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SHARP $\mathbf{L}^1$ STABILITY ESTIMATES FOR HYPERBOLIC CONSERVATION LAWS

P. Goatin and P.G. LeFloch

Abstract: In this paper, we introduce a generalization of Liu–Yang's weighted norm to linear and to nonlinear hyperbolic equations. Following an approach due to the second author for piecewise constant solutions, we establish sharp L^1 continuous dependence estimates for general solutions of bounded variation. Two different strategies are successfully investigated. On one hand, we justify passing to the limit in an L^1 estimate valid for piecewise constant wave-front tracking approximations. On the other hand, we use the technique of generalized characteristics and, following closely an approach by Dafermos, we derive the sharp L^1 estimate directly from the equation.

1 – Introduction

We are interested in the continuous dependence of entropy solutions to hyperbolic conservation laws

$$\partial_t u + \partial_x f(u) = 0\,, \qquad u(x,t) \in \mathbb{R}, \quad x \in \mathbb{R}, \quad t > 0\,, \tag{1.1}$$

where the flux $f\colon \mathbb{R} \to \mathbb{R}$ is a smooth and convex function. After works by Liu and Yang [22] and Dafermos [9], we aim at deriving sharp L^1 estimates of the form

$$\|u^{II}(t)-u^I(t)\|_{w(t)}+\int_s^t M(\tau;u^I,u^{II})\,d\tau \le \|u^{II}(s)-u^I(s)\|_{w(s)}\,, \quad 0\le s\le t\,, \tag{1.2}$$

for any two entropy solutions of bounded variation u^I and u^{II} of (1.1), where $\|\cdot\|_{w(t)}$ is a weighted L^1-norm equivalent to the standard L^1 norm on the real

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line. In (1.2), the positive term $M(\tau; u^I, u^{II})$ is intended to provide a sharp bound on the strict decrease of the L^1 norm. The estimate with $w \equiv 1$ and $M \equiv 0$ is of course well-known.

Recall that the fundamental problem of the continuous dependence of solutions for hyperbolic systems of conservation laws was solved recently by Bressan, Crasta and Piccoli [2]. A major contribution came from Liu and Yang [22, 23] who introduced a decreasing L^1 functional ensuring (1.2) for scalar conservation laws and systems of two equations. This research culminated in papers published simultaneously by Bressan, Liu and Yang [4], Hu and LeFloch [14], and Liu and Yang [24], which contain particularly simple proofs of the continuous dependence of entropy solutions for systems.

In the present paper, we restrict attention to scalar conservation laws and, following the approach developed by the second author (Hu and LeFloch [14] and LeFloch [18, 19]), we investigate the stability issue from the standpoint of Holmgren's and Haar's methods ([21] and the references therein). The problem under consideration is (essentially) equivalent to showing the uniqueness and L^1 stability for the following hyperbolic equation with discontinuous coefficient:

$$\partial_t \psi + \partial_x (a\,\psi) = 0\,, \qquad \psi(x,t) \in \mathbb{R}, \quad x \in \mathbb{R}, \quad t > 0\ . \tag{1.3}$$

That is, for solutions with bounded variation we aim at deriving an estimate like

$$\|\psi(t)\|_{w(t)} + \int_s^t \tilde{M}(\tau; a, \psi)\ d\tau \ \le\ \|\psi(s)\|_{w(s)}\,, \qquad 0 \le s \le t\ . \tag{1.4}$$

For the application to (1.1) one should define a by

$$a\ =\ a(u^I, u^{II})\ =\ \frac{f(u^{II}) - f(u^I)}{u^{II} - u^I}\ . \tag{1.5}$$

One may also consider the equation (1.3) for more general coefficients a.

Recall that the existence and uniqueness of solutions to the Cauchy problem associated with (1.3) was established in LeFloch [16] in the class of bounded measures, under the assumption $a_x \le E$ for some constant E. The latter holds when a is given by (1.5) (at least when u^I and u^{II} contain no rarefaction center on the line $t = 0$ which holds "generically"). See also [6, 18, 19, 21] for further existence results.

It must be observed that we restrict attention here to more regular solutions, having bounded total variation, as this is natural in view of the application to the conservation law (1.1). In this direction, recall that an L^1 stability result like (1.4) was established in [14] (see, therein, Section 5, and our Theorem 2.2 below) in the class of piecewise Lipschitz continuous solutions, with $\tilde{M} \equiv 0$ however. This uniqueness and stability result was achieved under the assumption that the coefficient a does not contain any rarefaction shock (see Section 2 below for the definition). In [14], the following essential observation was made:

$$\text{The linearized equation (1.3)–(1.5) based on two entropy solutions of (1.1) does not exhibit rarefaction shocks.} \tag{1.6}$$

(This is also true for systems of conservation laws, as far as solutions with small amplitude are concerned.) One of our aims here is to extend the L^1 stability result for (1.3) in [14] to arbitrary solutions of bounded variation.

The present paper relies also heavily on the contribution by Liu and Yang [22] who, for approximate solutions constructed by the Glimm scheme, discovered a weighted norm having a sharp decay of the form (1.2). Subsequently, the Liu–Yang's functional was extended by Dafermos ([9], Chapter 11) to arbitrary functions of bounded variation (BV) and, using the notion of generalized characteristics, Dafermos derived precisely an estimate of the form (1.2) valid for BV solutions.

The aim of this paper is to provide a new derivation and some generalization of this L^1 functional. Toward the derivation of bounds like (1.2) or (1.4), we make the following preliminary observations:

(1) As pointed out by Hu and LeFloch [14], the geometrical properties of the propagating discontinuities in a (Lax, fast or slow undercompressive, rarefaction shock, according to the terminology in [14]) play an essential role. It turns out that the (jump of the) weight $w(x,t)$ should be assigned precisely on each *undercompressive discontinuity*. On the other hand, Lax discontinuities are very stable and do not require weight, while (in exact entropy soutions) rarefaction shocks do not arise, according to (1.6).

(2) Certain (invariance) properties on the coefficient a are necessary to define the weight globally in space; see (2.9)–(2.10) in Section 2.

(3) The weight however is far from being unique and we believe that this flexibility in choosing the weight may be helpful in certain applications.

The content of this paper is as follows.

In Section 2, we consider piecewise constant solutions of (1.3) and introduce a class of weighted norms satisfying a sharp bound of the form (1.4). See Theorem 2.3 below. All undercompressive and Lax discontinuities contribute to the decrease of the L^1 norm. For the sake of comparison, we also consider the L^1 norm without weight; see Theorem 2.2.

In Section 3, we point out that the setting of Section 2 covers the case of the conservation law (1.1). Passing to the limit in wave front tracking approximations, in Theorem 3.5 we arrive to the sharp bound (1.2) for general BV solutions. The proof is based on fine convergence properties established earlier by Bressan and LeFloch [3] and on a technique of stability of nonconservative products developed by DalMaso, LeFloch, and Murat [10] and LeFloch and Liu [20].

Next, in Sections 4 and 5 we return to the equation (1.3) studied in Section 2 but, now, we deal with general BV solutions. We follow closely ideas developed by Dafermos [8, 9] for solutions of (1.1), and extend them to the linear equation (1.3). Using generalized characteristics we establish first a maximum principle in Theorem 4.5. Finally, in Theorem 5.1 using the technique of generalized characteristics, we establish the sharp L^1 stability property (1.4) directly, for general BV solutions of (1.3). The result applies in particular to the conservation law (1.1) and allows us to recover (1.2).

Throughout the paper, we always assume that all functions of bounded variation under consideration are normalized to be defined everywhere as right-continuous functions.

Finally let us recall that the undercompressive waves studied here also arise in [1, 5, 17, 19].

2 – Decreasing weighted norms for piecewise constant solutions

Given a piecewise constant function $a\colon \mathbb{R}\times\mathbb{R}_+ \to \mathbb{R}$, let us consider the linear hyperbolic equation

$$\partial_t\psi + \partial_x(a\,\psi) = 0\,, \qquad \psi(x,t) \in \mathbb{R}\;, \tag{2.1}$$

and restrict attention to piecewise constant solutions. By definition, the function a admits a set of jump points $\mathcal{J}(a)$, consisting of finitely many straightlines defined on open time intervals, together with a finite set of interaction points $\mathcal{I}(a)$, consisting of the end points of the lines in $\mathcal{J}(a)$. The function a is constant in each connected component of the complement $\mathcal{C}(a)$ of $\mathcal{I}(a)\cup\mathcal{J}(a)$. At a point $(x,t) \in \mathcal{J}(a)$ we denote by $\lambda^a = \lambda^a(x,t)$ the speed of the discontinuity and $a_\pm = a_\pm(x,t) = a(x\pm,t)$ the left- and right-hand traces. It is tacitly assumed that the discontinuity speeds λ^a remain uniformly bounded. Finally the function is normalized to be right-continuous. A similar notation is used for the function ψ.

The geometrical property of the coefficient a play a central role for the analysis of (2.1), so we recall the following terminology [14]:

Definition 2.1. A point $(x,t) \in \mathcal{J}(a)$ is called a Lax discontinuity iff

$$a_-(x,t) \;>\; \lambda^a(x,t) \;>\; a_+(x,t) \;,$$

a slow undercompressive discontinuity iff

$$\lambda^a(x,t) \;\le\; \min\Big(a_-(x,t),\, a_+(x,t)\Big) \;,$$

a fast undercompressive discontinuity iff

$$\lambda^a(x,t) \;\ge\; \max\Big(a_-(x,t),\, a_+(x,t)\Big) \;,$$

and a rarefaction-shock discontinuity iff

$$a_-(x,t) \;<\; \lambda^a(x,t) \;<\; a_+(x,t) \;. \;\square$$

For each $t > 0$, we denote by $\mathcal{L}(a)$, $\mathcal{S}(a)$, $\mathcal{F}(a)$, and $\mathcal{R}(a)$ the set of points $(x,t) \in \mathcal{J}(a)$ corresponding to Lax, slow undercompressive, fast undercompressive, and rarefaction-shock discontinuities, respectively.

Theorem 2.2. *Consider a piecewise constant speed $a = a(x,t)$. Let ψ be any piecewise constant solution of* (2.1). *Then we have for all $0 \le s \le t$*

$$\begin{aligned} \|\psi(t)\|_{L^1(\mathbb{R})} &+ \int_s^t \sum_{(x,\tau)\in\mathcal{L}(a)} 2\,\Big(a_-(x,\tau) - \lambda^a(x,\tau)\Big)\,|\psi_-(x,\tau)|\;d\tau \\ &= \|\psi(s)\|_{L^1(\mathbb{R})} + \int_s^t \sum_{(x,\tau)\in\mathcal{R}(a)} 2\,\Big(\lambda^a(x,\tau) - a_-(x,\tau)\Big)\,|\psi_-(x,\tau)|\;d\tau \;. \end{aligned} \tag{2.2}$$

In (2.2), the left-hand traces are chosen for definiteness only. Indeed it will be noticed in the proof below that for all $(x,\tau) \in \mathcal{L}(a) \cup \mathcal{R}(a)$

$$\Big(\lambda^a(x,\tau) - a_-(x,\tau)\Big)\,|\psi_-(x,\tau)| \;=\; -\Big(\lambda^a(x,\tau) - a_+(x,\tau)\Big)\,|\psi_+(x,\tau)| \;.$$

Observe that the Lax discontinuities contribute to the decrease of the L^1 norm, while the rarefaction-shocks increase it. On the other hand, the undercompressive discontinuities don't modify the L^1 norm. When a contains no rarefaction shocks (this is the case when (2.1) is a linearized equation derived from entropy solutions of a conservation law, as discovered in Hu and LeFloch [14]), Theorem 5.1 yields

$$\|\psi(t)\|_{L^1(\mathbb{R})} \;\le\; \|\psi(s)\|_{L^1(\mathbb{R})}\,, \qquad 0 \le s \le t \;, \tag{2.3}$$

where we neglected the favorable contribution of the Lax discontinuities appearing in the left-hand side of (2.2). In particular, (2.3) implies that the Cauchy problem for (2.1) admits a unique solution (in the class of piecewise constant functions at this stage), provided a has no rarefaction-shock discontinuities.

On the other hand, it is clear that the sign of the function ψ is important for the sake of deriving the L^1 stability of the solutions ψ of (2.1). For instance, if ψ has a constant sign for all (x,t), then (2.3) holds as an *equality*

$$\|\psi(t)\|_{L^1(\mathbb{R})} = \|\psi(s)\|_{L^1(\mathbb{R})}, \qquad 0 \le s \le t ,$$

which implies that the Cauchy problem for (2.1) admits at most one solution ψ of a given sign.

Proof: Denote by $\mathbf{P}(E)$ the projection of a subset E of the (x,t)-plane on the t-axis. By definition, any piecewise Lipschitz continuous solution ψ is also Lipschitz continuous in time with values in $L^1(\mathbb{R})$. So, it is enough to derive (2.2) for all $t \notin E := \mathbf{P}(\mathcal{I}(a) \cup \mathcal{I}(\psi))$. The latter is just a finite set. The following is valid in each open interval I such that $I \cap E = \emptyset$.

We denote by $x_j(t)$ for $t \in I$ and $j = 1, ..., m$ the discontinuity lines where the function $\psi(\cdot, t)$ changes sign, with the convention that

$$(-1)^j\, \psi(x,t) \ge 0 \quad \text{for } x \in [x_j(t), x_{j+1}(t)] . \tag{2.4}$$

Set $\psi_j^\pm(t) = \psi_\pm(x_j(t),t)$, $\lambda_j(t) = \lambda^a(x_j(t),t)$, etc. Then by using that ψ solves (2.1) we find (for all t in the interval I)

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}} |\psi(x,t)|\, dx &= \frac{d}{dt}\sum_{j=1}^m (-1)^j \int_{x_j(t)}^{x_{j+1}(t)} \psi(x,t)\, dx \\
&= \sum_{j=1}^m (-1)^j \left(\int_{x_j(t)}^{x_{j+1}(t)} \partial_t \psi(x,t)\, dx + \lambda_{j+1}(t)\, \psi_{j+1}^-(t) - \lambda_j(t)\, \psi_j^+(t) \right) \\
&= \sum_{j=1}^m (-1)^j \left(\int_{x_j(t)}^{x_{j+1}(t)} -\partial_x \Big(a(x,t)\, \psi(x,t) \Big)\, dx + \lambda_{j+1}(t)\, \psi_{j+1}^-(t) - \lambda_j(t)\, \psi_j^+(t) \right) \\
&= \sum_{j=1}^m (-1)^j \left(\Big(a_j^+(t) - \lambda_j(t) \Big)\, \psi_j^+(t) + \Big(a_j^-(t) - \lambda_j(t) \Big)\, \psi_j^-(t) \right) .
\end{aligned}$$

The Rankine–Hugoniot relation associated with (2.1) reads

$$\Big(a_j^+(t) - \lambda_j(t) \Big)\, \psi_j^+(t) = \Big(a_j^-(t) - \lambda_j(t) \Big)\, \psi_j^-(t) , \tag{2.5}$$

therefore by (2.4)

$$\frac{d}{dt}\int_{\mathbb{R}} |\psi(x,t)|\,dx \;=\; 2\sum_{j=1}^{m} \pm\Big(a_j^{\pm}(t)-\lambda_j(t)\Big)\,|\psi_j^{\pm}(t)|\;. \tag{2.6}$$

Consider each point $x_j(t)$ successively. If $x_j(t)$ is a Lax discontinuity, then $a_j^-(t) > \lambda_j(t) > a_+(t)$ and both coefficients $\pm(a_j^{\pm}(t) - \lambda_j(t))$ are negative. If $x_j(t)$ is a rarefaction-shock discontinuity, then $a_j^-(t) < \lambda_j(t) < a_+(t)$ and the coefficients $\pm(a_j^{\pm}(t) - \lambda_j(t))$ are positive. These two cases lead us to the two sums in (2.2). Indeed one just needs to observe the following: if (x,τ) correspond to a Lax or rarefaction-shock discontinuity of the speed a, but ψ does not change sign at (x,τ) (so it is not counted in (2.6)), then actually by the Rankine–Hugoniot relation (see (2.5)) we conclude easily that

$$\psi_-(x,\tau) \;=\; \psi_+(x,\tau) \;=\; 0\;,$$

and so it does not matter to include the point (x,τ) in the sums (2.2).

