will tend to zero and for each $H$ and $h$, let $\mathcal{T}_H$ and $\mathcal{T}_h$ be two families of triangulations satisfying (1.12), $\mathcal{T}_h$ being derived from $\mathcal{T}_H$ by a suitable mesh refinement. Let $X_H$ and $M_H$, and $X_h$ and $M_h$ be two pairs of finite-element spaces respectively defined on $\mathcal{T}_H$ and $\mathcal{T}_h$, in such a way that $X_H \subset X_h$ and $M_H \subset M_h$. We assume that the pair $(X_h, M_h)$ satisfies the approximation properties (1.16) and (1.17) and the inf-sup condition (1.13) with $\beta^* > 0$ independent of $h$. Thus the pair $(X_H, M_H)$ also satisfies (1.13) with the same constant $\beta^*$ and (1.16), (1.17) with $H$ instead of $h$.

**Remark 3.1.** For the theoretical analysis below, the pairs $(X_H, M_H)$ and $(X_h, M_h)$ can be chosen independently of each other, provided they satisfy (1.13), (1.16) and (1.17). For instance, $X_H$ and $X_h$ need not be finite-element spaces of the same degree. □

Recall the two-grid algorithm described in the Introduction.

- Find $(\mathbf{u}_H, p_H) \in X_H \times M_H$, solution of (0.17), (0.18):

$$\forall\,\mathbf{v}_H \in X_H\ , \quad \nu(\nabla\mathbf{u}_H, \nabla\mathbf{v}_H) + \Big((\mathbf{u}_H\cdot\nabla)\mathbf{u}_H, \mathbf{v}_H\Big) - (p_H, \operatorname{div}\mathbf{v}_H) = \langle\mathbf{f}, \mathbf{v}_H\rangle\ ,$$

$$\forall\,q_H \in M_H\ , \quad (q_H, \operatorname{div}\mathbf{u}_H) = 0\ .$$

- Find $(\mathbf{u}_h, p_h) \in X_h \times M_h$, solution of (0.19), (0.20):

$$\forall\,\mathbf{v}_h \in X_h\ , \quad \nu(\nabla\mathbf{u}_h, \nabla\mathbf{v}_h) + \Big((\mathbf{u}_H\cdot\nabla)\mathbf{u}_h, \mathbf{v}_h\Big) - (p_h, \operatorname{div}\mathbf{v}_h) = \langle\mathbf{f}, \mathbf{v}_h\rangle\ ,$$

$$\forall\,q_h \in M_h\ , \quad (q_h, \operatorname{div}\mathbf{u}_h) = 0\ .$$

Now, we assume that $\mathbf{f} \in L^{3/2}(\Omega)^3$ and $u = (\mathbf{u}, p/\nu)$ is a nonsingular solution of (0.14)–(0.16). Let $H \le \eta_1$, where $\eta_1 > 0$ is the constant of Theorem 2.4, and let $u_H = (\mathbf{u}_H, p_H/\nu)$ be the nonsingular solution of (0.17), (0.18). Then (2.13) becomes:

$$\|\mathbf{u} - \mathbf{u}_H\|_{L^3(\Omega)} \le C_1(\mathbf{f})\,H\ . \tag{3.1}$$

The next lemma establishes that the linearized problem (0.19), (0.20) is well-posed.

**Lemma 3.2.** *In addition to the above assumptions, we suppose that*

$$H \le \frac{\nu}{2\,C_2}\,, \tag{3.2}$$

*where $C_2 = C_1(\mathbf{f})\,S_6$ and $S_6$ is the Sobolev constant of (0.25). Then (0.19), (0.20) has a unique solution $(\mathbf{u}_h, p_h)$ and*

$$|\mathbf{u}_h|_{H^1(\Omega)} \le \frac{2}{\nu}\,\|\mathbf{f}\|_{H^{-1}(\Omega)}\,, \qquad \|p_h\|_{L^2(\Omega)} \le C(\mathbf{f},\nu,\beta^*)\,, \tag{3.3}$$

*where $C(\mathbf{f},\nu,\beta^*)$ is independent of $h$ and $H$.*

**Proof:** Let us show that the bilinear form in the left-hand side of (0.19) is uniformly elliptic. As $\operatorname{div}\mathbf{u} = 0$, we can write

$$a(\mathbf{v}_h,\mathbf{v}_h) = \nu\,|\mathbf{v}_h|^2_{H^1(\Omega)} + \Big(\big((\mathbf{u}_H - \mathbf{u})\cdot\nabla\big)\mathbf{v}_h, \mathbf{v}_h\Big)\,.$$

But

$$\begin{aligned}\Big|\Big(\big((\mathbf{u}_H - \mathbf{u})\cdot\nabla\big)\mathbf{v}_h, \mathbf{v}_h\Big)\Big| &\le \|\mathbf{u}_H - \mathbf{u}\|_{L^3(\Omega)}\,|\mathbf{v}_h|_{H^1(\Omega)}\,\|\mathbf{v}_h\|_{L^6(\Omega)}\\ &\le C_1(\mathbf{f})\,S_6\,H\,|\mathbf{v}_h|^2_{H^1(\Omega)}\,.\end{aligned}$$

Hence if (3.2) holds,

$$a(\mathbf{v}_h,\mathbf{v}_h) \ge \nu\,|\mathbf{v}_h|^2_{H^1(\Omega)} - C_2\,H\,|\mathbf{v}_h|^2_{H^1(\Omega)} \ge \frac{\nu}{2}\,|\mathbf{v}_h|^2_{H^1(\Omega)}\,.$$

Therefore, problem (0.19), (0.20) has a unique solution satisfying the first estimate of (3.3). The bound for $p_h$ stems from the following consequence of (1.13): for any $p_h \in M_h$, there exists a unique $\mathbf{v}_h \in V_h^{\perp}$, i.e.

$$\forall\,\mathbf{w}_h \in V_h\,, \qquad (\nabla\mathbf{v}_h, \nabla\mathbf{w}_h) = 0\,,$$

such that

$$(p_h, \operatorname{div}\mathbf{v}_h) = \|p_h\|^2_{L^2(\Omega)}\,, \qquad |\mathbf{v}_h|_{H^1(\Omega)} \le \frac{1}{\beta^*}\,\|p_h\|_{L^2(\Omega)}\,. \ \blacksquare \tag{3.4}$$

For estimating the error of (0.19), (0.20), we need to sharpen (1.20).

