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OSCILLATIONS AND NONOSCILLATIONS IN RETARDED EQUATIONS

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Abstract: This note is concerned with the oscillatory behavior of a scalar linear retarded equation. Criteria for oscillations are obtained either dependent or independently of the delays.

1 – Introduction

The aim of this paper is to study the oscillatory behavior of all solutions of the linear difference retarded functional equation

$$x(t) = \int_{-1}^{0} x\Big(t - r(\theta)\Big)\, d\nu(\theta) \ , \tag{1}$$

where $x(t) \in \mathbb{R}$, ν is a function of bounded variation on $[-1, 0]$ such that $\nu(-1) = 0$, and $r(\theta)$ is a positive continuous real function on $[-1, 0]$.

The most common general linear retarded functional equation studied in the literature (see for example [3]) is

$$x(t) = \int_{-r}^{0} x(t+\theta)\, d\eta(\theta) \ , \tag{2}$$

where r is a positive real number and η is a function of bounded variation on $[-1, 0]$ atomic at zero, that is, such that,

$$\lim_{s\to 0^+} \int_{-s}^{0} |d\eta(\theta)| \ = \ 0 \ ,$$

if one denotes by $\int_{-s}^{0} |d\eta(\theta)|$, the variation of η on the interval $[-s, 0]$. If we allowed in equation (1), $r(\theta)$ to be nonnegative, then with $r(\theta) = -r\,\theta$ and

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$\eta(\theta) = \nu(\theta/r)$ atomic at zero, we obtain the class of equations (2). So, there is actually no gain in generality of (1) with respect to (2), but as we will see there is some gain in convenience. On the other hand the restriction on $r(\theta)$ to be positive, makes unnecessary any atomicity assumption on ν.

In the case where ν has no singular part, (1) takes the form

$$x(t) = \sum_{k=1}^{\infty} a_k\, x(t-r_k) + \int_{-1}^{0} a(\theta)\, x(t-r(\theta))\, d\theta \,, \tag{2}$$

where, $0 < \sigma \le r_k \le \tau$ $(k = 1, 2, ...)$ and the sequence a_k and the function $a(\theta)$ are such that

$$\sum_{k=1}^{\infty} |a_k| + \int_{-1}^{0} |a(\theta)|\, d\theta < \infty \,; \tag{3}$$

if ν is a step function with a number q of jump points, then we obtain the delay difference equation

$$x(t) = \sum_{j=1}^{q} a_j\, x(t-r_j) \,, \tag{4}$$

where the a_j are nonzero real numbers and each r_j is a positive real number $(j = 1, ..., q)$. With respect to this equation, (1) corresponds to a more general situation where coefficients and lags are all included in the function ν.

Considering the value $\|r\| = \max\{r(\theta) : -1 \le \theta \le 0\}$, by a solution of (1) we mean a continuous function $x\colon [-\|r\|, +\infty[\to \mathbb{R}$, which satisfies (1) for every $t \ge 0$. A solution is said *oscillatory* whenever it has an infinite number of zeros; otherwise it will be said *nonoscillatory*. When all solutions are oscillatory, the equation is called *totally oscillatory*.

As, for any given ν and r, a solution of (1) exhibits an integral exponential boundedness (see [3]), according to [5] we can conclude that it has a totally oscillatory behavior if and only if the characteristic equation

$$1 - \int_{-1}^{0} e^{-\lambda r(\theta)}\, d\nu(\theta) = 0 \tag{5}$$

has no real roots.

Through this characterization, we study in section 2 under which conditions on ν, one can assure that equation (1) is totally oscillatory or exhibit nonoscillatory solutions. In [4] this problem is discussed in the more general framework of a functional differential system of neutral type, but all the results obtained seem to exclude the present situation. In section 3 we analyze the dependence on the delay function $r(\theta)$, of the totally oscillatory behavior of (1).

By C^+ we denote the subset of $C([-1,0],\mathbb{R})$ formed by all positive functions, $r(\theta)$; through $\|r\|$ a metric is introduced in C^+. With respect to a delay function, $r \in C^+$, it will be often considered the value

$$m(r) = \min_{-1\le\theta\le 0} r(\theta) > 0 .$$

We take the Banach space *NBV* of all (normalized) real functions of bounded variation, ν, on $[-1,0]$, such that $\nu(-1)=0$, with the norm

$$\|\nu\| = \int_{-1}^{0} |d\nu(\theta)| ,$$

where by $\int_{-1}^{0} |d\nu(\theta)|$ we mean the total variation of ν on $[-1,0]$. If for a given $\theta \in [-1,0]$ there exists a $\varepsilon > 0$ such that ν is constant in $[\theta-\varepsilon,\, \theta+\varepsilon]$ ($[-\varepsilon,0]$ if $\theta=0$, $[-1,-1+\varepsilon]$ if $\theta=-1$) we say that θ is a *point of constancy of* ν; if for every $\varepsilon > 0$, sufficiently small, ν is increasing (decreasing) and nonconstant in $[\theta-\varepsilon,\, \theta+\varepsilon]$ ($[-\varepsilon,0]$ if $\theta=0$, $[-1,-1+\varepsilon]$ if $\theta=-1$) θ will be said a *point of increase of* ν (respectively, a *point of decrease of* ν).