Suppose next that $x_j(t)$ is an undercompressive discontinuity. Then the two sides of (2.5) have different sign, therefore

$$\Big(a_j^+(t)-\lambda_j(t)\Big)\,\psi_j^+(t) \;=\; \Big(a_j^-(t)-\lambda_j(t)\Big)\,\psi_j^-(t) \;=\; 0\;,$$

and the corresponding term in (2.6) vanishes. ∎

Our objective now is to derive an improved version of Theorem 2.2, based on a weighted L^1 norm adapted to the equation (2.1). For piecewise constant functions, we set

$$\|\psi(t)\|_{w(t)} := \int_{\mathbb{R}} |\psi(x,t)|\,w(x,t)\;dx\;, \tag{2.7}$$

where $w = w(x,t) > 0$ is a piecewise constant and uniformly bounded function. We determine this function based on the following constraint on its jumps, at each discontinuity of the speed a,

$$w_+(x,t) - w_-(x,t) \begin{cases} \le 0 & \text{if } (x,t)\in\mathcal{S}(a),\\[4pt] \ge 0 & \text{if } (x,t)\in\mathcal{F}(a)\;. \end{cases} \tag{2.8}$$

The weight is chosen so that the left-hand trace of a slow undercompressive discontinuity and the right-hand trace of a fast one are weighted more. This is consistent with the immediate observation that the terms $(\lambda_j(t) - a_j^-(t))\,|\psi_j^-(t)|$ and $(a_j^+(t) - \lambda_j(t))\,|\psi_j^+(t)|$ have a favorable (negative) sign for slow and fast undercompressive discontinuities, respectively. On the other hand, the jumps of w at Lax or rarefaction-shock discontinuities will remain unconstrained. This choice is motivated by the two observations:

(i) Lax shocks already provide us with a good contribution in (2.2), and

(ii) rarefaction shocks are the source of instability and non-uniqueness and cannot be "fixed up".

The constraint in (2.8) is different for slow and for fast undercompressive discontinuities. To actually exhibit a (uniformly bounded) weight satisfying (2.8), we put a restriction on how the nature of the discontinuities changes in time as wave interactions take place. (An incoming wave may be a slow undercompressive one and become a fast one after the interaction, etc. A different constraint is placed before and after the interaction.)

Precisely, we suppose that, to the speed $a = a(x,t)$, we can associate on one hand a function $\kappa : \mathbb{R}\times\mathbb{R}_+ \to \mathbb{R}$ having bounded total variation and such that $\mathcal{J}(\kappa) \subset \mathcal{J}(a)$ and $\mathcal{I}(\kappa) \subset \mathcal{I}(a)$, and on the other hand a partition of the discontinuities

$$\mathcal{J}(a) = \mathcal{J}^I(a) \cup \mathcal{J}^{II}(a) , \tag{2.9}$$

so that, for each $(x,t) \in \mathcal{J}(a)$, the limits $\kappa_\pm = \kappa_\pm(x,t)$ determine if the wave is slow or fast on its left or right side, as follows:

$$\operatorname{sgn}\Big(a_\pm(x,t) - \lambda(x,t)\Big) = \begin{cases} \operatorname{sgn}\kappa_\mp & \text{if } (x,t) \in \mathcal{J}^I(a), \\ -\operatorname{sgn}\kappa_\mp & \text{if } (x,t) \in \mathcal{J}^{II}(a) . \end{cases} \tag{2.10}$$

Here we use $\operatorname{sgn}(y) = -1, 0, 1$ iff $y < 0$, $y = 0$, $y > 0$, respectively. Therefore a discontinuity $(x,t) \in \mathcal{J}^I(a)$ (for instance) is

$$\begin{aligned} \text{a Lax one iff } & \kappa_- < 0 \text{ and } \kappa_+ > 0 , \\ \text{a slow undercompressive one iff } & \kappa_- \geq 0 \text{ and } \kappa_+ \geq 0 , \\ \text{a fast undercompressive one iff } & \kappa_- \leq 0 \text{ and } \kappa_+ \leq 0 , \\ \text{a rarefaction-shock iff } & \kappa_- > 0 \text{ and } \kappa_+ < 0 . \end{aligned}$$

Furthermore, to measure the strength of the jumps, we introduce a piecewise constant function, $b = b(x,t)$, having the same jump points as the function a. For instance, we could assume that there exist constants $C_1, C_2 > 0$ such that at each discontinuity of a

$$C_1 \Big|a_+(y,t) - a_-(y,t)\Big| \leq \Big|b_+(y,t) - b_-(y,t)\Big| \leq C_2 \Big|a_+(y,t) - a_-(y,t)\Big| . \tag{2.11}$$

However, strictly speaking, this condition will not be used, in the present section at least.

Based on the functions κ and b and for t except wave interaction times, we can set

$$\begin{aligned} V^I(x,t) &= \sum_{\substack{(y,t)\in\mathcal{J}^I(a),\\ y<x}} \left|b_+(y,t)-b_-(y,t)\right| ,\\ V^{II}(x,t) &= \sum_{\substack{(y,t)\in\mathcal{J}^{II}(a),\\ y<x}} \left|b_+(y,t)-b_-(y,t)\right| , \end{aligned} \tag{2.12}$$

so that the total variation of $b(t)$ on the interval $(-\infty,x)$ decomposes into

$$TV_{-\infty}^x(b(t)) \;=\; V^I(x,t)+V^{II}(x,t) \;. \tag{2.13}$$

Fix some parameter $m\geq 0$. Consider now the weight-function defined for each $(x,t)\in\mathcal{C}(a)$ by

$$w(x,t) \;=\; \begin{cases} m+V^I(\infty,t)-V^I(x,t)+V^{II}(x,t) & \text{if } \kappa(x,t)>0,\\ m+V^I(x,t)+V^{II}(\infty,t)-V^{II}(x,t) & \text{if } \kappa(x,t)\leq 0 \;. \end{cases} \tag{2.14}$$

It is immediate to see that indeed (2.8) holds and that with (2.11)

$$m\;\leq\; w(x,t)\;\leq\; m+TV(b(t))\;\leq\; m+C_2\,TV(a(t))\,, \qquad x\in\mathbb{R}\;. \tag{2.15}$$

Note also that the weight depends on b and a, but not on the solution.

Theorem 2.3. *Consider a piecewise constant speed $a=a(x,t)$ admitting a decomposition (2.9)–(2.10) and satisfying the total variation estimate (2.15). Consider the weight function $w=w(x,t)$ defined by (2.13). Let ψ be any piecewise constant solution of the linear hyperbolic equation (2.1). Then the weighted norm (2.7) satisfies for all $0\leq s\leq t$*

$$\begin{aligned} &\|\psi(t)\|_{w(t)} + \int_s^t \sum_{(x,\tau)\in\mathcal{L}(a)} \Big(2m+TV(b)-\left|b_+(x,\tau)-b_-(x,\tau)\right|\Big)\\ &\qquad\qquad \cdot \left|a_-(x,\tau)-\lambda(x,\tau)\right| \left|\psi_-(x,\tau)\right|\,d\tau\\ &\quad+\int_s^t \sum_{(x,\tau)\in\mathcal{S}(a)\cup\mathcal{F}(a)} \left|b_+(x,\tau)-b_-(x,\tau)\right| \left|a_-(x,\tau)-\lambda(x,\tau)\right| \left|\psi_-(x,\tau)\right|\,d\tau\\ &= \|\psi(s)\|_{w(s)} + \int_s^t \sum_{(x,\tau)\in\mathcal{R}(a)} \Big(2m+TV(b)\Big) \left|a_-(x,\tau)-\lambda(x,\tau)\right| \left|\psi_-(x,\tau)\right|\,d\tau\\ &\quad+\int_s^t \sum_{(x,\tau)\in\mathcal{R}(a)} \left|b_+(x,\tau)-b_-(x,\tau)\right| \left|a_-(x,\tau)-\lambda(x,\tau)\right| \left|\psi_-(x,\tau)\right|\,d\tau\;. \end{aligned} \tag{2.16}$$

The statement (2.16) is sharper than (2.2), as *all* discontinuities contribute now to the decrease of the weighted L^1 norm. Note that as $m \to \infty$, we recover exactly (2.2) from (2.16).

Proof: We proceed similarly as in the proof of Theorem 2.2. However, $x_j(t)$ for $t \in I$ (some open interval avoiding the interaction points in a or ψ) denote now all the jump points in either a or ψ. We obtain as before the identity

$$\begin{aligned}
(2.17) \qquad & \frac{d}{dt}\int_{\mathbb{R}} |\psi(x,t)|\, w(x,t)\ dx \\
&= \sum_{j=1}^{m} \Big(\big(\lambda_j(t) - a_j^-(t)\big)\, |\psi_j^-(t)|\, w_j^-(t) \ + \ \big(a_j^+(t) - \lambda_j(t)\big)\, |\psi_j^+(t)|\, w_j^+(t) \Big) \\
&= \sum_{j=1}^{m} \Big(\operatorname{sgn}\big(\lambda_j(t) - a_j^-(t)\big)\, w_j^-(t) \ + \ \operatorname{sgn}\big(a_j^+(t) - \lambda_j(t)\big)\, w_j^+(t) \Big) \\
&\quad \cdot \Big|\lambda_j(t) - a_j^-(t)\Big|\, |\psi_j^-(t)| \ ,
\end{aligned}$$

where we used the Rankine–Hugoniot relation (2.5).

If $x_j(t)$ is a Lax discontinuity in $\mathcal{J}^I(a)$, then by (2.11) we have $\kappa_- < 0$ and $\kappa_+ > 0$. So by (2.14) we find

$$\begin{aligned}
w_j^- &= m + V^I(x_j(t)-) + V^{II}(\infty) - V^{II}(x_j(t)-) \ , \\
w_j^+ &= m + V^I(\infty) - V^I(x_j(t)+) + V^{II}(x_j(t)+) \ ,
\end{aligned}$$

and so

$$\begin{aligned}
(2.18\text{a}) \qquad & \operatorname{sgn}\big(\lambda_j(t) - a_j^-(t)\big)\, w_j^-(t) \ + \ \operatorname{sgn}\big(a_j^+(t) - \lambda_j(t)\big)\, w_j^+(t) \\
&= -w_j^-(t) - w_j^+(t) \\
&= -2\,m - TV(b) + \Big|b_j^+(t) - b_j^-(t)\Big| \ .
\end{aligned}$$

If $x_j(t)$ is a rarefaction-shock discontinuity in $\mathcal{J}^I(a)$, then by (2.11) we have $\kappa_- > 0$ and $\kappa_+ < 0$. By (2.13) we find

$$\begin{aligned}
w_j^- &= m + V^I(\infty) - V^I(x_j(t)-) + V^{II}(x_j(t)-) \ , \\
w_j^+ &= m + V^I(x_j(t)+) + V^{II}(\infty) - V^{II}(x_j(t)+) \ ,
\end{aligned}$$

and so

$$\begin{aligned}
(2.18\text{b}) \qquad & \operatorname{sgn}\big(\lambda_j(t) - a_j^-(t)\big)\, w_j^-(t) \ + \ \operatorname{sgn}\big(a_j^+(t) - \lambda_j(t)\big)\, w_j^+(t) \\
&= w_j^-(t) + w_j^+(t) \\
&= 2\,m + TV(b) + \Big|b_j^+(t) - b_j^-(t)\Big| \ .
\end{aligned}$$

If $x_j(t)$ is a fast undercompressive discontinuity in $\mathcal{J}^I(a)$, then by (2.11) we have $\kappa_- \le 0$ and $\kappa_+ \le 0$. By (2.13) we find

$$\begin{aligned}
\text{(2.18c)} \qquad \operatorname{sgn}\Big(\lambda_j(t) - a_j^-(t)\Big)\, w_j^-(t) \;+\; \operatorname{sgn}\Big(a_j^+(t) - \lambda_j(t)\Big)\, w_j^+(t) \\
&= w_j^-(t) - w_j^+(t) \\
&= m + V^I(x_j(t)-) + V^{II}(\infty) - V^{II}(x_j(t)-) \\
&\quad - m - V^I(x_j(t)+) - V^{II}(\infty) + V^{II}(x_j(t)+) \\
&= -\Big|b_j^+(t) - b_j^-(t)\Big| .
\end{aligned}$$

Similarly for slow undercompressive discontinuities in $\mathcal{J}^I(a)$ we obtain

$$\text{(2.18d)} \quad \operatorname{sgn}\Big(\lambda_j(t) - a_j^-(t)\Big)\, w_j^-(t) + \operatorname{sgn}\Big(a_j^+(t) - \lambda_j(t)\Big)\, w_j^+(t) \;=\; -|b_j^+(t) - b_j^-(t)| .$$

Using (2.18) in (2.17) we conclude that

$$\begin{aligned}
\|\psi(t)\|_{w(t)} \;&+\; \int_s^t \sum_{(x,\tau)\in\mathcal{L}(a)} \Big(2m + TV(b) - \Big|b_+(y,\tau) - b_-(y,\tau)\Big|\Big) \\
&\qquad \cdot \Big|a_-(x,\tau) - \lambda(x,\tau)\Big|\, |\psi_-(x,\tau)| \;d\tau \\
&+ \int_s^t \sum_{(x,\tau)\in\mathcal{S}(a)\cup\mathcal{F}(a)} \Big|b_+(y,\tau) - b_-(y,\tau)\Big|\,\Big|a_-(x,\tau) - \lambda(x,\tau)\Big|\,|\psi_-(x,\tau)| \;d\tau \\
= \;\|\psi(s)\|_{w(s)} \;&+\; \int_s^t \sum_{(x,\tau)\in\mathcal{R}(a)} \Big(2m + TV(b) + \Big|b_+(y,\tau) - b_-(y,\tau)\Big|\Big) \\
&\qquad \cdot \Big|a_-(x,\tau) - \lambda(x,\tau)\Big|\, |\psi_-(x,\tau)| \;d\tau \;,
\end{aligned}$$

which is equivalent to (2.16). ∎

Using that $\mathcal{R}(a)$ is included in the set of points where ψ changes sign, it is easy to deduce from (2.16) that:

Corollary 2.4. *Under the assumptions and notations in Theorem 2.3, we have for all* $0 \le s \le t$

$$\|\psi(t)\|_{w(t)} \le \|\psi(s)\|_{w(s)} + \Big(2m + TV(b)\Big) \sup_{\substack{(x,\tau)\in\mathcal{R}(a)\\ s\le\tau\le t}} \Big|b_+(x,\tau) - b_-(x,\tau)\Big| \int_s^t TV(\psi(\tau))\, d\tau$$

and, in particular, letting $m \to \infty$

$$\|\psi(t)\|_{L^1(\mathbb{R})} \le \|\psi(s)\|_{L^1(\mathbb{R})} + 2 \sup_{\substack{(x,\tau)\in\mathcal{R}(a)\\ s\le\tau\le t}} \Big|b_+(x,\tau) - b_-(x,\tau)\Big| \int_s^t TV(\psi(\tau))\,d\tau\,. \tag{2.19}$$

Finally, in view of Corollary 2.4, in case the function a contains no rarefaction shocks, we deduce that

$$\|\psi(t)\|_{w(t)} \le \|\psi(s)\|_{w(s)}\,, \qquad 0 \le s \le t\,.$$

Observe that this result is achieved, based on a weight that depends on an arbitrary function, b, and on the sole assumption that a decomposition (2.9)–(2.10) of the jumps of a is available. However, our result in this section covers only piecewise constant solutions. We will see in Section 5 that a stronger structure assumption on the coefficients a is necessary to handle general solutions of bounded variation.

3 – Sharp L^1 estimate for hyperbolic conservation laws

In this section, we apply Theorem 2.3 to the case that a is the averaging coefficient (1.5) based on two entropy solutions of (1.1). First, we check that the assumptions required in Section 2 on the coefficient a do hold in this situation. Therefore Theorem 2.3 applies to the piecewise constant solutions defined by the wave-front traking (also called polygonal approximation) algorithm proposed by Dafermos in [7]. Next, we observe that, with a suitable choice of the definition of the wave strengths, the weighted norm in Section 2 reduces to Liu–Yang's functional. Finally we rigorously justify the passage to the limit in the estimate of Theorem 2.3 when the number of wave fronts tends to infinity and exact entropy solutions of (1.1) are recovered.