**Lemma 3.3.** *Under the assumptions (1.12), (1.13), (1.16) and (1.17), there exists an operator* $P_h \in \mathcal{L}(H_0^1(\Omega)^3; X_h)$ *satisfying (1.19), (1.20) and for any* $\mathbf{v} \in H_0^1(\Omega)^3$*:*

$$\begin{aligned} \|P_h(\mathbf{v}) - \mathbf{v}\|_{L^3(\Omega)} \; &\leq \; \|\Pi_h(\mathbf{v}) - \mathbf{v}\|_{L^3(\Omega)} \\ &\quad + C\Big(\|\Pi_h(\mathbf{v}) - \mathbf{v}\|_{H^{1/2}(\Omega)} + h^{1/2}|\Pi_h(\mathbf{v}) - \mathbf{v}|_{H^1(\Omega)}\Big) \, . \end{aligned} \tag{3.5}$$

**Proof:** By virtue of (1.13), we can construct $P_h$ as follows:

$$P_h(\mathbf{v}) \; = \; \Pi_h(\mathbf{v}) - \mathbf{c}_h \; ,$$

where the correction $\mathbf{c}_h$ is the unique solution in $V_h^{\perp}$ of

$$\forall\, \mu_h \in M_h \, , \qquad (\mu_h, \operatorname{div} \mathbf{c}_h) = \Big(\mu_h, \operatorname{div}(\Pi_h(\mathbf{v}) - \mathbf{v})\Big) \; ,$$

and $\mathbf{c}_h$ satisfies:

$$|\mathbf{c}_h|_{H^1(\Omega)} \; \leq \; \frac{1}{\beta^*} \Big\| \operatorname{div}(\Pi_h(\mathbf{v}) - \mathbf{v}) \Big\|_{L^2(\Omega)} \, .$$

Now, we proceed by duality as in Proposition 1.5: owing to Theorem 1.1, for any $\mathbf{f} \in L^{3/2}(\Omega)^3$, the solution $(\mathbf{y}, \lambda) = T(\mathbf{f})$ belongs to $H^{3/2}(\Omega)^3 \times H^{1/2}(\Omega)$, with continuous dependence on $\mathbf{f}$. Therefore, we have, for any $\mathbf{y}_h \in V_h$ and any $\lambda_h \in M_h$:

$$(\mathbf{c}_h, \mathbf{f}) \; = \; \Big(\nabla \mathbf{c}_h, \nabla(\mathbf{y} - \mathbf{y}_h)\Big) - (\operatorname{div} \mathbf{c}_h, \lambda - \lambda_h) - \Big(\operatorname{div}(\Pi_h(\mathbf{v}) - \mathbf{v}), \lambda_h\Big) \, .$$

We write the last term as

$$\Big(\operatorname{div}(\Pi_h(\mathbf{v}) - \mathbf{v}), \lambda_h\Big) \; = \; \Big(\operatorname{div}(\Pi_h(\mathbf{v}) - \mathbf{v}), \lambda_h - \lambda\Big) - \langle \nabla \lambda, \Pi_h(\mathbf{v}) - \mathbf{v} \rangle \; ,$$

where the duality product is taken between $H^{-1}(\Omega)^3$ and $H_0^1(\Omega)^3$. Now, we choose $\lambda_h = r_h(\lambda)$, and since $\mathbf{y}$ belongs to $V$, we can choose $\mathbf{y}_h = P_h(\mathbf{y}) \in V_h$. Thus, (1.17), (1.16), (1.20) and (1.10) yield:

$$\|\lambda - r_h(\lambda)\|_{L^2(\Omega)} + |\mathbf{y} - P_h(\mathbf{y})|_{H^1(\Omega)} \; \leq \; C\, h^{1/2}\, \|\mathbf{f}\|_{L^{3/2}(\Omega)} \, .$$

Hence

$$\|\mathbf{c}_h\|_{L^3(\Omega)} \; \leq \; C\, h^{1/2} \Big(|\mathbf{c}_h|_{H^1(\Omega)} + \|\operatorname{div}(\Pi_h(\mathbf{v}) - \mathbf{v})\|_{L^2(\Omega)}\Big) + C\, \|\Pi_h(\mathbf{v}) - \mathbf{v}\|_{H^{1/2}(\Omega)} \; ,$$

whence (3.5). ∎

**Theorem 3.4.** *Under the assumptions of Lemma 3.2, the solution* $(\mathbf{u}_h, p_h)$ *of (0.19), (0.20) satisfies the error bound:*

$$
\begin{aligned}
|\mathbf{u}-\mathbf{u}_h|_{H^1(\Omega)} \;\le\; & 2\,|\mathbf{u}-P_h(\mathbf{u})|_{H^1(\Omega)} + \frac{2\,S_6}{\nu^2}\,\|\mathbf{f}\|_{H^{-1}(\Omega)}\,\|\mathbf{u}-\mathbf{u}_H\|_{L^3(\Omega)} \\
& + \frac{1}{\nu}\,\|\mathbf{u}\|_{L^6(\Omega)}\,\|\mathbf{u}-P_h(\mathbf{u})\|_{L^3(\Omega)} + \frac{\sqrt{3}}{\nu}\,\|p-r_h(p)\|_{L^2(\Omega)}\;,
\end{aligned}
\tag{3.6}
$$

$$
\begin{aligned}
\|p-p_h\|_{L^2(\Omega)} \;\le\; & \Big(1+\frac{\sqrt{3}}{\beta^*}\Big)\,\|p-r_h(p)\|_{L^2(\Omega)} \\
& + \frac{1}{\beta^*}\Big(\nu\,|\mathbf{u}-P_h(\mathbf{u})|_{H^1(\Omega)} + \frac{2\,S_6}{\nu}\,\|\mathbf{f}\|_{H^{-1}(\Omega)}\,\|\mathbf{u}-\mathbf{u}_H\|_{L^3(\Omega)} \\
& + \|\mathbf{u}\|_{L^6(\Omega)}\,\|\mathbf{u}-\mathbf{u}_h\|_{L^3(\Omega)}\Big)\;.
\end{aligned}
\tag{3.7}
$$

**Proof:** Taking the difference between (0.14) multiplied by a test function $\mathbf{v}_h \in V_h$ and (0.19), and inserting any $\mathbf{w}_h \in V_h$ and $q_h \in M_h$, we obtain

$$
\begin{aligned}
\nu\Big(\nabla(\mathbf{w}_h-\mathbf{u}_h),\nabla\mathbf{v}_h\Big) \;=\; & \nu\Big(\nabla(\mathbf{w}_h-\mathbf{u}),\nabla\mathbf{v}_h\Big) + \Big(\big((\mathbf{u}_H-\mathbf{u})\cdot\nabla\big)\mathbf{u}_h,\mathbf{v}_h\Big) \\
& + \Big((\mathbf{u}\cdot\nabla)(\mathbf{u}_h-\mathbf{w}_h),\mathbf{v}_h\Big) + \Big((\mathbf{u}\cdot\nabla)(\mathbf{w}_h-\mathbf{u}),\mathbf{v}_h\Big) \\
& + (\operatorname{div}\mathbf{v}_h,\, p-q_h)\;.
\end{aligned}
\tag{3.8}
$$

Therefore, the choice $\mathbf{v}_h = \mathbf{u}_h - \mathbf{w}_h \in V_h$ gives

$$
\begin{aligned}
\nu|\mathbf{w}_h-\mathbf{u}_h|_{H^1(\Omega)} \;\le\; & \nu\,|\mathbf{w}_h-\mathbf{u}|_{H^1(\Omega)} + S_6\,\|\mathbf{u}-\mathbf{u}_H\|_{L^3(\Omega)}\,|\mathbf{u}_h|_{H^1(\Omega)} \\
& + \|\mathbf{u}\|_{L^6(\Omega)}\,\|\mathbf{u}-\mathbf{w}_h\|_{L^3(\Omega)} + \sqrt{3}\,\|p-q_h\|_{L^2(\Omega)}\;.
\end{aligned}
$$

Then the choices $\mathbf{w}_h = P_h(\mathbf{u})$ and $q_h = r_h(p)$, together with (3.3) yield (3.6).

Similarly, we derive for any $\mathbf{v}_h \in X_h$ and $q_h \in M_h$:

$$
\begin{aligned}
(\operatorname{div}\mathbf{v}_h,\, q_h-p_h) \;=\; & (\operatorname{div}\mathbf{v}_h,\, q_h-p) + \nu\Big(\nabla(\mathbf{u}-\mathbf{u}_h),\nabla\mathbf{v}_h\Big) \\
& + \Big(\big((\mathbf{u}-\mathbf{u}_H)\cdot\nabla\big)\mathbf{u}_h,\,\mathbf{v}_h\Big) + \Big((\mathbf{u}\cdot\nabla)(\mathbf{u}-\mathbf{u}_h),\,\mathbf{v}_h\Big)\;.
\end{aligned}
\tag{3.9}
$$

Let us choose the function $\mathbf{v}_h \in V_h^{\perp}$ associated with $q_h - p_h$ by (3.4); this gives

$$
\begin{aligned}
\|p_h-q_h\|_{L^2(\Omega)} \;\le\; & \frac{\sqrt{3}}{\beta^*}\,\|p-q_h\|_{L^2(\Omega)} + \frac{\nu}{\beta^*}\,|\mathbf{u}-P_h(\mathbf{u})|_{H^1(\Omega)} \\
& + \frac{S_6}{\beta^*}\,|\mathbf{u}_h|_{H^1(\Omega)}\,\|\mathbf{u}-\mathbf{u}_H\|_{L^3(\Omega)} + \frac{1}{\beta^*}\,\|\mathbf{u}\|_{L^6(\Omega)}\,\|\mathbf{u}-\mathbf{u}_h\|_{L^3(\Omega)}\;.
\end{aligned}
$$

Then (3.7) follows from this inequality, the choice $q_h = r_h(p)$ and (3.3). ∎

The following corollary states the main result of this section. Its proof is an easy consequence of Theorem 3.4.

**Corollary 3.5.** *In addition to the hypotheses of Lemma 3.2, suppose that* $(\mathbf{u}, p)$ *belongs to* $H^2(\Omega)^3 \times H^1(\Omega)$. *Then the solution* $(\mathbf{u}_h, p_h)$ *of (0.19), (0.20) satisfies the error bound:*

$$
\begin{aligned}
|\mathbf{u}-\mathbf{u}_h|_{H^1(\Omega)} \;&\le\; C\,h\Big(|\mathbf{u}|_{H^2(\Omega)} + |p|_{H^1(\Omega)} + h^{1/2}\,\|\mathbf{u}\|_{L^6(\Omega)}\,|\mathbf{u}|_{H^2(\Omega)}\Big)\\
&\quad + 2\,\frac{S_6}{\nu^2}\,C(\mathbf{u},p)\,H^{3/2}\,\|\mathbf{f}\|_{H^{-1}(\Omega)}\;,
\end{aligned}
\tag{3.10}
$$

$$
\begin{aligned}
\|p-p_h\|_{L^2(\Omega)} \;&\le\; C\Big[h\Big(|\mathbf{u}|_{H^2(\Omega)} + |p|_{H^1(\Omega)}\Big) + \|\mathbf{u}\|_{L^6(\Omega)}\,\|\mathbf{u}-\mathbf{u}_h\|_{L^3(\Omega)}\Big]\\
&\quad + 2\,\frac{S_6}{\nu\,\beta^*}\,C(\mathbf{u},p)\,H^{3/2}\,\|\mathbf{f}\|_{H^{-1}(\Omega)}\;.
\end{aligned}
\tag{3.11}
$$

*Both errors are of the order of* $h$, *if we take* $h = H^{3/2}$.

**Remark 3.6.** The situation is simpler in two dimensions because we can replace Theorem 1.1 by the following result of Grisvard [16]: in a Lipschitz polygon $\Omega$,

$$
T \,\in\, \mathcal{L}\Big(L^{4/3}(\Omega)^2;\, W^{2,4/3}(\Omega)^2 \times W^{1,4/3}(\Omega)\Big)\;.
\tag{3.12}
$$

Since $W^{1,4/3}(\Omega)$ and $W^{2,4/3}(\Omega)$ are continuously imbedded into $H^{1/2}(\Omega)$ and $H^{3/2}(\Omega)$ respectively, we can extend the analysis of the previous sections to the two-dimensional case, provided we assume in Corollary 2.6 and in Section 3 that $\mathbf{f} \in L^{4/3}(\Omega)^2$. The analysis is somewhat simpler and the conclusions are the same. □

## 4 – The case of a convex polyhedron

On a convex polyhedron, the following regularity result, established by Dauge [12], will enable us to derive (0.24).

**Theorem 4.1.** *If* $\Omega$ *is a convex polyhedron, then*

$$
T \,\in\, \mathcal{L}\Big(L^2(\Omega)^3;\, H^2(\Omega)^3 \times H^1(\Omega)\Big)\;.
\tag{4.1}
$$

Therefore, by interpolating between (1.9) and (4.1), we obtain for any real number $s \in [0,1]$:

$$
T \,\in\, \mathcal{L}\Big(H^{s-1}(\Omega)^3;\, H^{s+1}(\Omega)^3 \times H^s(\Omega)\Big)\;.
\tag{4.2}
$$

Let $r \in [2,3]$ and $r'$ be its dual exponent: $1/r + 1/r' = 1$. For $s = 3/r - 1/2$, we have $H^{1-s}(\Omega) \subset L^r(\Omega)$ and hence $L^{r'}(\Omega) \subset H^{s-1}(\Omega)$. With these considerations, the argument of Proposition 1.5 gives:

**Proposition 4.2.** *Let $\Omega$ be a convex polyhedron and assume the triangulation and finite-element spaces satisfy (1.12), (1.13), (1.16) and (1.17). Let $\mathbf{u}$ be the solution of (1.5) and $\mathbf{u}_h$ the solution of (1.24). Then there exists a constant $C$, independent of $h$, such that*

$$\|\mathbf{u} - \mathbf{u}_h\|_{L^2(\Omega)} \leq C\, h\Big(|\mathbf{u} - \mathbf{u}_h|_{H^1(\Omega)} + \inf_{s_h \in M_h} \|p - s_h\|_{L^2(\Omega)}\Big)\,. \tag{4.3}$$

*More generally, for each real number $r \in [2,3]$, there exists a constant $C$ independent of $h$ such that*

$$\|\mathbf{u} - \mathbf{u}_h\|_{L^r(\Omega)} \leq C\, h^{3/r-1/2}\Big(|\mathbf{u} - \mathbf{u}_h|_{H^1(\Omega)} + \inf_{s_h \in M_h} \|p - s_h\|_{L^2(\Omega)}\Big)\,. \tag{4.4}$$