Consider the nonlinear scalar conservation law:

$$\partial_t u + \partial_x f(u) = 0\,, \qquad u(x,t) \in \mathbb{R}\,, \tag{3.1}$$

where the flux $f\colon \mathbb{R} \to \mathbb{R}$ is a smooth function. Let u^I and u^{II} be two bounded entropy solutions of (3.1) having bounded total variation. Given $h > 0$ let us approximate the data $u^I(0)$ and $u^{II}(0)$ by piecewise constant functions $u^{I,h}(0)$, $u^{II,h}(0)$, having finitely many jumps and such that as $h \to 0$

$$u^{I,h}(0) \to u^I(0), \quad u^{II,h}(0) \to u^{II}(0) \qquad \text{in the} \quad L^1 \text{ norm}\,, \tag{3.2}$$

$$TV(u^{I,h}(0)) \to TV(u^I(0)), \quad TV(u^{II,h}(0)) \to TV(u^{II}(0)) . \tag{3.3}$$

Applying Dafermos' scheme [7], we can construct corresponding, piecewise constant, approximate solutions $u^{I,h}$ and $u^{II,h}$ having finitely many jump lines and for $t \geq s \geq 0$ and $p \in [1, \infty]$

$$\|u^{I,h}(t)\|_{L^p(\mathbb{R})} \leq \|u^{I,h}(s)\|_{L^p(\mathbb{R})}, \quad \|u^{II,h}(t)\|_{L^p(\mathbb{R})} \leq \|u^{II,h}(s)\|_{L^p(\mathbb{R})} , \tag{3.4}$$

and for all $-\infty \leq A + M\,(t-s) \leq B - M\,(t-s)$

$$\begin{aligned} TV_{A+M\,(t-s)}^{B-M\,(t-s)}(u^{I,h}(t)) \ &\leq \ TV_A^B(u^{I,h}(s)) , \\ TV_{A+M\,(t-s)}^{B-M\,(t-s)}(u^{II,h}(t)) \ &\leq \ TV_A^B(u^{II,h}(s)) . \end{aligned} \tag{3.5}$$

More precisely, the functions $u^{I,h}$ and $u^{II,h}$ are exact solutions of (3.1) satisfying therefore the Rankine–Hugoniot relation at every jump. They contain two kinds of jump discontinuities: *Lax shocks* satisfy the so-called Oleinik entropy inequalities, while *rarefaction jumps* do not, but have small strength, that is

$$\left|u^{I,h}(x+,t) - u^{I,h}(x-,t)\right| \ \leq \ h, \quad \left|u^{II,h}(x+,t) - u^{II,h}(x-,t)\right| \ \leq \ h . \tag{3.6}$$

Furthermore, for a subsequence $h \to 0$ at least, we have for each time $t \geq 0$

$$u^{I,h}(t) \to u^I(t), \quad u^{II,h}(t) \to u^{II}(t) \qquad \text{in the} \quad L^1 \text{ norm} .$$

To study the L^1 distance between these approximate solutions, we set

$$\psi := u^{II,h} - u^{I,h} ,$$

which is a solution of the linear hyperbolic equation

$$\partial_t \psi + \partial_x(a^h\,\psi) \ = \ 0, \quad a^h(x,t) := \frac{f(u^{II,h}(x,t)) - f(u^{I,h}(x,t))}{u^{II,h}(x,t) - u^{I,h}(x,t)} . \tag{3.7}$$

First of all, based on Theorem 2.2 and (3.5)–(3.6), we obtain immediately:

Theorem 3.1. *The approximate solutions $u^{I,h}$ and $u^{II,h}$ satisfy the following L^1 stability estimate for all $0 \leq s \leq t$*

$$\begin{aligned} &\left\|u^{II,h}(t) - u^{I,h}(t)\right\|_{L^1(\mathbb{R})} \\ &\quad + \int_s^t \sum_{(x,\tau)\in\mathcal{L}(a)} 2\left(a^h(x-,\tau) - \lambda^{a^h}(x,\tau)\right)\left|u^{II,h}(x-,\tau) - u^{I,h}(x-,\tau)\right| d\tau \\ &\leq \left\|u^{II,h}(s) - u^{I,h}(s)\right\|_{L^1(\mathbb{R})} + 2\,h\,(t-s)\,\|f''\|_\infty \Big(TV(u^{I,h}(0)) + TV(u^{II,h}(0))\Big) . \end{aligned} \tag{3.8}$$

From the functions u^I and u^{II} we define the function a as in (3.7). Recall that the wave front tracking scheme converges locally uniformly (see the proof of Theorem 3.5 below for a definition), so that the BV solutions u^I and u^{II} are endowed with additional regularity properties. Consider for instance the function u^I. In particular, for all but countably many times t and for each x, either x is a point of continuity of u^I in the classical sense (say $(x,t) \in \mathcal{C}(u^I)$) or else it is a point of jump in the classical sense (say $(x,t) \in \mathcal{J}(u^I)$) and, to the discontinuity, one can also associate a shock speed, denoted by $\lambda^I(x,t)$.

From the properties shared by u^I and u^{II}, one deduces immediately a similar property for the coefficient a. Excluding countably many times at most, at each point of jump of a we can define the propagation speed $\lambda^a(x,t)$ of the discontinuity located at the point (x,t). Namely, we have

$$\lambda^a(x,t) = \begin{cases} \lambda^I(x,t) & \text{if } (x,t) \in \mathcal{J}(u^I), \\ \lambda^{II}(x,t) & \text{if } (x,t) \in \mathcal{J}(u^{II}) \ . \end{cases}$$

In the limit $h \to 0$ we deduce from (3.8) that:

Corollary 3.2. *For all $0 \le s \le t$ we have*

$$\begin{aligned} (3.9) \qquad & \left\| u^{II}(t) - u^I(t) \right\|_{L^1(\mathbb{R})} \\ & + \int_s^t \sum_{(x,\tau) \in \mathcal{L}(a)} 2\left(a(x-,\tau) - \lambda^a(x,\tau) \right) \left| u^{II}(x-,\tau) - u^I(x-,\tau) \right| d\tau \\ & \le \left\| u^{II}(s) - u^I(s) \right\|_{L^1(\mathbb{R})} . \end{aligned}$$

We omit the proof of Corollary 3.2 as (3.9) is a consequence of a stronger estimate proven in Theorem 3.5 below (by taking $m \to \infty$ in (3.15)). Note that (3.9) is a stronger statement than the standard L^1 contraction estimate

$$\left\| u^{II}(t) - u^I(t) \right\|_{L^1(\mathbb{R})} \le \left\| u^{II}(s) - u^I(s) \right\|_{L^1(\mathbb{R})} .$$

Proof: We apply the estimate (2.2) with ψ replaced with $u^{II,h} - u^{I,h}$. We just need to observe (see [14]) that all the rarefaction-shock discontinuities in a^h are due to rarefaction fronts in $u^{I,h}$ or in $u^{II,h}$, which have small strength according to (3.6). In other words we have

$$\begin{aligned}\int_s^t \sum_{(x,\tau)\in\mathcal{R}(a)} 2\left(\lambda^a(x,\tau) - a_-(x,\tau)\right)|\psi_-(x,\tau)|\, d\tau \\ &\le \sup_{\substack{(x,\tau)\in\mathcal{R}(a)\\ s\le\tau\le t}} 2\left|a_+(x,\tau) - a_-(x,\tau)\right| \int_s^t TV(\psi(\tau))\, d\tau \\ &\le 2\,\|f''\|_\infty\, h \int_s^t TV(\psi(\tau))\, d\tau \\ &\le 2\,\|f''\|_\infty\, h\,(t-s)\left(TV(u^{I,h}(0)) + TV(u^{II,h}(0))\right).\end{aligned}$$

This establishes (3.8). ∎

We now want to apply Theorem 2.3 and control a weighted norm of $u^{II,h} - u^{I,h}$. In this direction, our main observation is:

Lemma 3.3. *When the function f is strictly convex, the coefficient a^h satisfies all of the assumptions* (2.9)–(2.10).

Proof: The function a^h is piecewise constant, and we can associate to this function an obvious decomposition of the form (2.9). To establish (2.10), consider for instance a jump point $(x,t) \in \mathcal{J}(u^{I,h}) \cap \mathcal{C}(u^{II,h})$, together with its left- and right-hand traces u^I_- and u^I_+. Since $u^{I,h}$ is a solution of (3.1), the corresponding speed $\lambda = \lambda(x,t)$ satisfies the Rankine–Hugoniot relation:

$$-\lambda\,(u^I_+ - u^I_-) + f(u^I_+) - f(u^I_-) \;=\; 0\,.$$

Thus the term in the left-hand side of (2.10) takes the form

$$\begin{aligned} a_\pm(x,t) - \lambda(x,t) &= \frac{f(u^{II}) - f(u^I_\pm)}{u^{II} - u^I_\pm} - \frac{f(u^I_+) - f(u^I_-)}{u^I_+ - u^I_-} \\ &= \int_0^1 \left(f'\left(\theta\, u^{II} + (1-\theta)\, u^I_\pm\right) - f'\left(\theta\, u^I_\mp + (1-\theta)\, u^I_\pm\right)\right) d\theta\,.\end{aligned}$$

Thus we obtain

$$\begin{aligned} & a_\pm(x,t) - \lambda(x,t) \;=\; \mu\,(u^{II} - u^I_\mp)\,, \\ & \mu := \int_0^1\!\!\int_0^1 f''\left(\rho\left(\theta\, u^{II} + (1-\theta)\, u^I_\pm\right) + (1-\rho)\left(\theta\, u^I_\mp + (1-\theta)\, u^I_\pm\right)\right)\theta\; d\theta\, d\rho\,. \end{aligned} \tag{3.10}$$

Since f is strictly convex, the coefficient is bounded away from zero. In view of (3.10), if we now choose $\kappa(x,t) := u^{II,h} - u^{I,h}$, the desired property (2.10) holds true. ∎

Next, we define the weight $w^h = w^h(x,t)$ associated with the function a^h, by the formula (2.14) in which we specify

$$\kappa^h(x,t) := u^{II,h} - u^{I,h} \ . \tag{3.11}$$

It follows immediately from Theorem 2.3 that:

Theorem 3.4. *Suppose that the function f is strictly convex. The approximate solutions constructed by Dafermos scheme satisfy the L^1 stability estimate for all $0 \le s \le t$*

$$\begin{aligned}
&\left\| u^{II,h}(t) - u^{I,h}(t) \right\|_{w^h(t)} \\
&\quad + \int_s^t \sum_{(x,\tau)\in\mathcal{L}(a^h)} \Big(2\,m + TV(b^h) - \left| b^h(x+,\tau) - b^h(x-,\tau) \right| \Big) \\
&\qquad \cdot \left| a^h(x-,\tau) - \lambda^h(x,\tau) \right| \left| u^{II,h}(x-,\tau) - u^{I,h}(x-,\tau) \right| \ d\tau \\
&\quad + \int_s^t \sum_{(x,\tau)\in\mathcal{S}(a^h)\cup\mathcal{F}(a^h)} \left| b^h(x+,\tau) - b^h(x-,\tau) \right| \left| a^h(x-,\tau) - \lambda^h(x,\tau) \right| \\
&\qquad \cdot \left| u^{II,h}(x-,\tau) - u^{I,h}(x-,\tau) \right| \ d\tau \\
&= \left\| u^{II,h}(s) - u^{I,h}(s) \right\|_{w^h(s)} \\
&\quad + \int_s^t \sum_{(x,\tau)\in\mathcal{R}(a^h)} \Big(2\,m + TV(b^h) + \left| b^h(x+,\tau) - b^h(x-,\tau) \right| \Big) \\
&\qquad \cdot \left| a^h(x-,\tau) - \lambda^h(x,\tau) \right| \left| u^{II,h}(x-,\tau) - u^{I,h}(x-,\tau) \right| \ d\tau \ ,
\end{aligned} \tag{3.12}$$

where a^h is the averaging coefficient defined in (3.7) *and $\lambda^h(x,\tau)$ represents the speed of the discontinuity located at $(x,\tau) \in \mathcal{J}(a^h)$.*

We emphasize that (3.12) is an *equality* in which the contribution to the L^1 norm of each type of wave appears clearly. The coefficient a^h exhibits three types of waves: the Lax and undercompressive discontinuities in a^h contribute to the decay of the L^1 weighted distance. The statement (3.12) quantifies sharply this effect. On the other hand, the rarefaction-shocks appearing in the right-hand side of (3.12) increase the L^1 norm.

In the rest of this section, we assume that the function $b = b^h$ is chosen to be specifically

$$b^h(x+,t) - b^h(x-,t) = \begin{cases} u^{I,h}(x+,t) - u^{I,h}(x-,t) & \text{if } (x,t) \in \mathcal{J}(u^{I,h}), \\ u^{II,h}(x+,t) - u^{II,h}(x-,t) & \text{if } (x,t) \in \mathcal{J}(u^{II,h}), \end{cases} \tag{3.13}$$

but a more general definition is possible.

Our next purpose is to pass to the limit ($h \to 0$) in the statement established in Theorem 3.4 for piecewise constant approximate solutions. We recover here a result derived by Dafermos [9] via a different approach. Recall the notation $\mathcal{C}(u^I)$, $\mathcal{S}(u^I)$, etc introduced earlier. Denote by $\mathcal{I}(u^I)$ the countable set of interactions times. Let $V^I(t)$ be the total variation function associated with $u^I(t)$. Based on the functions $V^I(t)$ and $V^{II}(t)$, we then define the weight w as in (2.14) but with (2.12) replaced by the total variation functions of $u^I(t)$ and $u^{II}(t)$, with $\kappa := u^{II} - u^I$ and

$$b(x+,t) - b(x-,t) = \begin{cases} u^I(x+,t) - u^I(x-,t) & \text{if } (x,t) \in \mathcal{J}(u^I), \\ u^{II}(x+,t) - u^{II}(x-,t) & \text{if } (x,t) \in \mathcal{J}(u^{II}). \end{cases} \tag{3.14}$$

Furthermore, to any functions of bounded variation u, v, w in the space variable x (the time variable being fixed) we associate the measure on $\mathbb{R}$

$$\mu = \Big(a(u,v) - f'(u)\Big)\,(v-u)\;dw$$

understood as the nonconservative product in the sense of Dal Maso, LeFloch and Murat [10] and characterized by the following two conditions:

(**1**) If B is a Borel set included in the set of continuity points of w

$$\mu(B) = \int_B \Big(a(u,v) - f'(u)\Big)\,(v-u)\;dw\;, \tag{3.15a}$$

where the integral is defined in a classical sense;

(**2**) If x is a point of jump of w, then

$$\begin{aligned} \mu(\{x\}) = \frac{1}{2}\bigg(&\Big(a(u_+,v_+) - a(u_-,u_+)\Big)\,(v_+ - u_+) \\ &+ \Big(a(u_-,v_-) - a(u_-,u_+)\Big)\,(v_- - u_-)\bigg)\,|w_+ - w_-| \end{aligned} \tag{3.15b}$$

with $u_\pm = u(x\pm)$, etc.

Note that, if $u = u^I$ and $v = u^{II}$, the two terms $(a(u_\pm, v_\pm) - a(u_-,u_+)) \cdot$ $\cdot\, (v_\pm - u_\pm)$ in fact coincide.

Theorem 3.5. *Let the function f be strictly convex and let u^I and u^{II} be two entropy solutions of bounded variation of the conservation law* (1.1). *For all $0 \le s \le t$ we have*

$$
\begin{aligned}
(3.16)\qquad & \left\| u^{II}(t) - u^I(t) \right\|_{w(t)} \\
& + \int_s^t \sum_{(x,\tau)\in\mathcal{L}(a)\cap\mathcal{J}(u^I)} q \left| a\Big(u^I(x-), u^{II}(x-)\Big) - a\Big(u^I(x+), u^I(x-)\Big) \right| \\
& \qquad \cdot \left| u^{II}(x) - u^I(x) \right| \, d\tau \\
& + \int_s^t \sum_{(x,\tau)\in\mathcal{L}(a)\cap\mathcal{J}(u^{II})} q \left| a\Big(u^I(x-), u^{II}(x-)\Big) - a\Big(u^{II}(x+), u^{II}(x-)\Big) \right| \\
& \qquad \cdot \left| u^{II}(x) - u^I(x) \right| \, d\tau \\
& + \int_s^t \int_{\mathbb{R}} \Big(a(u^I, u^{II}) - f'(u^I) \Big) \, (u^{II} - u^I) \, dV^I \, d\tau \\
& + \int_s^t \int_{\mathbb{R}} \Big(a(u^I, u^{II}) - f'(u^{II}) \Big) \, (u^I - u^{II}) \, dV^{II} \, d\tau \quad \le \quad \left\| u^{II}(s) - u^I(s) \right\|_{w(s)} .
\end{aligned}
$$

where $q = q(\tau) = 2m + TV(u^I(\tau)) + TV(u^{II}(\tau))$.

Observe that the terms in integrals in (3.16) globally contribute to the decrease of weighted norm, as is better seen rewriting the formula as follows (V_c^I and V_c^{II} being the continuous parts of the measures V^I and V^{II}):

$$
\begin{aligned}
(3.16')\qquad & \left\| u^{II}(t) - u^I(t) \right\|_{w(t)} \\
& + \int_s^t \sum_{(x,\tau)\in\mathcal{L}(a)\cap\mathcal{J}(u^I)} \Big(q - |u_+^I - u_-^I| \Big) \left| a(u_-^I, u_+^{II}) - a(u_+^I, u_-^I) \right| |u^{II} - u^I| \, d\tau \\
& + \int_s^t \sum_{(x,\tau)\in\mathcal{L}(a)\cap\mathcal{J}(u^{II})} \Big(q - |u_+^{II} - u_-^{II}| \Big) \left| a(u_-^I, u_-^{II}) - a(u_+^{II}, u_-^{II}) \right| |u^{II} - u^I| \, d\tau \\
& + \int_s^t \sum_{(x,\tau)\in\big(\mathcal{S}(a)\cup\mathcal{F}(a)\big)\cap\mathcal{J}(u^I)} \left| a(u_-^I, u_-^{II}) - a(u_+^I, u_-^I) \right| |u^{II} - u^I| \, |u_+^I - u_-^I| \, d\tau \\
& + \int_s^t \sum_{(x,\tau)\in\big(\mathcal{S}(a)\cup\mathcal{F}(a)\big)\cap\mathcal{J}(u^{II})} \left| a(u_-^I, u_-^{II}) - a(u_+^{II}, u_-^{II}) \right| |u^{II} - u^I| \, |u_+^{II} - u_-^{II}| \, d\tau
\end{aligned}
$$

$$+\int_s^t\int_{\mathbb{R}}\Big|a(u^I,u^{II})-f'(u^I)\Big|\,|u^{II}-u^I|\;dV_c^I\,d\tau$$

$$+\int_s^t\int_{\mathbb{R}}\Big|a(u^I,u^{II})-f'(u^{II})\Big|\,|u^I-u^{II}|\;dV_c^{II}\,d\tau\quad\le\quad\Big\|u^{II}(s)-u^I(s)\Big\|_{w(s)}\ .$$

The following estimate is a direct consequence of the definition (3.15):

Lemma 3.6. *There exists a constant $C>0$ such that for all functions of bounded variation $u,\tilde u,v,\tilde v,w$ defined on some interval $[\alpha,\beta]$*

$$\left|\int_\alpha^\beta\Big(a(u,v)-f'(u)\Big)\,(v-u)\;dw\;-\;\int_\alpha^\beta\Big(a(\tilde u,\tilde v)-f'(\tilde u)\Big)\,(\tilde v-\tilde u)\;dw\right|$$
$$\le\;C\,\Big(\|\tilde u-u\|_{L^\infty(\alpha,\beta)}+\|\tilde v-v\|_{L^\infty(\alpha,\beta)}\Big)\,TV_{[\alpha,\beta]}(w)\ . \tag{3.17}$$

Proof of Theorem 3.5:

Step 1: *Preliminaries.*

For each $t\ge 0$, the functions $V^{I,h}(t)$ and $V^{II,h}(t)$ associated with the wave front tracking approximations $u^{I,h}(t)$ and $u^{II,h}(t)$ are of uniformly bounded variation as $h\to 0$. The measures $dV^{I,h}$ and $dV^{II,h}$ are also Lipschitz continuous in time (with constant independent of h) for the weak convergence, except at interaction points. On the other hand, interaction times in the limiting solutions are at most countable. Therefore, extracting subsequences if necessary, the measures $dV^{I,h}$ and $dV^{II,h}$ converge to some limiting (non-negative) measures, say:

$$dV^{I,h}(t)\to d\bar V^I(t)\,,\qquad dV^{II,h}(t)\to d\bar V^{II}(t)\ . \tag{3.18}$$

By lower semi-continuity, we have at each time t

$$dV^I(t)\le d\bar V^I(t)\,,\qquad dV^{II}(t)\le d\bar V^{II}(t)\ , \tag{3.19}$$

and, in particular, at each (x,t)

$$V^I(x,t)\le \bar V^I(x,t)\,,\qquad V^{II}(x,t)\le \bar V^{II}(x,t)\ . \tag{3.20a}$$

$$\begin{aligned}V^I(+\infty,t)-V^I(x,t)\;&\le\;\bar V^I(+\infty,t)-\bar V^I(x,t)\ ,\\ V^{II}(+\infty,t)-V^{II}(x,t)\;&\le\;\bar V^{II}(+\infty,t)-\bar V^{II}(x,t)\ .\end{aligned} \tag{3.20b}$$

Based on the functions $\bar V^I(t)$ and $\bar V^{II}(t)$, on the coefficient $\kappa:=u^{II}-u^I$ and on the function in (3.14), we can define a weight denoted by $\bar w$, along the same lines as in (2.14). We will show that the left-hand side of (3.16) is bounded above by

$$
\begin{aligned}
(3.21)\qquad & \left\| u^{II}(t) - u^{I}(t) \right\|_{\bar w(t)} \\
&+ \int_s^t \sum_{(x,\tau)\in\mathcal{L}(a)\cap\mathcal{J}(u^I)} \bar q \left| a\Big(u^I(x-),u^{II}(x-)\Big) - a\Big(u^I(x+),u^I(x-)\Big) \right| \\
&\qquad \cdot \left| u^{II}(x) - u^I(x) \right| \, d\tau \\
&+ \int_s^t \sum_{(x,\tau)\in\mathcal{L}(a)\cap\mathcal{J}(u^{II})} \bar q \left| a\Big(u^I(x-),u^{II}(x-)\Big) - a\Big(u^{II}(x+),u^{II}(x-)\Big) \right| \\
&\qquad \cdot \left| u^{II}(x) - u^I(x) \right| \, d\tau \\
&+ \int_s^t \int_{\mathbb{R}} \Big(a(u^I,u^{II}) - f'(u^I) \Big) \, (u^{II} - u^I) \, d\bar V^I(y,\tau)\, d\tau \\
&+ \int_s^t \int_{\mathbb{R}} \Big(a(u^I,u^{II}) - f'(u^{II}) \Big) \, (u^{I} - u^{II}) \, d\bar V^{II}(y,\tau)\, d\tau
\end{aligned}
$$

where $\bar q := 2\,m + \bar V^I(+\infty) + \bar V^{II}(+\infty)$, and that (3.21) coincides with the desired upper bound $\|u^{II}(s) - u^I(s)\|_{w(s)}$. The former statement is postponed to Step 5 below and we focus now on the latter.

Fix some $t \ge s \ge 0$ and rewrite (3.12) in the equivalent form

$$
\begin{aligned}
(3.22)\qquad & \left\| u^{II,h}(t) - u^{I,h}(t) \right\|_{w^h(t)} \\
&+ \int_s^t \sum_{(x,\tau)\in\mathcal{L}(a^h)} \Big(2m + TV(b^h) \Big) \left| a^h(x-,\tau) - \lambda^h(x,\tau) \right| \left| u^{II,h}(x-,\tau) - u^{I,h}(x-,\tau) \right| \, d\tau \\
&+ \int_s^t \sum_{(x,\tau)\in\mathcal{J}^I(a^h)} \left| b^h(x+,\tau) - b^h(x-,\tau) \right| \Big(a^h(x-,\tau) - \lambda^h(x,\tau) \Big) \\
&\qquad \cdot \Big(u^{II,h}(x-,\tau) - u^{I,h}(x-,\tau) \Big) \, d\tau \\
&+ \int_s^t \sum_{(x,\tau)\in\mathcal{J}^{II}(a^h)} \left| b^h(x+,\tau) - b^h(x-,\tau) \right| \Big(a^h(x-,\tau) - \lambda^h(x,\tau) \Big) \\
&\qquad \cdot \Big(u^{I,h}(x-,\tau) - u^{II,h}(x-,\tau) \Big) \, d\tau
\end{aligned}
$$

$$= \left\| u^{II,h}(s) - u^{I,h}(s) \right\|_{w^h(s)}$$

$$+ \int_s^t \sum_{(x,\tau)\in\mathcal{R}(a^h)} \Big(2m + TV(b^h)\Big) \left| a^h(x-,\tau) - \lambda^h(x,\tau) \right| \left| u^{II,h}(x-,\tau) - u^{I,h}(x-,\tau) \right| d\tau \,,$$

or, with obvious notations,

$$\left\| u^{II,h}(t) - u^{I,h}(t) \right\|_{w^h(t)} + \Omega_1^h + \Omega_2^h \;=\; \left\| u^{II,h}(s) - u^{I,h}(s) \right\|_{w^h(s)} + \Omega_3^h \,. \tag{3.23}$$

As the maximum strength of rarefaction fronts in $u^{I,h}$ and $u^{II,h}$ vanishes with h (see (3.6)) and rarefaction shocks in a^h arise only from these rarefaction fronts (see (1.6)), we have

$$\Omega_3^h \to 0 \quad \text{as} \;\; h \to 0 \,. \tag{3.24}$$

On the other hand, we can always choose the (initial) approximations at time s in such a way that

$$\bar{w}(s) \;=\; w(s) \tag{3.25}$$

and

$$\lim_{h\to 0} \left\| u^{II,h}(s) - u^{I,h}(s) \right\|_{w^h(s)} = \left\| u^{II}(s) - u^{I}(s) \right\|_{w(s)} \,. \tag{3.26}$$

It remains to prove that the limit of the left-hand side of (3.22) is exactly (3.21). This will be established in the following three steps.

Step 2: We will rely on the local uniform convergence of the front tracking approximations (see Bressan and LeFloch [3]). For all but countably many times τ we have the following properties for u^I (as well as for u^{II}):

(1) For each point of jump z of u^I there exists a sequence $z^h \to z$ such that for each $\epsilon > 0$ there exists $\delta > 0$ such that

$$\begin{aligned} &\left| u^{I,h}(x) - u^I(z+) \right| + \left| u^I(x) - u^I(z+) \right| \;<\; \epsilon \quad \text{for all} \;\; x - z^h \in (0,\delta) \,, \\ &\left| u^{I,h}(x) - u^I(z-) \right| + \left| u^I(x) - u^I(z-) \right| \;<\; \epsilon \quad \text{for all} \;\; x - z^h \in (-\delta,0) \end{aligned} \tag{3.27a}$$

and (clearly)

$$\begin{aligned} &\left| u^I(x) - u^I(z+) \right| \;<\; \epsilon \quad \text{for all} \;\; x - z \in (0,\delta) \,, \\ &\left| u^I(x) - u^I(z-) \right| \;<\; \epsilon \quad \text{for all} \;\; x - z \in (-\delta,0) \,. \end{aligned} \tag{3.27b}$$

(**2**) For each point of continuity z of u^I and for each $\epsilon > 0$, there exists $\delta > 0$ such that

$$\left|u^{I,h}(x) - u^I(z)\right| + \left|u^I(x) - u^I(z)\right| < \epsilon \quad \text{for all } \; x - z \in (-\delta, \delta) \; . \tag{3.28}$$

We also recall from [3] that, for all but countably many times t, the atomic parts of the measures $\bar V^I$ and $\bar V^{II}$ coincide with the one of V^I and V^{II}, that is for each $y \in \mathbb{R}$

$$\begin{aligned} \bar V^I(y+,t) - \bar V^I(y-,t) &= V^I(y+,t) - V^I(y-,t) \; , \\ \bar V^{II}(y+,t) - \bar V^{II}(y-,t) &= V^{II}(y+,t) - V^{II}(y-,t) \; . \end{aligned} \tag{3.29}$$

Following LeFloch and Liu [20] who established the weak stability of nonconservative products under local uniform convergence, we want to show that

$$\begin{aligned} \Omega_2^h(\tau) \; := \; & \int_{\mathbb{R}} \Big(a\Big(u^{I,h}(y,\tau), u^{II,h}(y,\tau)\Big) - f'(u^{I,h}(y,\tau))\Big) \\ & \qquad \cdot \Big(u^{II,h}(y,\tau) - u^{I,h}(y,\tau)\Big) \; dV^{I,h}(y) \\ & + \int_{\mathbb{R}} \Big(a\Big(u^{I,h}(y,\tau), u^{II,h}(y,\tau)\Big) - f'(u^{II,h}(y,\tau))\Big) \\ & \qquad \cdot \Big(u^{I,h}(y,\tau) - u^{II,h}(y,\tau)\Big) \; dV^{II,h}(y) \end{aligned} \tag{3.30}$$

$$\begin{aligned} \longrightarrow \; & \int_{\mathbb{R}} \Big(a\Big(u^I(y,\tau), u^{II}(y,\tau)\Big) - f'(u^I(y,\tau))\Big) \Big(u^{II}(y,\tau) - u^I(y,\tau)\Big) \; d\bar V^I(y) \\ & + \int_{\mathbb{R}} \Big(a\Big(u^I(y,\tau), u^{II}(y,\tau)\Big) - f'(u^{II}(y,\tau))\Big) \Big(u^I(y,\tau) - u^{II}(y,\tau)\Big) \; d\bar V^{II}(y) \; . \end{aligned}$$

By Lebesgue dominated convergence theorem and since a uniform bound in τ and h is available, it will follow from (3.29) that

$$\begin{aligned} \Omega_2^h \; = \; \int_s^t \Omega_2^h(\tau)\, d\tau \; \longrightarrow \; & \int_s^t \int_{\mathbb{R}} \Big(a(u^I, u^{II}) - f'(u^I)\Big) \, (u^{II} - u^I) \; d\bar V^I(y,\tau)\, d\tau \\ & + \int_s^t \int_{\mathbb{R}} \Big(a(u^I, u^{II}) - f'(u^{II})\Big) \, (u^I - u^{II}) \; d\bar V^{II}(y,\tau)\, d\tau \; . \end{aligned} \tag{3.30'}$$

Given $\epsilon > 0$, select finitely many (large) jumps in u^I or u^{II}, located at $y_1, y_2, ..., y_n$, so that

$$\sum_{\substack{x \neq y_j \\ j=1,2,\dots,n}} \left|u^I(x+) - u^I(x-)\right| + \left|u^{II}(x+) - u^{II}(x-)\right| \; < \; \epsilon \; . \tag{3.31}$$

To each y_j we associate the corresponding discontinuity point y_j^h in $u^{I,h}$ or $u^{II,h}$. To simplify the presentation we will focus on the case where $y_j < y_j^h < y_{j+1} < y_{j+1}^h$ for all j. The other cases can be treated similarly. In view of the local convergence property (3.27)–(3.28) and by extracting a covering of the interval $[y_0, y_n]$, we have also

$$\Big|u^{I,h}(x) - u^I(x)\Big| + \Big|u^{II,h}(x) - u^{II}(x)\Big| \;\le\; 2\,\epsilon\,, \quad x \in (y_j^h, y_{j+1}) \subseteq (y_j, y_{j+1})\,. \tag{3.32}$$

In view of (3.30) we can construct functions u_ϵ^I and u_ϵ^{II} that are continuous everywhere except possibly at the points y_j and such that the following conditions hold with u replaced by either u^I or u^{II}:

$$\begin{aligned} & TV\Big(u_\epsilon;\, \mathbb{R}\backslash\{y_1, ..., y_n\}\Big) \;\le\; C\, TV\Big(u;\, \mathbb{R}\backslash\{y_1, ..., y_n\}\Big)\,, \\ & \|u - u_\epsilon\|_\infty \le C\,\epsilon\,, \qquad TV\Big(u - u_\epsilon;\, \mathbb{R}\backslash\{y_1, ..., y_n\}\Big) \le C\,\epsilon\,, \end{aligned} \tag{3.33}$$

where C is independent of ϵ.