As a consequence, (2.11) is replaced by:

$$\|T - T_h\|_{\mathcal{L}(H^{-1}(\Omega)^3; L^r(\Omega)^3)} \leq C\, h^{3/r-1/2}\,, \qquad 2 \leq r \leq 3\,. \tag{4.5}$$

Then the statement of Theorem 2.4 is replaced by:

**Theorem 4.3.** *Let $\Omega$ be a convex polyhedron and let $\mathbf{f} \in L^2(\Omega)^3$. Suppose that the triangulation $\mathcal{T}_h$ satisfies (1.12) and the finite element spaces $X_h$ and $M_h$ satisfy (1.13), (1.16) and (1.17). Let $u$ be a nonsingular solution of (0.14)–(0.16) and let $u_h$ be the nonsingular solution of (2.4) in a neighborhood of $u$. Then, for each real number $r \in [2,3]$, there exists $\eta_1$ with $0 < \eta_1 \leq \eta_0$, such that for all $h \leq \eta_1$, the first argument of $u - u_h$ satisfies, with a constant $C$ independent of $h$:*

$$\|\mathbf{u} - \mathbf{u}_h\|_{L^r(\Omega)} \leq C\Big(|\mathbf{u} - \mathbf{u}_h|^2_{H^1(\Omega)} + \|(T - T_h)\, G(\mathbf{u})\|_{L^r(\Omega)}\Big)\,. \tag{4.6}$$

*Thus, there exists a constant $C_r(\mathbf{f})$, independent of $h$, such that*

$$\|\mathbf{u} - \mathbf{u}_h\|_{L^r(\Omega)} \leq C_r(\mathbf{f})\, h^{1/2+3/r}\,, \qquad 2 \leq r \leq 3\,. \tag{4.7}$$

The only difference with the proof of Theorem 2.4 is that, when $r = 2$, $G'(\mathbf{u})$ must belong to $\mathcal{L}(L^2(\Omega)^3; H^{-1}(\Omega)^3)$. For this reason, we suppose from the onset that $\mathbf{f}$ belongs to $L^2(\Omega)^3$, thus ensuring that $\mathbf{u} \in H^2(\Omega)^3$. Of course, that much smoothness is not necessary for proving (4.6) and this is only a simplification.

Now, let us analyze again the error of (0.19), (0.20). For the analysis below, we need sharper properties of the approximation operator $P_h$. To this end, we assume that there exists an operator $P_h \in \mathcal{L}(H_0^1(\Omega)^3; X_h)$ that satisfies (1.19), next for $k = 0$ or 1,

$$\forall \mathbf{v} \in [H^{1+k}(\Omega) \cap H_0^1(\Omega)]^3, \quad \|P_h(\mathbf{v}) - \mathbf{v}\|_{L^2(\Omega)} \leq C\, h^{1+k}\, |\mathbf{v}|_{H^{1+k}(\Omega)}\ , \tag{4.8}$$

and for any real number $p \geq 2$, $k = 0$ or 1,

$$\forall \mathbf{v} \in [W^{1+k,p}(\Omega) \cap H_0^1(\Omega)]^3, \quad |P_h(\mathbf{v}) - \mathbf{v}|_{W^{1,p}(\Omega)} \leq C_p\, h^k\, |\mathbf{v}|_{W^{1+k,p}(\Omega)}\ , \tag{4.9}$$

with constants independent of $h$. The reader will find at the end of this section a construction of $P_h$ for examples of low-degree finite-elements. With this operator $P_h$, we can extend the statement of Theorem 3.4.

**Theorem 4.4.** *In addition to the hypotheses of Theorem 4.3, assume that $P_h$ satisfies (1.19), (4.8) and (4.9). Let $u = (\mathbf{u}, p/\nu)$ be a nonsingular solution of (0.14)–(0.16); let $H \leq \eta_1$, where $\eta_1 > 0$ is the constant of Theorem 4.3 for $r = 2, 3$, and let $u_H$ be the nonsingular solution of Step One, (0.17), (0.18), in a neighborhood of $u$. If in addition,*

$$H^{3/2} \leq \frac{\nu}{2\, S_6\, C_3(\mathbf{f})}\ , \tag{4.10}$$

*where $C_r(\mathbf{f})$ is the constant of (4.7), then Step Two, (0.19), (0.20) has a unique solution $(\mathbf{u}_h, p_h)$ and*

$$\begin{aligned} |\mathbf{u} - \mathbf{u}_h|_{H^1(\Omega)} \leq\ & 3\, |\mathbf{u} - P_h(\mathbf{u})|_{H^1(\Omega)} + \frac{2\sqrt{3}}{\nu}\, \|p - r_h(p)\|_{L^2(\Omega)} \\ & + \frac{2}{\nu} \Big( C_2(\mathbf{f})\, C_3\, S_6\, H^2\, |\mathbf{u}|_{W^{1,3}(\Omega)} + C\, h^2\, \|\mathbf{u}\|_{L^\infty(\Omega)}\, |\mathbf{u}|_{H^2(\Omega)} \Big), \end{aligned} \tag{4.11}$$

*where $C_3$ is derived from (4.9) with $k = 0$ and $p = 3$ and $C$ is derived from (4.8),*

$$\begin{aligned} \|p - p_h\|_{L^2(\Omega)} \leq\ & \Big(1 + \frac{\sqrt{3}}{\beta^*}\Big)\, \|p - r_h(p)\|_{L^2(\Omega)} + \frac{1}{\beta^*} \Big( \nu\, |\mathbf{u} - P_h(\mathbf{u})|_{H^1(\Omega)} \\ & + S_6 \Big( C_3(\mathbf{f})\, H^{3/2}\, |\mathbf{u} - \mathbf{u}_h|_{H^1(\Omega)} + C_2(\mathbf{f})\, H^2\, |\mathbf{u}|_{W^{1,3}(\Omega)} \Big) \\ & + \|\mathbf{u}\|_{L^\infty(\Omega)}\, \|\mathbf{u} - \mathbf{u}_h\|_{L^2(\Omega)} \Big)\ . \end{aligned} \tag{4.12}$$

*Both errors are of the order of $h$ when $h = H^2$.*

**Proof:** Let us revert to the proof of Theorem 3.4 and estimate again the two nonlinear terms in (3.8), with $\mathbf{v}_h = \mathbf{u}_h - \mathbf{w}_h$ and $\mathbf{w}_h = P_h(\mathbf{u})$. The first term can be split as follows:

$$\Big(\big((\mathbf{u}_H - \mathbf{u})\cdot\nabla\big)\mathbf{u}_h, \mathbf{v}_h\Big) = \Big(\big((\mathbf{u}_H - \mathbf{u})\cdot\nabla\big)(\mathbf{u}_h - \mathbf{w}_h), \mathbf{v}_h\Big) + \Big(\big((\mathbf{u}_H - \mathbf{u})\cdot\nabla\big)\mathbf{w}_h, \mathbf{v}_h\Big),$$

$$\begin{aligned}\left|\Big(\big((\mathbf{u}_H - \mathbf{u})\cdot\nabla\big)\mathbf{u}_h, \mathbf{v}_h\Big)\right| \;\le\;& S_6\, \|\mathbf{u}_H - \mathbf{u}\|_{L^3(\Omega)}\, |\mathbf{v}_h|^2_{H^1(\Omega)} \\ &+ S_6\, \|\mathbf{u}_H - \mathbf{u}\|_{L^2(\Omega)}\, |P_h(\mathbf{u})|_{W^{1,3}(\Omega)}\, |\mathbf{v}_h|_{H^1(\Omega)}\,.\end{aligned}$$

Therefore, applying (4.7) with $r = 3$ and (4.10), and next (4.7) with $r = 2$ and (4.9) with $k = 0$ and $p = 3$, we obtain

$$\left|\Big(\big((\mathbf{u}_H - \mathbf{u})\cdot\nabla\big)\mathbf{u}_h, \mathbf{v}_h\Big)\right| \le \frac{\nu}{2}\, |\mathbf{v}_h|^2_{H^1(\Omega)} + C_2(\mathbf{f})\, C_3\, S_6\, H^2\, |\mathbf{u}|_{W^{1,3}(\Omega)}\, |\mathbf{v}_h|_{H^1(\Omega)}\,.$$

The second term is bounded using (4.8):

$$\begin{aligned}\Big|\big((\mathbf{u}\cdot\nabla)(\mathbf{w}_h - \mathbf{u}), \mathbf{v}_h\big)\Big| \;&\le\; \|\mathbf{u}\|_{L^\infty(\Omega)}\, \|\mathbf{u} - P_h(\mathbf{u})\|_{L^2(\Omega)}\, |\mathbf{v}_h|_{H^1(\Omega)} \\ &\le\; C\, h^2\, \|\mathbf{u}\|_{L^\infty(\Omega)}\, |\mathbf{u}|_{H^2(\Omega)}\, |\mathbf{v}_h|_{H^1(\Omega)}\,,\end{aligned}$$

and (4.11) follows from these two inequalities. Finally, the estimate (4.12) follows easily by writing the first nonlinear term of (3.9) as:

$$\begin{aligned}&\left|\Big(\big((\mathbf{u}_H - \mathbf{u})\cdot\nabla\big)\mathbf{u}_h, \mathbf{v}_h\Big)\right| = \\ &= \left|\Big(\big((\mathbf{u}_H - \mathbf{u})\cdot\nabla\big)(\mathbf{u}_h - \mathbf{u}), \mathbf{v}_h\Big) + \Big(\big((\mathbf{u}_H - \mathbf{u})\cdot\nabla\big)\mathbf{u}, \mathbf{v}_h\Big)\right| \\ &\le S_6\Big(\|\mathbf{u}_H - \mathbf{u}\|_{L^3(\Omega)}\, |\mathbf{u}_h - \mathbf{u}|_{H^1(\Omega)} + \|\mathbf{u}_H - \mathbf{u}\|_{L^2(\Omega)}\, |\mathbf{u}|_{W^{1,3}(\Omega)}\Big)\, |\mathbf{v}_h|_{H^1(\Omega)} \\ &\le S_6\Big(C_3(\mathbf{f})\, H^{3/2}\, |\mathbf{u} - \mathbf{u}_h|_{H^1(\Omega)} + C_2(\mathbf{f})\, H^2\, |\mathbf{u}|_{W^{1,3}(\Omega)}\Big)\, |\mathbf{v}_h|_{H^1(\Omega)}\,. \blacksquare\end{aligned}$$

As applications, let us consider two examples of pairs of low-order finite-element spaces $(X_h, M_h)$: on one hand, the "mini-element" introduced by Arnold, Brezzi and Fortin in [2], that uses a space of continuous functions for the pressure, and on the other hand, the Bernardi–Raugel element, introduced by Bernardi and Raugel in [4], that uses a space of piecewise constant functions for the pressure. On a regular triangulation, both elements satisfy (1.13), (1.16) and (1.17) (cf. [2], Brezzi & Fortin [7], [4] and [15]). Therefore, for applying the above analysis to the two-grid algorithm with these elements, it suffices to construct for each one an approximation operator $P_h$ satisfying (1.19), (4.8) and (4.9).

First, we consider the "mini-element". Let $\mathbb{P}_k$ denote the space of polynomials in three variables with total degree less than or equal to $k$. In each tetrahedron $K$, the pressure $p$ is a polynomial of $\mathbb{P}_1$ and each component of the velocity is the sum of a polynomial of $\mathbb{P}_1$ and a "bubble" function. Denoting the vertices of $K$ by $\mathbf{a}_i$, $1 \leq i \leq 4$, and its corresponding barycentric coordinate by $\lambda_i$, the basic bubble function $b_K$ is the polynomial of degree four

$$b_K(\mathbf{x}) \;=\; \lambda_1(\mathbf{x})\,\lambda_2(\mathbf{x})\,\lambda_3(\mathbf{x})\,\lambda_4(\mathbf{x}) \;,$$

that vanishes on the boundary of $K$. Thus, we take

$$X_h \;=\; \Big\{\mathbf{v}_h \in H_0^1(\Omega)^3\,;\;\; \forall K \in \mathcal{T}_h\,,\;\mathbf{v}_h|_K \in (\mathbb{P}_1 \oplus \mathrm{Vect}(b_K))^3\Big\}\;, \tag{4.13}$$

$$M_h \;=\; \Big\{q_h \in H^1(\Omega) \cap L_0^2(\Omega)\,;\;\; \forall K \in \mathcal{T}_h\,,\; q_h|_K \in \mathbb{P}_1\Big\}\;. \tag{4.14}$$

We define similarly $X_H$ and $M_H$ on $\mathcal{T}_H$, replacing $h$ by $H$. The next lemma constructs $P_h$ for the "mini-element".

**Lemma 4.5.** *Let $\Omega$ be a Lipschitz polyhedron. If the triangulation $\mathcal{T}_h$ satisfies (1.12), there exists an operator $P_h \in \mathcal{L}(H_0^1(\Omega)^3; X_h)$ satisfying (1.19), (4.8) and (4.9), with constants independent of $h$.*

**Proof:** Take $\mathbf{v}$ in $H_0^1(\Omega)^3$ and let $\Pi_h$ be a regularization operator such as the Scott & Zhang [30] operator that is globally continuous and is a polynomial of $\mathbb{P}_1$ in each tetrahedron. We choose

$$P_h(\mathbf{v}) \;=\; \Pi_h(\mathbf{v}) \;-\; \sum_{K \in \mathcal{T}_h} \mathbf{c}_K\, b_K \;, \tag{4.15}$$

where the constants $\mathbf{c}_K$ are adjusted so that $P_h$ satisfies (1.19):

$$\forall\, q_h \in M_h\,, \qquad \int_\Omega \mathrm{div}(P_h(\mathbf{v}) - \mathbf{v})\, q_h \;d\mathbf{x} \;=\; 0\;.$$

But $q_h$ belongs to $M_h$ and by construction, $P_h(\mathbf{v}) - \mathbf{v}$ vanishes on the boundary of $\Omega$, therefore, this amounts to

$$\forall\, q_h \in M_h\,, \qquad \int_\Omega (P_h(\mathbf{v}) - \mathbf{v}) \cdot \nabla q_h \;d\mathbf{x} \;=\; 0\;. \tag{4.16}$$