Consider the decompositions

$$\int_{\mathbb{R}} \Big(a(u^{I,h}, u^{II,h}) - f'(u^{I,h})\Big)\,(u^{II,h} - u^{I,h})\; dV^{I,h} \;=\; \sum_{j=0}^{n} \int_{(y_j^h, y_{j+1}^h)} \cdots \;+\; \sum_{j=1}^{n} \int_{\{y_j^h\}} \cdots$$

and

$$\int_{\mathbb{R}} \Big(a(u^I, u^{II}) - f'(u^I)\Big)\,(u^{II} - u^I)\; d\bar{V}^I \;=\; \sum_{j=0}^{n} \int_{(y_j, y_{j+1})} \cdots \;+\; \sum_{j=1}^{n} \int_{\{y_j\}} \cdots\,.$$

Here $y_0^h = y_0 = -\infty$ and $y_{n+1}^h = y_{n+1} = +\infty$. Thus in (3.30) we have to estimate

$$\begin{aligned} \Omega_2^h(\tau) \;&=\; \int_{\mathbb{R}} \Big(a(u^{I,h}, u^{II,h}) - f'(u^{I,h})\Big)\,(u^{II,h} - u^{I,h})\; dV^{I,h} \\ &\quad - \int_{\mathbb{R}} \Big(a(u^I, u^{II}) - f'(u^I)\Big)\,(u^{II} - u^I)\; d\bar{V}^I \\ &=\; T_1^h + T_2^h \end{aligned} \tag{3.34}$$

with

$$\begin{aligned} T_1^h \;:=\; & \sum_{j=1}^{n} \int_{\{y_j^h\}} \Big(a(u^{I,h}, u^{II,h}) - f'(u^{I,h})\Big)\,(u^{II,h} - u^{I,h})\; dV^{I,h} \\ & - \sum_{j=1}^{n} \int_{\{y_j\}} \Big(a(u^I, u^{II}) - f'(u^I)\Big)\,(u^{II} - u^I)\; d\bar{V}^I \end{aligned}$$

and

$$T_2^h := \sum_{j=0}^{n} \int_{(y_j^h, y_{j+1}^h)} \Big(a(u^I, u^{II}) - f'(u^{I,h}) \Big) \, (u^{II,h} - u^{I,h}) \; dV^{I,h}$$
$$- \sum_{j=0}^{n} \int_{(y_j, y_{j+1})} \Big(a(u^I, u^{II}) - f'(u^I) \Big) \, (u^{II} - u^I) \; d\bar{V}^I \; .$$

First, relying on the convergence property (3.29) we have immediately

$$\begin{aligned} T_1^h = \sum_{j=1}^{n} & \Big(a\Big(u^{I,h}(y_j^h-), u^{II}(y_j^h-)\Big) - \lambda^{I,h}(y_j^h-) \Big) \, \Big(u^{II,h}(y_j^h-) - u^{I,h}(y_j^h-) \Big) \\ & \cdot \Big| u^{I,h}(y_j^h+) - u^{I,h}(y_j^h-) \Big| \\ - & \Big(a\Big(u^I(y_j-), u^{II}(y_j-)\Big) - \lambda^I(y_j-) \Big) \, \Big(u^{II}(y_j-) - u^I(y_j-) \Big) \\ & \cdot \Big| u^I(y_j+) - u^I(y_j-) \Big| \; , \end{aligned}$$

so that

$$|T_1^h| \; \leq \; C \sum_{j=1}^{n} \sum_{\pm} \Big| u^{I,h}(y_j^h \pm) - u^I(y_j \pm) \Big| + \Big| u^{II,h}(y_j^h \pm) - u^{II}(y_j \pm) \Big| \; .$$

Thus, in view of the local convergence at jump points (3.27a), for h small enough we obtain

$$|T_1^h| \; \leq \; C \, \epsilon \; . \tag{3.35}$$

Relying on the simplifying assumption $y_j < y_j^h < y_{j+1} < y_{j+1}^h$ for all j, we can decompose T_2^h as follows:

$$\begin{aligned} T_2^h = \sum_{j=0}^{n} \int_{(y_j^h, y_{j+1})} & \Big(a(u^{I,h}, u^{II,h}) - f'(u^{I,h}) \Big) \, (u^{II,h} - u^{I,h}) \; dV^{I,h} \\ & - \Big(a(u^I, u^{II}) - f'(u^I) \Big) \, (u^{II} - u^I) \; d\bar{V}^I \\ & - \sum_{j=0}^{n} \int_{(y_j, y_j^h]} \Big(a(u^I, u^{II}) - f'(u^I) \Big) \, (u^{II} - u^I) \; d\bar{V}^I \\ & + \sum_{j=0}^{n} \int_{[y_{j+1}, y_{j+1}^h)} \Big(a(u^{I,h}, u^{II,h}) - f'(u^{I,h}) \Big) \, (u^{II,h} - u^{I,h}) \; dV^{I,h} \\ := \; & T_{2,1}^h + T_{2,2}^h + T_{2,3}^h \; . \end{aligned} \tag{3.36}$$

We first consider $T^h_{2,2}$:

$$
\begin{aligned}
T^h_{2,2} \;=\; & -\sum_{j=0}^{n} \int_{(y_j,y_j^h]} \Big(a\Big(u^I(y_j+),u^{II}(y_j+)\Big) - f'(u^I(y_j+))\Big)\cdot \\
&\qquad\qquad \cdot \Big(u^{II}(y_j+) - u^I(y_j+)\Big)\; d\bar V^I(y) \\
& + \sum_{j=0}^{n} \int_{(y_j,y_j^h]} \Big\{ \Big(a\Big(u^I(y),u^{II}(y)\Big) - f'(u^I(y))\Big)\Big(u^{II}(y) - u^I(y)\Big) \\
& - \Big(a\Big(u^I(y_j+),u^{II}(y_j+)\Big) - f'(u^I(y_j+))\Big)\Big(u^{II}(y_j+) - u^I(y_j+)\Big)\Big\}\; d\bar V^I(y)\ .
\end{aligned}
$$

Therefore, with (3.17), we obtain

$$
\begin{aligned}
|T^h_{2,2}| \;\le\; & C\sum_j \Big|\bar V^I(y_j+) - \bar V^I(y_j^h+)\Big| \\
& + C\,V^I(+\infty)\left(\sup_{y\in(y_j,y_j^h]}\Big|u^I(y) - u^I(y_j+)\Big| + \sup_{x\in(y_j,y_j^h]}\Big|u^{II}(y) - u^{II}(y_j+)\Big|\right)\ .
\end{aligned}
$$

Since $y_j^h \to y_j$, we have $|\bar V^I(y_j+) - \bar V^I(y_j^h+)| \to 0$, therefore for h sufficiently small

$$
|T^h_{2,2}| \;\le\; C\,\epsilon\ . \tag{3.37}
$$

A similar argument for $T^h_{2,3}$ shows that

$$
|T^h_{2,3}| \;\le\; C\,\epsilon\ . \tag{3.38}
$$

Next consider the decomposition

$$
\begin{aligned}
&\Big(a(u^{I,h},u^{II,h}) - f'(u^{I,h})\Big)\,(u^{II,h} - u^{I,h})\,dV^{I,h} \;-\; \Big(a(u^I,u^{II}) - f'(u^I)\Big)\,(u^{II} - u^I)\,d\bar V^I \\
&= \Big(a(u^{I,h},u^{II,h}) - f'(u^{I,h})\Big)\,(u^{II,h} - u^{I,h})\,dV^{I,h} - \Big(a(u^I,u^{II}) - f'(u^I)\Big)\,(u^{II} - u^I)\,dV^{I,h} \\
&\quad + \Big(a(u^I,u^{II}) - f'(u^I)\Big)\,(u^{II} - u^I)\,dV^{I,h} \;-\; \Big(a(u^I_\epsilon,u^{II}_\epsilon) - f'(u^I_\epsilon)\Big)\,(u^{II}_\epsilon - u^I_\epsilon)\,dV^{I,h} \\
&\quad + \Big(a(u^I_\epsilon,u^{II}_\epsilon) - f'(u^I_\epsilon)\Big)\,(u^{II}_\epsilon - u^I_\epsilon)\,dV^{I,h} \;-\; \Big(a(u^I_\epsilon,u^{II}_\epsilon) - f'(u^I_\epsilon)\Big)\,(u^{II}_\epsilon - u^I_\epsilon)\,d\bar V^I \\
&\quad + \Big(a(u^I_\epsilon,u^{II}_\epsilon) - f'(u^I_\epsilon)\Big)\,(u^{II}_\epsilon - u^I_\epsilon)\,d\bar V^I \;-\; \Big(a(u^I,u^{II}) - f'(u^I)\Big)\,(u^{II} - u^I)\,d\bar V^I\ ,
\end{aligned}
$$

which, with obvious notation, yields a decomposition for $T^h_{2,1}$

$$
T^h_{2,1} \;=\; M^h_1 + M^h_2 + M^h_3 + M^h_4\ . \tag{3.39}
$$

Using (3.17) and the local convergence property (3.31), we obtain

$$
\begin{aligned}
|M_1^h| &\le C \sum_{j=0}^{n} \int_{(y_j^h, y_{j+1})} |dV^{I,h}| \left(\sup_{(y_j^h, y_{j+1})} |u^{I,h} - u^I| + \sup_{(y_j^h, y_{j+1})} |u^{II,h} - u^{II}| \right) \\
&\le C\,\epsilon \,.
\end{aligned} \tag{3.40}
$$

Similarly using (3.17) and (3.33) we obtain

$$
\begin{aligned}
|M_2^h| &\le C \sum_{j=0}^{n} \int_{(y_j^h, y_{j+1})} |dV^{I,h}| \left(\sup_{(y_j^h, y_{j+1})} |u^I - u_\epsilon^I| + \sup_{(y_j^h, y_{j+1})} |u^{II} - u_\epsilon^{II}| \right) \\
&\le C\,\epsilon \,.
\end{aligned} \tag{3.41}
$$

Dealing with M_4^h is similar:

$$
\begin{aligned}
|M_4^h| &\le C \sum_{j=0}^{n} \int_{(y_j^h, y_{j+1})} |d\bar{V}^I| \left(\sup_{(y_j^h, y_{j+1})} |u^I - u_\epsilon^I| + \sup_{(y_j^h, y_{j+1})} |u^{II} - u_\epsilon^{II}| \right) \\
&\le C\,\epsilon \,.
\end{aligned} \tag{3.42}
$$

Finally to treat M_3^h we observe that, since u_ϵ^I and u_ϵ^{II} are continuous functions on each interval (y_j^h, y_{j+1}) and since $dV^{I,h}$ is sequence of bounded measures converging weakly-star toward $d\bar{V}^I$, we have for all h sufficiently small

$$
|M_3^h| \le \epsilon \,. \tag{3.43}
$$

Combining (3.39)–(3.43) we get

$$
|T_{2,1}^h| \le C\,\epsilon \,. \tag{3.44}
$$

Combining (3.36)–(3.38) and (3.44) we obtain

$$
|T_2^h| \le C\,\epsilon
$$

and thus with (3.34)–(3.35)

$$
|\Omega_2^h(\tau)| \le C\,\epsilon \qquad \text{for all } h \text{ sufficiently small} \,.
$$

Since ϵ is arbitrary, this completes the proof of (3.30).

Step 3: Consider now the term

$$
\begin{aligned}
\Omega_1(\tau) \; = & \sum_{(x,\tau)\in\mathcal{L}(a^h)\cap\mathcal{J}(u^{I,h})} \Big(2m + TV(b^h)\Big)\left|a^h(x-,\tau) - \lambda^h(x,\tau)\right| \\
& \cdot \left|u^{II,h}(x-,\tau) - u^{I,h}(x-,\tau)\right| .
\end{aligned} \tag{3.45}
$$

On one hand, observe that

$$
\begin{aligned}
TV(b^h(\tau)) \; = \; & TV(u^{I,h}(\tau)) + TV(u^{II,h}(\tau)) \\
& \longrightarrow \; \bar V^I(+\infty,\tau) + \bar V^{II}(+\infty,\tau) \; .
\end{aligned} \tag{3.46}
$$

For all but countably many τ the following holds. Extracting a subsequence if necessary we can always assume that for each j either $(y_j^h,\tau)\in\mathcal{L}(a^h)$ for all h, or else $(y_j^h,\tau)\notin\mathcal{L}(a^h)$ for all h. Then consider the following three sets: denote by J_1 the set of indices j such that $(y_j^h,\tau)\in\mathcal{L}(a^h)$ and $(y_j,\tau)\in\mathcal{L}(a)$. Let J_2 the set of indices j such that $(y_j^h,\tau)\notin\mathcal{L}(a^h)$ and $(y_j,\tau)\in\mathcal{L}(a)$. Finally J_3 is the set of indices j such that $(y_j^h,\tau)\in\mathcal{L}(a^h)$ and $(y_j,\tau)\notin\mathcal{L}(a)$.

The local convergence property (3.27) implies

$$
\begin{aligned}
& \sum_{j\in J_1} \left|a\Big(u^{I,h}(y_j^h-),u^{II,h}(y_j^h-)\Big) - a\Big(u^{I,h}(y_j^h-),u^{I,h}(y_j^h+)\Big)\right| \\
& \qquad\qquad \cdot \left|u^{II,h}(y_j^h-) - u^{I,h}(y_j^h-)\right| \\
\longrightarrow & \sum_{j\in J_1} \left|a\Big(u^I(y_j-),u^{II}(y_j-)\Big) - a\Big(u^I(y_j-),u^I(y_j+)\Big)\right| \left|u^{II}(y_j-) - u^I(y_j-)\right| .
\end{aligned} \tag{3.47}
$$

(Indeed, given $\epsilon>0$, choose finitely many jump points as in (3.31) and use (3.27) with ϵ replaced with $\epsilon\,|u^I(z+)-u^I(z+)|$).

On the other hand for indices in J_2 or J_3 we have

$$
\begin{aligned}
& \sum_{j\in J_2\cup J_3} \left|a\Big(u^{I,h}(y_j^h-),u^{II,h}(y_j^h-)\Big) - a\Big(u^{I,h}(y_j^h-),u^{I,h}(y_j^h+)\Big)\right| \\
& \qquad\qquad \cdot \left|u^{II,h}(y_j^h-) - u^{I,h}(y_j^h-)\right| \; \longrightarrow \; 0
\end{aligned} \tag{3.48}
$$

but

$$
\begin{aligned}
& \sum_{j\in J_2\cup J_3} \left|a\Big(u^I(y_j-),u^{II}(y_j-)\Big) - a\Big(u^I(y_j-),u^I(y_j+)\Big)\right| \\
& \qquad\qquad \cdot \left|u^{II}(y_j-) - u^I(y_j-)\right| \; = \; 0 \; .
\end{aligned} \tag{3.49}
$$

Indeed, for each $j \in J_2$, y_j is a Lax shock but y_j^h is not. Extracting a subsequence if necessary, it must be that the Lax inequalities are violated on the left or on the right side of y_j^h for all h. So it must be that, assuming that it is the case on the left side, $a(u^I(y_j-), u^{II}(y_j-)) - a(u^I(y_j-), u^I(y_j+)) \geq 0$ while $a(u^I(y_j^h-), u^{II}(y_j^h-)) - a(u^I(y_j^h-), u^I(y_j^h+)) \leq 0$ for all h. But the latter converges toward the former by the local uniform convergence, which proves that $a(u^I(y_j-), u^{II}(y_j-)) - a(u^I(y_j-), u^I(y_j+)) = 0$.

Combining (3.45)–(3.49) yields

$$\Omega_1^h \longrightarrow \int_0^t \sum_{(x,\tau)\in\mathcal{L}(a)} \bar{q}(\tau) \left|a(x-,\tau) - \lambda(x,\tau)\right| \left|u^{II}(x,\tau) - u^I(x,\tau)\right| d\tau \,, \tag{3.50}$$

where $\bar{q} := 2m + \bar{V}^I(+\infty) + \bar{V}^{II}(+\infty)$.

Step 4: *Continuity of the weighted norm.*

Fix some time t. Recall that the weight $\bar{w}(t)$ is defined based on the total variation functions $\bar{V}^{II}$ and $\bar{V}^I$ and on the function $u^{II}(t) - u^I(t)$. The weight $w^h(t)$ is defined based on the total variation functions $V^{II,h}$ and $V^{I,h}$ and on the function $u^{II,h}(t) - u^{I,h}(t)$. On the other hand, $u^{II,h} - u^{I,h}(t) \to u^{II} - u^I(t)$, $V^{II,h} \to \bar{V}^{II}$ and $V^{I,h} \to \bar{V}^I$. Therefore we have

$$w(x,t) = \bar{w}(x,t) \quad \text{whenever} \quad u^{II}(x,t) - u^I(x,t) \neq 0 \,. \tag{3.51}$$

Combining (3.51) and the L^1 convergence $u^{II,h} - u^{I,h}(t) \to u^{II} - u^I(t)$, we have

$$\left\|u^{II}(t) - u^I(t)\right\|_{\bar{w}(t)} = \lim_{h\to 0}\left\|u^{II,h}(t) - u^{I,h}(t)\right\|_{w^h(t)} \,. \tag{3.52}$$

Step 5: *The left-hand side of* (3.16) *is bounded above by* (3.21).

First of all, the inequality

$$\begin{aligned} &\int_0^t \int_{\mathbb{R}} \Big(a(u^I, u^{II}) - f'(u^I)\Big)\,(u^{II} - u^I)\; dV^I(y,\tau)\, d\tau \\ &\qquad\qquad \leq \int_0^t \int_{\mathbb{R}} \Big(a(u^I, u^{II}) - f'(u^I)\Big)\,(u^{II} - u^I)\; d\bar{V}^I(y,\tau)\, d\tau \end{aligned} \tag{3.53}$$

is a direct consequence of (3.19) and the definition of the nonconservative product in (3.15).

On the other hand, by the definition of the weighted norm and because of (3.20), similarly to (3.51) we have the inequality

$$w(x,t) \le \bar{w}(x,t) \quad \text{whenever} \quad u^{II}(x,t) - u^{I}(x,t) \neq 0 \,. \tag{3.54}$$

Hence, by (3.53) and (3.54) the left-hand side of (3.16) is bounded above by (3.21). This completes the proof of Theorem 3.5. ∎

4 – Generalized characteristics and maximum principle

We now return to the setting in Section 2 and aim at extending the analysis therein to arbitrary functions of bounded variation. For exact solutions of the hyperbolic equation

$$\partial_t \psi + \partial_x (a\, \psi) \;=\; 0 \,, \tag{4.1}$$

we will establish a maximum principle: Any solution of (4.1) remains non-negative for all times if it is so initially. For a more precise (local) statement, our proof will make use of Dafermos–Filippov theory of generalized characteristics.

Our main assumption throughout this section is the following:

$$\text{There exists a constant } E \text{ such that } \partial_x a \le \frac{E}{t} \,. \tag{4.2}$$

This is nothing but a generalization of the well-known Oleinik's entropy inequality. To motivate (4.2), let us recall the following result.

Let f be a strictly convex function and u be an entropy solution (with bounded variation for all times) of the conservation law

$$\partial_t u + \partial_x f(u) \;=\; 0\,, \quad u(x,t) \in \mathbb{R} \,. \tag{4.3}$$

Then is is known that there exists a constant $C = C(u)$ such that

$$\partial_x u \;\le\; \frac{C}{t} \,. \tag{4.4}$$

Lemma 4.1. *If u^I and u^{II} are two entropy solutions of the conservation law* (4.3), *then the averaging speed*

$$a \;=\; a(u^I, u^{II}) := \frac{f(u^{II}) - f(u^I)}{u^{II} - u^I} \,. \tag{4.5}$$

satisfies our assumption (4.2), *with $E = \sup f''\,(C(u^I) + C(u^{II}))/2$.*

Proof: Let us fix some time $t > 0$. On each Borel set consisting of points of continuity of both u^I and u^{II}, the following holds:

$$\begin{aligned} \partial_x a &= \partial_x \int_0^1 f'\Big(\theta\, u^I + (1-\theta)\, u^{II}\Big)\, d\theta \\ &= \int_0^1 f''\Big(\theta\, u^I + (1-\theta)\, u^{II}\Big) \Big(\theta\, \partial_x u^I + (1-\theta)\, \partial_x u^{II}\Big)\, d\theta \\ &\le \int_0^1 \sup f'' \left(\theta\, \frac{C(u^I)}{t} + (1-\theta)\, \frac{C(u^{II})}{t}\right) d\theta \\ &\le \sup f''\, \frac{C(u^I) + C(u^{II})}{2\,t} \; . \end{aligned}$$

On the other hand, at a point x where one of u^I or u^{II} is discontinuous, we have with an obvious notation

$$a_+ - a_- = \int_0^1 f'\Big(\theta\, u_+^I + (1-\theta)\, u_+^{II}\Big)\, d\theta - \int_0^1 f'\Big(\theta\, u_-^I + (1-\theta)\, u_-^{II}\Big)\, d\theta \le 0 \; ,$$

since f' is an increasing function and (for instance by (4.4)) both u^I and u^{II} satisfy $u_+^I \le u_-^I$ and $u_+^{II} \le u_-^{II}$. ∎

By definition, a generalized characteristic $y = y(t)$ associated with the coefficient a must satisfy for almost every t (in its domain of definition)

$$a_+(y(t), t) \le y'(t) \le a_-(y(t), t) \; . \tag{4.6}$$

According to Filippov's theory of differential equations [12], through each point $(\bar{x}, \bar{t})$ there pass a maximal and a minimal generalized characteristic.

Definition 4.2. A generalized characteristic is said to be genuine iff for almost every t it satisfies

$$y'(t) \in \Big\{a_-(y(t), t),\; a_+(y(t), t)\Big\} \; . \; \square \tag{4.7}$$

Proposition 4.3. *Any minimal backward generalized characteristic is genuine and for almost every t satisfies*

$$y'(t) = a_-(y(t), t) \; . \tag{4.8}$$

Similarly, for a maximal backward generalized characteristic we have $y'(t) = a_+(y(t), t)$.

Proof: Here we only rely on the following consequence of (4.2): $a_+ \le a_-$ at each discontinuity point of the function a. Geometrically, this condition prevents the existence of rarefaction-shocks in a. On the other hand, rarefaction centers (also prevented by (4.2) for $t > 0$) could still be allowed for the present purpose.

Consider $(\bar{x}, \bar{t}) \in (-\infty, +\infty) \times (0, \infty)$, and let $y(t) := y(t; \bar{x}, \bar{t})$ be the minimal backward characteristic through $(\bar{x}, \bar{t})$. We prove that it is genuine on its domain $(s, \bar{t}]$. We proceed as in [8] and assume by contradiction that there is a measurable set J, $\bar{J} \subset (s, \bar{t}]$ of positive Lebesgue measure, and $\varepsilon > 0$ such that

$$a_-(y(t), t) - y'(t) \; > \; 2\,\varepsilon\,, \qquad t \in J\ . \tag{4.9}$$

For each $t \in J$ there exists $\delta(t) > 0$ with the property

$$a_+(x, t) \;\ge\; a_-(y(t), t) - \varepsilon\,, \qquad x \in \Big(y(t) - \delta(t),\, y(t)\Big)\ . \tag{4.10}$$

Finally, there is a subset $I \subset J$ with $\mu^*(I) > 0$ (here μ^* denotes the outer measure) and $\bar{\delta} > 0$ such that $\delta(t) > \bar{\delta}$ for $t \in I$.

Let τ be a density point of I, with respect to μ^*. Thus there exists $\bar{r}$, $0 < \bar{r} < \bar{t} - \tau$, so that

$$\frac{\mu^*(I \cap [\tau, \tau + r])}{r} \;>\; \frac{2\,|\alpha| + \varepsilon}{2\,|\alpha| + 2\varepsilon}\,, \qquad 0 < r \le \bar{r}\ , \tag{4.11}$$

where

$$\alpha \;:=\; \inf\Big\{a_+(x, t) - a_-(y(t), t) \colon\;\; s < t \le \bar{t},\;\; y(t) - \bar{\delta} \le x < y(t)\Big\}\ .$$

Now take a point $y \in (y(\tau) - \bar{\delta},\, y(\tau))$ with the property $y > y(\tau) - \frac{1}{2}\,\varepsilon\,\bar{r}$, and consider a forward characteristic $z(\cdot)$ through (y, τ). We first observe that

$$z(t) < y(t)\,, \qquad t > \tau\ ,$$

since $y(t)$ is the minimal backward characteristic through $(\bar{x}, \bar{t})$.

In addition, we have

$$z(t) \;>\; y(t) - \bar{\delta}\,, \qquad t >\in [\tau,\, \tau + \bar{r}]\ .$$

Indeed, suppose by contradiction that for some $r \in (0, \bar{r}]$, $z(t) > y(t) - \bar{\delta}$ for $t >\in [\tau,\, \tau + r)$, but $z(\tau + r) = y(\tau + r) - \bar{\delta}$. Then

$$\begin{aligned} 0 \;&=\; z(\tau + r) - y(\tau + r) - \bar{\delta} \\ &=\; y \,+ \int_\tau^{\tau + r} z'(t)\,dt \;-\; y(t) \;-\; \int_\tau^{\tau + r} y'(t)\,dt \;+\; \bar{\delta} \end{aligned}$$

$$
\begin{aligned}
&> \int_\tau^{\tau+r} \Big(z'(t) - y'(t)\Big)\, dt \\
&= \int_{I\cap[\tau,\tau+r]} \Big(z'(t) - a_-(y(t),t) + a_-(y(t),t) - y'(t)\Big)\, dt \\
&\quad + \int_{[\tau,\tau+r]\setminus I} \Big(z'(t) - a_-(y(t),t) + a_-(y(t),t) - y'(t)\Big)\, dt \\
&\geq \varepsilon\, \mu^* \Big(I \cap [\tau,\tau+r]\Big) \;+\; \alpha \Big(r - \mu^*\Big(I\cap[\tau,\tau+r]\Big)\Big) \;>\; 0\,,
\end{aligned}
$$

by (4.9)–(4.11), which leads to a contradiction. In the same way one obtains

$$
\begin{aligned}
0 \;&>\; z(\tau+\bar r) - y(\tau+\bar r) \\
&=\; y + \int_\tau^{\tau+\bar r} z'(t)\, dt \;-\; y(t) \;-\; \int_\tau^{\tau+\bar r} y'(t)\, dt \\
&>\; \varepsilon\, \mu^* \Big(I\cap[\tau,\tau+\bar r]\Big) \;+\; \alpha\Big(\bar r - \mu^*\Big(I\cap[\tau,\tau+r]\Big)\Big) \;-\; \frac{1}{2}\,\varepsilon\, \bar r \;>\; 0\,,
\end{aligned}
$$

which gives another contradiction. For the maximal backward characteristic the proof is similar. ∎

Proposition 4.4. *Forward characteristics leaving from some $(\bar x, \bar t)$ are unique when $\bar t > 0$.*

Proof: Suppose there were two forward characteristics $y(\cdot)$ and $z(\cdot)$ through $(\bar x, \bar t)$ with $y(\tau) < z(\tau)$ for some $\tau > \bar t$. By (4.2) we have

$$
z'(\tau) - y'(\tau) \;\leq\; a_-(z(t),t) - a_+(y(t),t) \;\leq\; C_{\bar t}(z(\tau) - y(\tau))\,. \tag{4.12}
$$

Integrating (4.12) from $\bar t$ to τ one gets $z(\tau) - y(\tau) = 0$, which gives a contradiction. ∎

Theorem 4.5. *Let $\psi = \psi(x,t)$ be a solution of* (4.1) *such that on some interval $[\xi_0, \zeta_0]$ we have*

$$
\psi(x,0) \;\geq\; 0\,, \qquad x \in [\xi_0, \zeta_0]\,. \tag{4.13}
$$

Let $\xi = \xi(t)$ be any forward generalized characteristic leaving from $(\xi_0, 0)$, and $\zeta = \zeta(t)$ be any forward generalized characteristic leaving from $(\zeta_0, 0)$.

Then we have for all $t \geq 0$

$$
\psi(x,t) \;\geq\; 0\,, \qquad x \in (\xi(t), \zeta(t))\,. \tag{4.14}
$$

Note that it may happen that $\xi(t) = \zeta(t)$ for t large enough.

Proof: Observe that the two characteristics cannot cross and fix any time $t > 0$ such that $\xi(t) < \zeta(t)$. Fix also any two points such that $\xi(t) < \bar{y} < \bar{z} < \zeta(t)$. Let $y(t)$ and $z(t)$ be the maximal and minimal backward characteristics emanating from $\bar{y}$ and $\bar{z}$, respectively. These characteristics can not leave the region limited by $\xi(t)$ and $\zeta(t)$.

Integrating (4.1) in the domain bounded by the characteristics $y(t)$ and $z(t)$, and using that these characteristics are genuine, so that the flux terms along the vertical boundaries vanish identically, we arrive at

$$\int_{\bar{y}}^{\bar{z}} \psi(x,t)\,dx \;=\; \int_{y(0)}^{z(0)} \psi(x,0)\,dx \;\geq\; 0\;. \tag{4.15}$$

The last inequality is due to the fact that $\psi(\cdot,0) \geq 0$ and the inequalities $\xi_0 = \xi(0) \leq y(0) \leq z(0) \leq \zeta(0) = \zeta_0$. Since $\bar{y}$ and $\bar{z}$ are arbitrary, we obtain (4.14). ∎

5 – A sharp L^1 estimate for hyperbolic linear equations

Based on the maximum principle established in Section 4, we now derive a sharp estimate for the weighted norm introduced in Section 2. We restrict attention again to the situation where u^I and u^{II} are two entropy solutions of the conservation law (4.3) and a is the averaging speed given in (4.5). We define a weight by analogy with what was done in Section 2 in the special case of piecewise constant solutions.

Given a solution ψ of the equation (4.1), we introduce weighted L^1 norm in the following way. Set

$$V^I(x,t) = TV_{-\infty}^x(u^I(t))\,, \qquad V^{II}(x,t) = TV_{-\infty}^x(u^{II}(t)) \tag{5.1}$$

and fix some parameter $m \geq 0$. Then consider the weight-function defined, for each $t \geq 0$ and each point of continuity x for $u^I(t)$ and $u^{II}(t)$, by

$$w(x,t) \;=\; \begin{cases} m + V^I(\infty,t) - V^I(x,t) + V^{II}(x,t) & \text{if } \psi(x,t) > 0, \\ m + V^I(x,t) + V^{II}(\infty,t) - V^{II}(x,t) & \text{if } \psi(x,t) \leq 0\;. \end{cases} \tag{5.2}$$

It is immediate to see that

$$m \;\leq\; w(x,t) \;\leq\; m + TV(u^I(t)) + TV(u^{II}(t))\,, \qquad x \in \mathbb{R}\;. \tag{5.3}$$

Finally the weighted norm on the solutions ψ of (4.1) is defined by

$$\|\psi(t)\|_{w(t)} \;:=\; \int_{\mathbb{R}} |\psi(x,t)|\, w(x,t)\;dx\;.$$

Note that the weight depends on the fixed solutions u^I and u^{II}, but also on the solution ψ.