Now, $\nabla q_h$ is a constant vector in each tetrahedron $K$. Therefore, (4.16) holds provided

$$\forall\, K \in \mathcal{T}_h\,, \qquad \int_K (P_h(\mathbf{v}) - \mathbf{v}) \;d\mathbf{x} \;=\; \mathbf{0}\;.$$

From the definition (4.15) of $P_h$ and the disjoint supports of the bubble functions, this last equation determines the constants $\mathbf{c}_K$:

$$\forall K \in \mathcal{T}_h\,, \quad \mathbf{c}_K = \frac{1}{\int_K b_K\, d\mathbf{x}} \int_K (\Pi_h(\mathbf{v}) - \mathbf{v})\, d\mathbf{x}\,. \tag{4.17}$$

Let us estimate first $\|\mathbf{c}_K\|$ and next $\|b_K\|_{L^2(\Omega)}$ and $|b_K|_{W^{1,p}(K)}$. Let $\hat{K}$ be the unit reference tetrahedron, $B_K$ the matrix of the affine transformation that maps $\hat{K}$ onto $K$ and $\hat{b}_{\hat{K}}$ the bubble function on $\hat{K}$. On one hand, for any $p \geq 2$,

$$\|\mathbf{c}_K\| \leq \hat{c}\,|K|^{-1/p}\,\|\Pi_h(\mathbf{v}) - \mathbf{v}\|_{L^p(K)}\,,$$

where $\hat{c}$ denote various constants that depend only on $\hat{K}$ and the exponent $p$. On the other hand,

$$\|b_K\|_{L^2(K)} \leq \hat{c}\,|K|^{1/2}\,, \quad \|b_K\|_{W^{1,p}(K)} \leq \hat{c}\,|K|^{1/p}\,\|B_K^{-1}\|\,.$$

Therefore

$$\begin{aligned} \|\mathbf{c}_K\|\,\|b_K\|_{L^2(K)} &\leq \hat{c}\,\|\Pi_h(\mathbf{v}) - \mathbf{v}\|_{L^2(K)}\,,\\ \|\mathbf{c}_K\|\,|b_K|_{W^{1,p}(K)} &\leq \hat{c}\,\|B_K^{-1}\|\,\|\Pi_h(\mathbf{v}) - \mathbf{v}\|_{L^p(K)}\,. \end{aligned}$$

From the disjoint support of the bubble functions $b_K$, we infer that

$$\begin{aligned} \|P_H(\mathbf{v}) - \mathbf{v}\|_{L^2(K)} &\leq (1+\hat{c})\,\|\Pi_h(\mathbf{v}) - \mathbf{v}\|_{L^2(K)}\,,\\ |P_h(\mathbf{v}) - \mathbf{v}|_{W^{1,p}(K)} &\leq |\Pi_h(\mathbf{v}) - \mathbf{v}|_{W^{1,p}(K)} + \hat{c}\,\|B_K^{-1}\|\,\|\Pi_h(\mathbf{v}) - \mathbf{v}\|_{L^p(K)}\,. \end{aligned} \tag{4.18}$$

Then (4.8) (in fact, a sharper result) and (4.9) follow from (4.18), the regularity (1.12) of the triangulation and the local approximation properties of $\Pi_h$. ∎

Now we turn to the Bernardi–Raugel finite element [4]. Let $f_i$ denote the face of $K$ opposite $\mathbf{a}_i$ and let $\mathbf{n}_i$ be the unit normal vector to $f_i$ pointing outside $K$. We define

$$\mathbf{p}_{1,K} = \mathbf{n}_1\lambda_2\lambda_3\lambda_4\,, \quad \mathbf{p}_{2,K} = \mathbf{n}_2\lambda_1\lambda_3\lambda_4\,, \quad \mathbf{p}_{3,K} = \mathbf{n}_3\lambda_1\lambda_2\lambda_4\,, \quad \mathbf{p}_{4,K} = \mathbf{n}_4\lambda_1\lambda_2\lambda_3\,,$$

and we set

$$\mathcal{P}_1(K) = \mathbb{P}_1^3 \oplus \mathrm{Vect}\{\mathbf{p}_{1,K}, \mathbf{p}_{2,K}, \mathbf{p}_{3,K}, \mathbf{p}_{4,K}\}\,.$$

The finite-element spaces for the Bernardi–Raugel element are

$$\begin{aligned} X_h &= \left\{\mathbf{v}_h \in H_0^1(\Omega)^3\,;\ \forall K \in \mathcal{T}_h\,,\ \mathbf{v}_h|_K \in \mathcal{P}_1(K)\right\}\,,\\ M_h &= \left\{q_h \in L_0^2(\Omega)\,;\ \forall K \in \mathcal{T}_h\,,\ q_h|_K \in \mathbb{P}_0\right\}\,. \end{aligned}$$

**Lemma 4.6.** *Let $\Omega$ be a Lipschitz polyhedron. If the triangulation $\mathcal{T}_h$ satisfies (1.12), there exists an operator $P_h \in \mathcal{L}(H_0^1(\Omega)^3; X_h)$ satisfying (1.19), (4.8) and (4.9), with constants independent of $h$.*

**Proof:** We only sketch the proof; it is similar to that of Lemma 4.5. For $\mathbf{v}$ in $H_0^1(\Omega)^3$, we choose

$$P_h(\mathbf{v}) = \Pi_h(\mathbf{v}) - \sum_{K\in\mathcal{T}_h} \sum_{i=1}^{4} \alpha_{i,K}\, \mathbf{p}_{i,K} \ . \tag{4.19}$$

It can be easily checked that, for satisfying (1.19), it suffices to take

$$\alpha_{i,K} = \frac{1}{\int_{f_i} \lambda_j \lambda_k \lambda_l \ d\sigma} \int_{f_i} (\Pi_h(\mathbf{v}) - \mathbf{v}) \cdot \mathbf{n}_i \ d\sigma \ .$$

On one hand, passing to $\hat{K}$, applying a trace theorem on $\hat{K}$ and reverting to $K$, we find

$$|\alpha_{i,K}| \ \leq\ \hat{c}\, |K|^{-1/p} \Big( \|\Pi_h(\mathbf{v}) - \mathbf{v}\|_{L^p(K)} + \|B_K\|\, |\Pi_h(\mathbf{v}) - \mathbf{v}|_{W^{1,p}(K)} \Big) \ .$$

On the other hand,

$$|\mathbf{p}_{i,K}|_{W^{1,p}(K)} \ \leq\ \hat{c}\, |K|^{1/p}\, \|B_K^{-1}\| \ .$$

Therefore,

$$|\alpha_{i,K}|\, \|\mathbf{p}_{i,K}\|_{L^2(K)} \ \leq\ \hat{c}\Big( \|\Pi_h(\mathbf{v}) - \mathbf{v}\|_{L^2(K)} + \|B_K\|\, |\Pi_h(\mathbf{v}) - \mathbf{v}|_{H^1(K)} \Big) \ ,$$

$$|\alpha_{i,K}|\, |\mathbf{p}_{i,K}|_{W^{1,p}(K)} \ \leq\ \hat{c}\Big( |\Pi_h(\mathbf{v}) - \mathbf{v}|_{W^{1,p}(K)} + \|B_K^{-1}\|\, \|\Pi_h(\mathbf{v}) - \mathbf{v}\|_{L^p(K)} \Big) \ .$$

The proof finishes as in Lemma 4.5. ∎

**Remark 4.7.** The results of this section are valid in two dimensions on a convex polygon. The proof of the analogue of Theorem 4.1 is due to Kellog & Osborn [19] and [16]. □

## 5 – Appendix

We propose to prove that the two grid-algorithm (0.17)–(0.20), combined with successive approximations for computing $(\mathbf{u}_H, p_H)$, is always convergent. First we consider the general case of (2.5), (2.6):

$$\forall\, \mathbf{v}_h \in X_h\,, \qquad \nu(\nabla \mathbf{u}_h, \nabla \mathbf{v}_h) + \Big( (\mathbf{u}_h \cdot \nabla)\mathbf{u}_h, \mathbf{v}_h \Big) - (p_h, \operatorname{div} \mathbf{v}_h) \ =\ \langle \mathbf{f}, \mathbf{v}_h \rangle \ ,$$

$$\forall\, q_h \in M_h\,, \qquad (q_h, \operatorname{div} \mathbf{u}_h) = 0 \ .$$

The proof of convergence is based on the following upper bound for the nonlinear term.

**Lemma 5.1.** *Let $\mathcal{T}_h$ satisfy (1.12). There exists a constant $\hat{C}$, independent of $h$, such that*

$$\forall\, \mathbf{u}_h \in V_h\,, \quad \Big|\Big((\mathbf{u}_h\cdot\nabla)\mathbf{u}_h, \mathbf{u}_h\Big)\Big| \le \hat{C}\, h^r\, \|\mathrm{div}\, \mathbf{u}_h\|_{L^2(\Omega)}\, |\mathbf{u}_h|^2_{H^1(\Omega)}\,, \tag{5.1}$$

*where $r = 1-\varepsilon$, $\varepsilon > 0$, if the dimension $d = 2$ and $r = \frac{1}{2}$ if $d = 3$.*

**Proof:** Let us consider first the case where the functions of $M_h$ are globally continuous (as in the case of the "mini-element"). Since the functions of $M_h$ must be polynomials of degree at least one in each triangle or tetrahedron, the functions of $V_h$ satisfy for all functions in $\{q_h \in H^1(\Omega)\,;\, \forall\, K \in \mathcal{T}_h\,,\ q_h|_K \in \mathbb{P}_1\}$:

$$(\mathrm{div}\, \mathbf{u}_h, q_h) = 0\,. \tag{5.2}$$

Note that the mean-value of $q_h$ is not necessarily zero because $(\mathrm{div}\, \mathbf{u}_h, 1) = 0$. Then Green's formula and (5.2) imply for any such $q_h$:

$$\Big((\mathbf{u}_h\cdot\nabla)\mathbf{u}_h, \mathbf{u}_h\Big) = -\frac{1}{2}\,(\mathrm{div}\, \mathbf{u}_h, \mathbf{u}_h\cdot \mathbf{u}_h) = -\frac{1}{2}\,(\mathrm{div}\, \mathbf{u}_h,\, \mathbf{u}_h\cdot \mathbf{u}_h - q_h)\,.$$

We choose in each element $K$, with vertices $\mathbf{a}_i$:

$$q_h = \sum_{i=1}^{d+1} \lambda_i\, \mathbf{u}_h(\mathbf{a}_i)\cdot \mathbf{u}_h(\mathbf{a}_i)\,, \tag{5.3}$$

that is globally continuous and is a polynomial of degree one in each $K$. Since $\sum_{i=1}^{d+1}\lambda_i = 1$, we can write

$$\begin{aligned}\mathbf{u}_h\cdot \mathbf{u}_h - \sum_{i=1}^{d+1} \lambda_i\, \mathbf{u}_h(\mathbf{a}_i)\cdot \mathbf{u}_h(\mathbf{a}_i) &= \sum_{i=1}^{d+1}\Big(\mathbf{u}_h\cdot \mathbf{u}_h - \mathbf{u}_h(\mathbf{a}_i)\cdot \mathbf{u}_h(\mathbf{a}_i)\Big)\,\lambda_i\\ &= \sum_{i=1}^{d+1}(\mathbf{u}_h - \mathbf{u}_h(\mathbf{a}_i))\cdot(\mathbf{u}_h + \mathbf{u}_h(\mathbf{a}_i))\,\lambda_i\,.\end{aligned}$$

Therefore Hölder's inequality yields for any real number $q > 2$:

$$\begin{aligned}&\Big\|\mathbf{u}_h\cdot \mathbf{u}_h - \sum_{i=1}^{d+1} \lambda_i\, \mathbf{u}_h(\mathbf{a}_i)\cdot \mathbf{u}_h(\mathbf{a}_i)\Big\|_{L^2(K)} \le\\ &\qquad \le \sum_{i=1}^{d+1}\Big\|(\mathbf{u}_h - \mathbf{u}_h(\mathbf{a}_i))\cdot(\mathbf{u}_h + \mathbf{u}_h(\mathbf{a}_i))\Big\|_{L^2(K)}\\ &\qquad \le \sum_{i=1}^{d+1}\|\mathbf{u}_h - \mathbf{u}_h(\mathbf{a}_i)\|_{L^q(K)}\,\|\mathbf{u}_h + \mathbf{u}_h(\mathbf{a}_i)\|_{L^{\frac{2q}{q-2}}(K)}\,.\end{aligned}$$

Passing to the reference element $\hat{K}$, we obtain

$$\|\mathbf{u}_h - \mathbf{u}_h(\mathbf{a}_i)\|_{L^q(K)} \;\le\; C_1\,|K|^{1/q}\,\|\hat{\mathbf{u}} - \hat{\mathbf{u}}(\hat{\mathbf{a}}_i)\|_{L^q(\hat{K})} \;\le\; C_2\,|K|^{1/q}\,|\hat{\mathbf{u}}|_{H^1(\hat{K})}\ ,$$

owing, on one hand that the linear mapping $\hat{\mathbf{u}} \mapsto \hat{\mathbf{u}} - \hat{\mathbf{u}}(\hat{\mathbf{a}}_i)$ vanishes on the constant functions, and on the other hand that $\hat{\mathbf{u}}$ belongs to a finite-dimensional space on $\hat{K}$ in which all norms are equivalent. Hence, reverting to $K$ and using (1.12):

$$\|\mathbf{u}_h - \mathbf{u}_h(\mathbf{a}_i)\|_{L^q(K)} \;\le\; C_3\,h_K\,|K|^{1/q-1/2}\,|\mathbf{u}_h|_{H^1(K)}\ .$$

When $d = 2$, we choose $q > 2$ and almost equal to 2; when $d = 3$, we take $q = 3$ so that $\frac{2\,q}{q-2} = 6$, the maximum exponent for Sobolev's imbedding. Hence

$$\|\mathbf{u}_h - \mathbf{u}_h(\mathbf{a}_i)\|_{L^q(K)} \;\le\; C_4\,h_K^r\,|\mathbf{u}_h|_{H^1(K)}\ .$$