Our sharp estimate will involve the nonconservative product

$$\mu^I_\psi(t) \;=\; \Big(a - f'(u^I(t))\Big)\,\psi(t)\;dV^I(t)$$

defined for all almost every $t \geq 0$ by

(1) If B is a Borel set included in the set of continuity points of $u^I(t)$ then

$$\mu^I_\psi(t)(B) \;= \int_B \Big(a(t) - f'(u^I(t))\Big)\,\psi(t)\;dV^I(t)\;, \tag{5.4a}$$

where the integral is defined in a classical sense;

(2) If x is a point of jump of $u^I(t)$, then

$$\mu^I_\psi(t)(\{x\}) \;=\; \Big(a(x-,t) - \lambda^I(x,t)\Big)\,\psi(x-,t)\,\Big|u^I(x+,t) - u^I(x-,t)\Big|\;. \tag{5.4b}$$

Here $\lambda^I(x,t)$ is a the shock speed of the discontinuity in u^I located at (x,t). The measure $\mu^{II}_\psi(t)$ is defined similarly. Regarding the expression (5.4b), it is worth noting that if (x,t) is a point of approximate jump of u^I and ψ, then the jump relation for the equation (4.1) reads

$$\Big(a(x-,t) - \lambda^I(x,t)\Big)\,\psi(x-,t) \;=\; \Big(a(x+,t) - \lambda^I(x,t)\Big)\,\psi(x+,t)\;. \tag{5.5}$$

In the same way we define

$$\mu^{II}_\psi(t) \;=\; \Big(f'(u^{II}(t)) - a\Big)\,\psi(t)\;dV^{II}(t)\;.$$

We now prove:

Theorem 5.1. *Let u^I and u^{II} be two entropy solutions of* (1.1) *such that $u^{II} - u^I$ admits finitely many changes of sign. Let ψ be any solution of bounded variation of the hyperbolic equation* (4.1) *satisfying the constraint*

$$\psi\,(u^{II} - u^I) \;\geq\; 0\;. \tag{5.6}$$

Then for all $0 \leq s \leq t$

$$\begin{aligned}\|\psi(t)\|_{w(t)} &+ \int_s^t \sum_{(x,\tau)\in\mathcal{L}(a)} \Big(2\,m + TV(a)\Big)\,\Big|a_-(x,\tau) - \lambda(x,\tau)\Big|\,|\psi_-(x,\tau)|\;d\tau \\ &+ \int_s^t\int_{\mathbb{R}} \Big(a(\tau) - f'(u^I(\tau))\Big)\,\psi(\tau)\;dV^I(\tau)\,d\tau \\ &+ \int_s^t\int_{\mathbb{R}} \Big(a(\tau) - f'(u^{II}(\tau))\Big)\,\psi(\tau)\;dV^{II}(\tau)\,d\tau \;\;\leq\;\; \|\psi(s)\|_{w(s)}\;.\end{aligned} \tag{5.7}$$

The assumption (5.6) is clearly satisfied with the choice $\psi = u^{II} - u^I$. Therefore our previous result in Theorem 3.5 (derived via a completely different proof) can be regarded as a corollary of Theorem 5.1.

It is interesting to observe that, when $u^{II} = u^I$, the weight (5.2) becomes constant, and therefore (5.7) reduces to the L^1 estimate.

$$\|\psi(t)\|_{L^1(\mathbb{R})} + \int_s^t \sum_{(x,\tau)\in\mathcal{L}(a)} \Big(2m + TV(a)\Big) \Big|a_-(x,\tau) - \lambda(x,\tau)\Big| \, |\psi_-(x,\tau)| \; d\tau$$
$$\leq \|\psi(s)\|_{L^1(\mathbb{R})} \, .$$

Also, note that under the assumption (5.6) $\mu^I_\psi(t)$ and $\mu^{II}_\psi(t)$ are positive except at points $(x,t) \in \mathcal{L}(a) \cup \mathcal{R}(a)$. However, these negative terms are offset in (5.7) by the positve terms under the first integral.

Proof: Fix any positive time t. By assumption we have finitely many points $-\infty = y_0 < y_1 < ... < y_n < y_{n+1} = +\infty$ such that, on each interval (y_i, y_{i+1}), we have $\psi(t) \geq 0$ when i is odd and $\psi(t) \geq 0$ when i is even. For every $i = 1, ..., n$, consider the (unique by Proposition 4.4) forward characteristic $y_i(\cdot)$ associated with the coefficient a and issuing from the initial point (y_i, t).

We will focus attention on some interval (y_i, y_{i+1}) with i odd, say, and with $-\infty < y_i < y_{i+1} < +\infty$. Except when specified differently, all of the characteristics to be considered from now on are associated with the solution u^{II}. For definiteness we will first study the case that the forward characteristic $\chi_0(\cdot)$ (associated with u^{II} and) issuing from the point (y_i, t) is located on the right-side of the curve y_i, that is,

$$y_i(\tau) \leq \chi_0(\tau) \, , \qquad t \leq \tau \leq t + \delta$$

for some $\delta > 0$ sufficiently small.

Fix some (sufficiently small) $\epsilon > 0$ and denote by $y_i < z_1 < ... < z_N < y_{i+1}$ the points where u^I has a jump larger or equal to ϵ, that is,

$$u^{II}_-(z_I, t) - u^{II}_+(z_I, t) \geq \epsilon \, , \qquad I = 1, ..., N \, . \tag{5.8}$$

For each $I = 1, ..., N$, consider also the forward characteristic $\chi_I(\cdot)$ issuing from the point (z_I, t). For definiteness, we will also assume that the forward characteristic $\chi_{N+1}(\cdot)$ issuing from (y_{i+1}, t) satisfies

$$\chi_{N+1}(\tau) \leq y_{i+1}(\tau) \, , \qquad t \leq \tau \leq t + \delta$$

for some $\delta > 0$ sufficiently small.

Next, let us select a time $s > t$ with $s - t$ so small that the following properties hold:

(**a**) No intersection among the characteristics y_i, χ_0, χ_1, ..., χ_N, χ_{N+1}, y_{i+1} occurs in the time interval $[t, s]$.

(**b**) For $I = 1, ..., N$, let $\zeta_I(\cdot)$ and $\xi_I(\cdot)$ be the minimal and the maximal backward characteristics emanating from the point $(\chi_I(s), s)$. Then the total variation of $u^{II}(\cdot, t)$ over the intervals $(\zeta_I(t), z_I)$ and $(z_I, \xi_I(t))$ should not exceed $\frac{\epsilon}{N}$.

(**c**) Let $\zeta_0(\cdot)$ be the minimal backward characteristic emanating from $(y_i(s), s)$ and $\xi_0(\cdot)$ be the maximal backward characteristic emanating from $(\chi_0(s), s)$. Then the total variation of $u^{II}(\cdot, t)$ over the intervals $(y_i, \xi_0(t))$ and $(\zeta_0(t), y_i)$ should not exceed ϵ.

(**d**) Let $\zeta_{N+1}(\cdot)$ be the minimal backward characteristic emanating from the point $(\chi_{N+1}(s), s)$ and $\xi_{N+1}(\cdot)$ be the maximal backward characteristic emanating from $(y_{i+1}(s), s)$. Then the total variation of $u^{II}(\cdot, t)$ over the intervals $(\zeta_{N+1}(t), y_{i+1})$ and $(y_{i+1}, \xi_{N+1}(t))$ should not exceed ϵ.

For $I = 0, ..., N$, and some integer k to be fixed later, consider a mesh of the form

$$\chi_I(s) = x_I^0 < x_I^1 < ... < x_I^k < x_I^{k+1} = \chi_{I+1}(s) \ . \tag{5.9}$$

For $I = 0, ..., N$ and $j = 1, ..., k$, consider also the maximal backward characteristic $\xi_I^j(\cdot)$ emanating from the point (x_I^j, s) and identify its intercept $z_I^j = \xi_I^j(t)$ by the horizontal line at time t. Finally set also

$$z_0^0 = y_i \,, \quad z_N^{k+1} = y_{i+1} \,, \quad z_{I-1}^{k+1} = z_I^0 = z_I, \quad I = 1, ..., N \ .$$

To start the proof, we integrate the equation (4.1) satisfied by the function ψ, successively in each domain limited by the characteristics introduced above. Applying Green's theorem, we arrive at the following five formulas:

(**i**) Integrating (4.1) on the region

$$\Big\{(x, \tau) \ / \ t < \tau < s, \quad y_i(\tau) < x < \chi_0(\tau)\Big\}$$

and multiplying by $V^{II}(y_i, t)$ one gets

$$\int_{y_i(s)}^{\chi_0(s)} \psi(x, s)\, V^{II}(y_i, t)\ dx \ + \int_t^s (y_i' - a_+)\, \psi_+(y_i(\tau), \tau)\, V^{II}(y_i, t)\ d\tau + \int_t^s (a_- - \lambda_0)\, \psi_-(\chi_0(\tau), \tau)\, V^{II}(y_i, t)\ d\tau \ = \ 0 \ . \tag{5.10i}$$

(ii) Integrating (4.1) on each of the regions

$$\Big\{(x,\tau)\ /\ \ t<\tau<s,\quad \xi_I^j(\tau)<x<\xi_I^{j+1}(\tau)\Big\}$$

for $I=0,...,N$ and $j=1,...,k$, and then multiplying by $V^{II}(z_I^j+,t)$, one gets

$$\begin{aligned}(5.10\text{ii})\qquad & \int_{x_I^j}^{x_I^{j+1}} \psi(x,s)\,V^{II}(z_I^j+,t)\ dx\ -\int_{z_I^j}^{z_I^{j+1}} \psi(x,t)\,V^{II}(z_I^j+,t)\ dx \\ &+\int_t^s (\lambda_I^j-a_+)\,\psi_+(\xi_I^j(\tau),\tau)\,V^{II}(z_I^j+,t)\ d\tau \\ &+\int_t^s (a_- -\lambda_I^{j+1})\,\psi_-(\xi_I^{j+1}(\tau),\tau)\,V^{II}(z_I^j+,t)\ d\tau \quad=\quad 0\ .\end{aligned}$$

(iii) Integrating (4.1) on each of the regions

$$\Big\{(x,\tau)\ /\ \ t<\tau<s,\quad \chi_I(\tau)<x<\xi_I^1(\tau)\Big\}$$

for $I=0,...,N$, and multiplying by $V^{II}(z_I+,t)$ one gets

$$\begin{aligned}(5.10\text{iii})\qquad & \int_{\chi_I(s)}^{x_I^1} \psi(x,s)\,V^{II}(z_I+,t)\ dx\ -\int_{z_I}^{z_I^1} \psi(x,t)\,V^{II}(z_I+,t)\ dx \\ &+\int_t^s (\lambda_I-a_+)\,\psi_+(\chi_I(\tau),\tau)\,V^{II}(z_I+,t)\ d\tau \\ &+\int_t^s (a_- -\lambda_I^1)\,\psi_-(\xi_I^1(\tau),\tau)\,V^{II}(z_I+,t)\ d\tau \quad=\quad 0\ .\end{aligned}$$

(iv) Integrating (4.1) on the regions

$$\Big\{(x,\tau)\ /\ \ t<\tau<s,\quad \xi_I^k(\tau)<x<\chi_{I+1}(\tau)\Big\}$$

for $I=0,...,N$, and multiplying by $V^{II}(z_I^k+,t)$ one gets

$$\begin{aligned}(5.10\text{iv})\qquad & \int_{x_I^k}^{\chi_{I+1}(s)} \psi(x,s)\,V^{II}(z_I^k+,t)\ dx\ -\int_{z_I^k}^{z_{I+1}} \psi(x,t)\,V^{II}(z_I^k+,t)\ dx \\ &+\int_t^s (\lambda_I^k-a_+)\,\psi_+(\xi_I^k(\tau),\tau)\,V^{II}(z_I^k+,t)\ d\tau \\ &+\int_t^s (a_- -\lambda_{I+1})\,\psi_-(\chi_{I+1}(\tau),\tau)\,V^{II}(z_I^k+,t)\ d\tau \quad=\quad 0\ .\end{aligned}$$

(**v**) Finally integrating (4.1) on the last region

$$\Big\{(x,\tau)\ /\ \ t<\tau<s,\quad \chi_{N+1}(\tau)<x<y_{i+1}(\tau)\Big\}$$

and multiplying by $V^{II}(y_{i+1},t)$ one gets

$$\begin{aligned}(5.10\text{v})\qquad &\int_{\chi_{N+1}(s)}^{y_{i+1}(s)}\psi(x,s)\,V^{II}(y_{i+1},t)\ dx\\ &\qquad+\int_t^s(\lambda_{N+1}-a_+)\,\psi_+(\chi_{N+1}(\tau),\tau)\,V^{II}(y_{i+1},t)\ d\tau\\ &\qquad+\int_t^s(a_--y'_{i+1})\,\psi_-(y_{i+1}(\tau),\tau)\,V^{II}(y_{i+1},t)\ d\tau\ =\ 0\ .\end{aligned}$$

Next, summing all of the formulas (5.10) leads us to the general identity:

$$\begin{aligned}(5.11)\qquad &\int_{y_i(s)}^{\chi_0(s)}\psi(x,s)\,V^{II}(y_i,t)\ dx\ +\sum_{I=0}^{N}\sum_{j=0}^{k}\int_{x_I^j}^{x_I^{j+1}}\psi(x,s)\,V^{II}(z_I^j+,t)\ dx\\ &+\int_{\chi_{N+1}(s)}^{y_{i+1}(s)}\psi(x,s)\,V^{I}(y_{i+1},t)\ dx\ -\sum_{I=0}^{N}\sum_{j=0}^{k}\int_{z_I^j}^{z_I^{j+1}}\psi(x,t)\,V^{II}(z_I^j+,t)\ dx\\ =\ &-\sum_{I=0}^{N}\sum_{j=1}^{k}\int_t^s\Big[V^{II}(z_I^j+,t)-V^{II}(z_I^{j-1}+,t)\Big]\,(\lambda_I^j-a_-)\,\psi_-(\xi_I^j(\tau),\tau)\ d\tau\\ &-\sum_{I=0}^{N}\int_t^s\Big[V^{II}(z_I+,t)-V^{II}(z_{I-1}^k+,t)\Big]\,(\lambda_I^j-a_-)\,\psi_-(\chi_I(\tau),\tau)\ d\tau\\ &-\int_t^s\Big[V^{II}(y_i+,t)-V^{II}(y_i,t)\Big]\,(\lambda_0-a_-)\,\psi_-(\chi_0(\tau),\tau)\ d\tau\\ &-\int_t^s\Big[V^{II}(y_{i+1}+,t)-V^{II}(z_N^k,t)\Big]\,(\lambda_{N+1}-a_-)\,\psi_-(\chi_{N+1}(\tau),\tau)\ d\tau\\ &-\int_t^s(y_i'-a_+)\,\psi_+(y_i(\tau),\tau)\,V^{II}(y_i,t)\ d\tau\\ &-\int_t^s(a_--y'_{i+1})\,\psi_-(y_{i+1}(\tau),\tau)\,V^{II}(y_{i+1},t)\ d\tau\ .\end{aligned}$$

To estimate the right-hand side of (5.11), we recall that the solution u^I of a scalar conservation law satisfies

$$V^{II}(y_i,t)\ \geq\ V^{II}(\chi_0(s),s)\,,\qquad V^{II}(z_I^j+,t)\ \geq\ V^{II}(x_I^j+,s)\ ,$$

for $I=0,...,N$ and $j=0,...,k$. Hence, choosing the difference $x_I^{j+1}-x_I^j$ in (5.9) sufficiently small and since the function $V^{II}(\cdot,t)$ is nondecreasing, we conclude that the left-hand side of (5.11) can be bounded from below, as follows:

$$\begin{aligned}\text{L.H.S.} \ \geq & \int_{y_i(s)}^{y_{i+1}(s)} \psi(x,s)\, V^{II}(x,s)\ dx \ - \ (s-t)\,\varepsilon \\ & - \int_{y_i(t)}^{y_{i+1}(t)} \psi(x,t)\, V^{II}(x,t)\ dx\ .\end{aligned} \tag{5.12}$$

Estimating the right-hand side of (5.11) is more involved. First note that each term arising in the left-hand side of (5.11) is non-positive. This follows from our condition (5.6). Indeed, consider a point (x,s) of approximate jump or approximate continuity of u^I, u^{II} and ψ. If all of these functions are continuous, the result is trivial. Call λ the discontinuity speed. Based on the jump relation (5.5), we see that either $\psi_-\,(\lambda - a_-) = \psi_+\,(\lambda - a_+) = 0$, or else all of the terms ψ_-, $\lambda - a_-$, ψ_+, and $\lambda - a_+$ are distinct from zero.

Suppose first that (x,s) is a point in the interior of the region limited by the two curves $y_i(\cdot)$ and $y_{i+1}(\cdot)$. In the latter case, since $\psi \geq 0$ in the region under consideration, we deduce that $\psi_- > 0$ and $\psi_+ > 0$, while the terms $\lambda - a_-$ and $\lambda - a_+$ are either both negative or both positive. Actually, in view of the sign condition (5.6), we have $u^{II}_\pm - u^I_\pm \geq 0$ and, therefore, $\lambda - a_\pm \geq 0$ as follows from (3.10) (here we are dealing with a jump of u^{II}).