Similarly

$$\|\mathbf{u}_h + \mathbf{u}_h(\mathbf{a}_i)\|_{L^{\frac{2q}{q-2}}(K)} \;\le\; C_5\,\|\mathbf{u}_h\|_{L^{\frac{2q}{q-2}}(K)}\ .$$

These two inequalities imply:

$$\Big(\sum_{K\in\mathcal{T}_h}\Big\|\mathbf{u}_h\cdot\mathbf{u}_h - \sum_{i=1}^{d+1}\lambda_i\,\mathbf{u}_h(\mathbf{a}_i)\cdot\mathbf{u}_h(\mathbf{a}_i)\Big\|^2_{L^2(K)}\Big)^{1/2} \;\le$$
$$\le\; C_6\,h^r\Big(\sum_{K\in\mathcal{T}_h}|\mathbf{u}_h|^2_{H^1(K)}\,\|\mathbf{u}_h\|^2_{L^{\frac{2q}{q-2}}(K)}\Big)^{1/2}\ ,$$

and (5.1) with $\hat{C} = \frac{1}{2}\,C_6\,S_{2q/(q-2)}$, follows from this inequality, Hölder's inequality, Jensen's inequality and Sobolev imbedding applied to the second factor.

When the functions of $M_h$ are allowed to be discontinuous from one element to the next, the proof is simpler, because in each $K$, $M_h$ must contain at least the constant functions. Thus we can choose $q_h = \mathbf{c}_h\cdot\mathbf{c}_h$ with

$$\mathbf{c}_h \;=\; \frac{1}{|K|}\int_K \mathbf{u}_h\;d\mathbf{x}\ ,$$

and we arrive at the same conclusion. ∎

We construct a solution of (2.5), (2.6) by the following successive approximation algorithm: starting with a given $\mathbf{u}_h^0 \in V_h$, compute for $n \ge 1$,

(5.4) $\quad \forall\,\mathbf{v}_h \in X_h\,,\;\; \nu(\nabla\mathbf{u}_h^n, \nabla\mathbf{v}_h) + \Big((\mathbf{u}_h^{n-1}\cdot\nabla)\mathbf{u}_h^n, \mathbf{v}_h\Big) - (p_h^n, \operatorname{div}\mathbf{v}_h) \;=\; \langle\mathbf{f}, \mathbf{v}_h\rangle\ ,$

(5.5) $\quad \forall\,q_h \in M_h\,,\;\; (q_h, \operatorname{div}\mathbf{u}_h^n) = 0\ .$

**Theorem 5.2.** *If $\mathcal{T}_h$ satisfies (1.12), $\mathbf{u}_h^0 \in V_h$ satisfies*

$$\|\operatorname{div} \mathbf{u}_h^0\|_{L^2(\Omega)} < \sqrt{d}\, \frac{2}{\nu}\, \|\mathbf{f}\|_{H^{-1}(\Omega)} , \tag{5.6}$$

*(for example, $\mathbf{u}_h^0 = \mathbf{0}$) and*

$$h^r \le \frac{\nu^2}{4\sqrt{d}\, \hat{C}\, \|\mathbf{f}\|_{H^{-1}(\Omega)}} , \tag{5.7}$$

*where $r$ and $\hat{C}$ are the constants of (5.1), then (5.4), (5.5) defines a unique sequence $(\mathbf{u}_h^n, p_h^n)$ and*

$$|\mathbf{u}_h^n|_{H^1(\Omega)} < \frac{2}{\nu}\, \|\mathbf{f}\|_{H^{-1}(\Omega)} , \qquad n \ge 1 , \tag{5.8}$$

*with a similar uniform bound for $p_h^n$.*

**Proof:** First observe that for any $n \ge 0$, if

$$\|\operatorname{div} \mathbf{u}_h^n\|_{L^2(\Omega)} < \sqrt{d}\, \frac{2}{\nu}\, \|\mathbf{f}\|_{H^{-1}(\Omega)} , \tag{5.9}$$

then $\mathbf{u}_h^{n+1}$ satisfies (5.8). Indeed, (5.9), (5.1) and (5.7) imply that

$$\left|\Big((\mathbf{u}_h^n \cdot \nabla)\mathbf{u}_h^{n+1}, \mathbf{u}_h^{n+1}\Big)\right| < \hat{C}\, h^r \sqrt{d}\, \frac{2}{\nu}\, \|\mathbf{f}\|_{H^{-1}(\Omega)}\, |\mathbf{u}_h^{n+1}|^2_{H^1(\Omega)} \le \frac{\nu}{2}\, |\mathbf{u}_h^{n+1}|^2_{H^1(\Omega)} .$$

Therefore

$$|\mathbf{u}_h^{n+1}|_{H^1(\Omega)} < \frac{2}{\nu}\, \|\mathbf{f}\|_{H^{-1}(\Omega)} ,$$

and $\operatorname{div} \mathbf{u}_h^{n+1}$ also satisfies (5.9). Then the theorem follows immediately by induction. ∎

Since $\mathbf{u}_h^n$ and $p_h^n$ belong to finite-dimensional spaces, the uniform bound (5.8) shows that (a subsequence of) $\mathbf{u}_h^n$ and $p_h^n$ converge respectively in $H^1(\Omega)^d$ and $L^2(\Omega)$ to $\mathbf{u}_h \in X_h$, $p_h \in M_h$ and it is easy to prove that $(\mathbf{u}_h, p_h)$ is a solution of (2.5), (2.6). Moreover $\mathbf{u}_h$ satisfies

$$|\mathbf{u}_h|_{H^1(\Omega)} \le \frac{2}{\nu}\, \|\mathbf{f}\|_{H^{-1}(\Omega)} , \tag{5.10}$$

with a similar bound for $p_h$. The uniform bound (5.10) shows that (a subsequence of) $\mathbf{u}_h$ and $p_h$ converge weakly in $H^1(\Omega)^d$ and $L^2(\Omega)$ respectively to $\mathbf{u} \in H^1(\Omega)^d$, $p \in L^2(\Omega)$. Then a standard argument shows first that $(\mathbf{u}, p)$ satisfies (0.14)–(0.16) and next that this is a strong convergence in $H^1(\Omega)^d \times L^2(\Omega)$.

Let us apply the above conclusions to the two-grid algorithm. On one hand, the nonlinear scheme (0.17), (0.18) on the coarse grid has at least one solution for $H$ sufficiently small and this solution satisfies the bound (5.10). Therefore, it converges as $H$ tends to zero. In particular, $\operatorname{div} \mathbf{u}_H$ tends to zero. Therefore, for $H$ small enough, we have

$$\|\operatorname{div} \mathbf{u}_H\|_{L^2(\Omega)} \leq \frac{\nu}{S_4^2}\,.$$

Substituting this inequality into the linearized scheme (0.19) on the fine grid, we derive as above that $\mathbf{u}_h$ satisfies (5.10). This implies the strong convergence of the solution of the two-grid algorithm in $H^1(\Omega)^d \times L^2(\Omega)$.

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