Consider next a point of the boundary y_i, for instance. So we now have $\psi_- < 0$ and $\psi_+ > 0$, while the terms $\lambda - a_-$ and $\lambda - a_+$ have opposite sign. Since no rarefaction-shock can arise, the discontinuity must be a Lax shock and so $\lambda - a_- < 0$ and $\lambda - a_+ > 0$. Again the corresponding term in (5.11) has a favorable sign. (Observe that the condition (5.6) was not used in this second case.)

Then, for all $I = 0, \dots, N$ and $j = 1, \dots, k$, let $\theta_I^j(\cdot)$ be the (maximal, for definiteness) backward characteristic associated with u^I and issuing from the point $(\xi_I^j(\tau), \tau)$. Denote also by $\theta(z_I^j; \tau)$ its intercept with the horizontal line at time t. Setting

$$\tilde a(x,t;\tau) := \frac{f(u^{II}(x,t)) - f(u^I(\theta(x,\tau),t))}{u^{II}(x,t) - u^I(\theta(x;\tau),t)}$$

and using that the solution u^I remains constant along the characteristic $\theta_I^j(\cdot)$, we obtain

$$(\lambda_I^j - a_-)(\xi_I^j(\tau)) \ = \ \lambda_I^j(z_I^j) - \tilde a(z_I^j, t; \tau)\ . \tag{5.13}$$

Then consider the (maximum, for definiteness) backward characteristic $y_I^j(\cdot)$ associated with a and issuing from the point $(\xi_I^j(\tau), \tau)$. By integrating ψ along the characteristic $y_I^j(\cdot)$ and using the inequality (4.4), we arrive at a lower bound for

ψ

$$\psi(\xi_I^j(\tau),\tau) \ \geq \ \psi(y_I^j(t),t) \left(\frac{t}{\tau}\right)^E , \qquad t < \tau < s \ . \tag{5.14}$$

Upon choosing $x_I^{j+1} - x_I^j$ in (5.9) so small that the oscillation of $V_c^{II}(\cdot)$ over each interval $(z_I^j - z_I^{j+1})$ does not exceed ε and recalling the standard estimates on Stieltjes integrals we deduce from (5.11)–(5.13) that

$$\begin{aligned}
&\sum_{I=0}^{N}\sum_{j=1}^{k}\int_t^s \Big[V^{II}(z_I^j+,t) - V^{II}(z_I^{j-1}+,t)\Big]\Big((\lambda_I^j - a_-)\,\psi_-(\xi_I^j(\tau),\tau)\Big)\,d\tau \qquad (5.15)\\
&\geq \sum_{I=0}^{N}\sum_{j=1}^{k}\int_t^s \Big[V_c^{II}(z_I^j,t) - V_c^{II}(z_I^{j-1},t)\Big]\Big(\lambda_I^j(z_I^j) - \tilde a(z_I^j,t,\tau)\Big)\,\psi(y_I^j(t),t)\left(\frac{t}{\tau}\right)^E d\tau\\
&\geq \int_t^s \sum_{I=0}^{N}\sum_{j=1}^{k}\left(\int_{z_I^{j-1}}^{z_I^j}\Big(\lambda_I^j(x) - \tilde a(x,t,\tau)\Big)\,\psi(x,t)\ dV_c^{II}(x,t) \ - \ c\,\varepsilon\right)\left(\frac{t}{\tau}\right)^E d\tau\\
&= \int_t^s \left(\int_{y_i}^{y_{i+1}}\Big(\lambda_I^j(x) - \tilde a(x,t,\tau)\Big)\,\psi(x,t)\ dV_c^{II}(x,t) \ - \ c(y_{i+1} - y_i)\,\varepsilon\right)\left(\frac{t}{\tau}\right)^E d\tau \ .
\end{aligned}$$

We now combine (5.10), (5.11) and (5.15), divide the resulting inequality by $s-t$, and let $s \searrow t$, $\epsilon \to 0$, obtaining the following inequality:

$$\begin{aligned}
&\frac{d^+}{dt}\int_{y_i(t)}^{y_{i+1}(t)} \psi(x,t)\,V^{II}(x,t)\ dx \qquad (5.16)\\
&\quad\leq \ - \ \int_{y_i}^{y_{i+1}}\Big(\lambda_I^j(x) - a(x,t)\Big)\,\psi(x,t)\ dV_c^{II}(x,t)\\
&\quad\quad - \sum_{(x,t)\in\mathcal{J}(u^{II})}\Big(u_-^{II}(x,t) - u_+^{II}(x,t)\Big)\,(\lambda^I - a_-)(x,t)\,\psi_-(x,t)\\
&\quad\quad - \Big(u_-^{II}(y_i,t) - u_+^{II}(y_i,t)\Big)\,(\lambda^I - a_-)(y_i,t)\,\psi_-(y_i,t)\\
&\quad\quad - \Big(u_-^{II}(y_{i+1},t) - u_+^{II}(y_{i+1},t)\Big)\,(\lambda^I - a_-)(y_{i+1},t)\,\psi_-(y_{i+1},t)\\
&\quad\quad - (y_i' - a_+)\,\psi_+(y_i,t)\,V^{II}(y_i,t) \ - \ (a_- - y_{i+1}')\,\psi_-(y_{i+1},t)\,V^{II}(y_{i+1},t) \ .
\end{aligned}$$

The third and fourth terms in the right-hand side of (5.16) are due to the fact that χ_0 and χ_{N+1} lie inside the region limited by y_i and y_{i+1}.

We can next focus on the intervals (y_i, y_{i+1}) with i even. Based on a completely symmetric argument and using now the weight $m + V^{II}(\infty,t) - V^{II}(\cdot,t)$

instead of $V^{II}(\cdot,t)$, we obtain

$$\begin{aligned}
(5.17)\qquad \frac{d^+}{dt}\int_{y_i(t)}^{y_{i+1}(t)} &(-\psi(x,t))\Big(m+V^{II}(\infty,t)-V^{II}(x,t)\Big)\,dx\\
&\le \int_{y_i}^{y_{i+1}}\Big(\lambda_I^j(x)-a(x,t)\Big)(-\psi(x,t))\;dV_c^{II}(x,t)\\
&\quad+\sum_{(x,t)\in\mathcal{J}(u^{II})}\Big(u_-^{II}(x,t)-u_+^{II}(x,t)\Big)\,(\lambda^I-a_-)(-\psi_-)(x,t)\\
&\quad+\Big(u_-^{II}(y_i,t)-u_+^{II}(y_i,t)\Big)\,(\lambda^I-a_-)(-\psi_-)(y_i,t)\\
&\quad+\Big(u_-^{II}(y_{i+1},t)-u_+^{II}(y_{i+1},t)\Big)\,(\lambda^I-a_-)(-\psi_-)(y_{i+1},t)\\
&\quad-(y_i'-a_+)(-\psi_+)(y_i,t)\Big(m+V^{II}(\infty,t)-V^{II}(y_i,t)\Big)\\
&\quad-(a_--y_{i+1}')(-\psi_-)(y_{i+1},t)\Big(m+V^{II}(\infty,t)-V^{II}(y_{i+1},t)\Big)\,.
\end{aligned}$$

By summation over $i=1,...,n$ in (5.16) for i odd and in (5.17) for i even respectively, we obtain

$$\begin{aligned}
\frac{d^+}{dt}\int_{-\infty}^{+\infty}&[\psi(x,t)]^+\,V^{II}(x,t)\;+\;[-\psi(x,t)]^+\Big(m+V^{II}(\infty,t)-V^{II}(x,t)\Big)\,dx\\
&\le\;-\sum_{(x,t)\in\mathcal{L}(a)\cap\mathcal{J}(u^{II})}\Big(m+V^{II}(\infty,t)\Big)\Big|\lambda(x,t)-a_-(x,t)\Big|\,|\psi_-(x,t)|\\
(5.18)\qquad&\quad-\sum_{(x,t)\in\mathcal{J}(u^{II})}\Big(u_-^{II}(x,t)-u_+^{II}(x,t)\Big)\Big(\lambda^I(x,t)-a_-(x,t)\Big)\,\psi_-(x,t)\\
&\quad-\int_{\mathbb{R}}\Big(f'(u^{II}(y,t))-a(y,t)\Big)\,\psi(y,t)\;dV_c^{II}(y,t)\;,
\end{aligned}$$

where the superscript $+$ denotes the positive part of the functions ψ and $-\psi$ respectively.

Consider now the case where

$$\chi_0(\tau)\;\le\;y_i(\tau)\,,\qquad t\le\tau\le t+\delta\;,$$

and

$$y_{i+1}(\tau)\;\le\;\chi_{N+1}(\tau)\,,\qquad t\le\tau\le t+\delta\;.$$

Assume that there exists a time $\bar\tau>t$ such that

$$\chi_0(\bar\tau)<y_i(\bar\tau)\,,\qquad y_{i+1}(\bar\tau)<\chi_{N+1}(\bar\tau)$$

(otherwise the curves of the two pairs will coincide, and we can reduce to the previous case). Let now $\xi_0(\cdot)$ be the maximal backward characteristic emanating from $(y_i(\bar\tau),\bar\tau)$, and $\zeta_{N+1}(\cdot)$ be the minimal backward characteristic emanating from the point $(y_{i+1}(\bar\tau),\bar\tau)$. Since characteristics cannot cross, we have that

$$y_i(t) < \xi_0(t)\,, \qquad \zeta_{N+1}(t) < y_{i+1}(t)\ ,$$

Then, by finite propagation speed, there exists a time $s > t$ such that

$$y_i(\tau) < \xi_0(\tau)\,, \quad \zeta_{N+1}(\tau) < y_{i+1}(\tau)\,, \qquad t \le \tau < s\ ,$$

$$y_i(s) = \xi_0(s)\,, \qquad \zeta_{N+1}(s) = y_{i+1}(s)\ .$$

Instead of properties (c), (d), we will require that s satisfies the following:

- **(c′)** Let $\zeta_0(\cdot)$ be the minimal backward characteristic emanating from $(\chi_0(s),s)$. Then the total variation of $u^{II}(\cdot,t)$ over the intervals $(y_i,\xi_0(t))$ and $(\zeta_0(t),y_i)$ should not exceed ϵ.
- **(d′)** Let $\xi_{N+1}(\cdot)$ be the maximal backward characteristic emanating from $(\chi_{N+1}(s),s)$. Then the total variation of $u^{II}(\cdot,t)$ over the intervals $(\zeta_{N+1}(t),y_{i+1})$ and $(y_{i+1},\xi_{N+1}(t))$ should not exceed ϵ.

From then on we can proceed as before. Finally we write the inequality in (5.18) exchanging the roles of u^I and u^{II}, and combining it with (5.17) we arrive exactly at the desired inequality (5.7) and the proof of Theorem 5.1 is completed. ∎

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REFERENCES

[1] Baiti, P.; LeFloch, P.G. and Piccoli, B. – Nonclassical shocks and the Cauchy problem: General conservation laws, *Contemporary Math.,* 238 (1999), 1–25.

[2] Bressan, A.; Crasta, G. and Piccoli, B. – Well-posedness of the Cauchy problem for $n \times n$ systems of conservation laws, *Mem. Amer. Math. Soc.,* 146 (2000).

[3] Bressan, A. and LeFloch, P.G. – Structural stability and regularity of entropy solutions to systems of conservation laws, *Indiana Univ. Math. J.,* 48 (1999), 43–84.

[4] Bressan, A.; Liu, T.P. and Yang, T. – L^1 stability estimate for $n \times n$ conservation laws, *Arch. Rational Mech. Anal.,* 149 (1999), 1–22.

[5] Chalons, C. and LeFloch, P.G. – High-order entropy-conservative schemes and kinetic relations for van der Waals fluids, *J. Comput. Phys.,* 167 (2001), 1–23.

[6] Crasta, G. and LeFloch, P.G. – *Existence theory for a class of strictly hyperbolic systems*, in preparation.

[7] Dafermos, C.M. – Polygonal approximations of solutions of the initial value problem for a conservation law, *J. Math. Anal. Appl.,* 38 (1972), 33–41.

[8] Dafermos, C.M. – *Generalized characteristics in hyperbolic conservation laws: a study of the structure and the asymptotic behavior of solutions*, in "Nonlinear Analysis and Mechanics: Heriot-Watt symposium" (R.J. Knops, Ed.), Pitman, London, Vol. 1 (1977), 1–58.

[9] Dafermos, C. – *Hyperbolic Conservation Laws in Continuum Physics*, Grundlehren Math. Wissen., Vol. 325, Springer Verlag, 2000.

[10] Dal Maso, G.; LeFloch, P.G. and Murat, F. – Definition and weak stability of nonconservative products, *J. Math. Pures Appl.,* 74 (1995), 483–548.

[11] Evans, L.C. and Gariepy, R.F. – *Measure Theory and Fine Properties of Functions*, Studies in Advanced Mathematics, CRC Press, 1992.

[12] Filippov, A.F. – Differential equations with discontinuous right hand-side, *Math. USSR-Sb.,* 51 (1960), 99–128. English transl. in *A.M.S. Transl., Ser. 2*, 42, 199–231.

[13] Goatin, P. and LeFloch, P.G. – Sharp L^1 continuous dependence of solutions of bounded variation for hyperbolic systems of conservation laws, *Arch. Rational Mech. Anal.,* (2001), to appear.

[14] Hu, J.-X. and LeFloch, P.G. – L^1 continuous dependence property for systems of conservation laws, *Arch. Rational Mech. Anal.,* 151 (2000), 45–93.

[15] Lax, P.D. – *Shock wave and entropy*, in "Contributions to Nonlinear Functional Analysis" (E. Zarantonello, Ed.), Acad. Press, New York, 1971, 603–634.

[16] LeFloch, P.G. – *An existence and uniqueness result for two nonstrictly hyperbolic systems*, IMA Volumes in Math. and its Appl. 27, "Nonlinear Evolution Equations that Change Type", (B.L. Keyfitz and M. Shearer, Eds.), Springer Verlag (1990), 126–138.

[17] LeFloch, P.G. – *An introduction to nonclassical shocks of systems of conservation laws*, Proc. International School on Hyperbolic Problems, Freiburg, Germany, Oct. 97 (D. Kröner, M. Ohlberger and C. Rohde, Eds.), Lect. Notes Comput. Eng., Vol. 5, Springer Verlag, 1998, 28–72.

[18] LeFloch, P.G. – L^1 *continuous dependence property for systems of conservation laws*, to appear.

[19] LeFloch, P.G. – *Hyperbolic Systems of Conservation Laws: The Theory of Classical and Nonclassical Shock Waves*, ETH Lecture Notes Series, Birkhauser, 2001, to appear.

[20] LeFloch, P.G. and Liu, T.P. – Existence theory for nonlinear hyperbolic systems in nonconservative form, *Forum Math.,* 5 (1993), 261–280.

[21] LeFloch, P.G. and Xin, Z.P. – Uniqueness via the adjoint problems for systems of conservation laws, *Comm. Pure Appl. Math.,* 46 (1993), 1499–1533.

[22] Liu, T.P. and Yang, T. – A new entropy functional for scalar conservation laws, *Comm. Pure Appl. Math.,* 52 (1999), 1427–1442.

[23] Liu, T.P. and Yang, T. – L^1 stability of weak solutions for 2x2 systems of hyperbolic conservation laws, *J. Amer. Math. Soc.,* 12 (1999), 729–774.

[24] Liu, T.P. and Yang, T. – Well-posedness theory for hyperbolic conservation laws, *Comm. Pure Appl. Math.,* 52 (1999), 1553–1580.

[25] Volpert, A.I. – The space BV and quasilinear equations, *Math. USSR Sbornik,* 73 (1967), 225–267.

P. Goatin
Centre de Mathématiques Appliquées & Centre National de la Recherche Scientifique
U.M.R. 7641, Ecole Polytechnique, 91128 Palaiseau Cedex – FRANCE
E-mail: `goatin@cmap.polytechnique.fr`

and

P.G. LeFloch
Centre de Mathématiques Appliquées & Centre National de la Recherche Scientifique
U.M.R. 7641, Ecole Polytechnique, 91128 Palaiseau Cedex – FRANCE
E-mail: `lefloch@cmap.polytechnique.fr`