If ν is increasing on $[-1,0]$, for a given $\theta \in [-1,0]$ we shall consider the right hand oscillation of ν at θ, as the nonnegative real value $\omega_\nu^+(\theta) = \nu(\theta^+) - \nu(\theta)$; analogously the left hand oscillation, $\omega_\nu^-(\theta)$, and the oscillation, $\omega_\nu(\theta)$, of ν at θ, will be given, respectively, by $\omega_\nu^-(\theta) = \nu(\theta) - \nu(\theta^-)$ and $\omega_\nu(\theta) = \omega_\nu^+(\theta) + \omega_\nu^-(\theta) = \nu(\theta^+) - \nu(\theta^-)$.

As is well known, any function $\nu \in NBV$, can be decomposed as the difference of two increasing functions α and β: $\nu = \alpha - \beta$. However, this decomposition is not unique. A particular decomposition of ν is given by

$$\nu = p - n \,, \tag{6}$$

where by p and n we denote, respectively, the *positive* and *negative variation* of ν, which are defined as follows. For each $\theta \in [-1,0]$, let $\mathcal{P}_\theta$ be the set of all partitions $P = \{-1 = \theta_0, \theta_1, ..., \theta_n = \theta\}$ of the interval $[-1,\theta]$ and to each $P \in \mathcal{P}_\theta$ associate the sets

$$A(P) = \Big\{j\colon\ \nu(\theta_j) - \nu(\theta_{j-1}) > 0\Big\} \quad \text{and} \quad B(P) = \Big\{j\colon\ \nu(\theta_j) - \nu(\theta_{j-1}) < 0\Big\} .$$

Then p and n are defined as

$$p(\theta) = \sup\bigg\{ \sum_{j\in A(P)} \Big(\nu(\theta_j) - \nu(\theta_{j-1})\Big)\colon\ P \in \mathcal{P}_\theta \bigg\} ,$$

$$n(\theta) = \sup\bigg\{ \sum_{j\in B(P)} \Big|\nu(\theta_j) - \nu(\theta_{j-1})\Big|\colon\ P \in \mathcal{P}_\theta \bigg\} ,$$

(whenever $A(P)$ or $B(P)$ are empty, we make $p(\theta) = 0$, $n(\theta) = 0$). One easily sees that both p and n are increasing functions such that $\nu(\theta) = p(\theta) - n(\theta)$, for every $\theta \in [-1, 0]$. Moreover, with respect to the total variation of ν on the interval $[-1, \theta]$,

$$V(\theta) = \int_{-1}^{\theta} |d\nu(s)| ,$$

we have $V(\theta) = p(\theta) + n(\theta)$, for every $\theta \in [-1, 0]$.

2 – Oscillations and nonoscillations

For given $\nu \in NBV$ and $r \in C^+$, define the function

$$g(\lambda) = \int_{-1}^{0} e^{-\lambda r(\theta)}\, d\nu(\theta) .$$

This function has some general features which will be important in the sequel. For example, by uniform convergence of the exponential series, we have

$$g(\lambda) = \sum_{k=0}^{\infty} m_k \, \frac{(-\lambda)^k}{k!} ,$$

where, for $k = 1, 2, \ldots$,

$$m_k = \int_{-1}^{0} (r(\theta))^k \, d\nu(\theta) .$$

As $|m_k| \le \|r\|^k \, \|\nu\|$, for $k = 1, 2, \ldots$, this series converges absolutely and uniformly for λ on compact sets of the complex plane. Thus $g(\lambda)$ is analytic in $\mathbb{R}$ and term-by-term differentiation implies that

$$g'(\lambda) = -\int_{-1}^{0} r(\theta)\, e^{-\lambda r(\theta)} \, d\nu(\theta) .$$

Moreover, since for $\lambda \ge 0$,

$$|g(\lambda)| \le e^{-\lambda m(r)} \, \|\nu\| ,$$

we have that,

$$\lim_{\lambda \to +\infty} g(\lambda) = 0 .$$

These facts enable us to conclude the following characterization of total oscillatory behavior.

Lemma 1. *Equation (1) is totally oscillatory if and only if* $g(\lambda) < 1$, *for every real* λ.

Proof: If $g(\lambda) < 1$ for every $\lambda \in \mathbb{R}$, then (5) has no real roots. Conversely, assuming that for some real number, λ_0, we have $g(\lambda_0) \geq 1$, by continuity, there exists a real λ_1 such that $g(\lambda_1) = 1$, which is contradictory. ∎

In order for (1) to be totally oscillatory, by the same arguments, one cannot have, $g(\lambda) \to +\infty$, as $\lambda \to -\infty$. This situation would give entail the existence of at least one nonoscillatory solution.

This is the case in the following theorems, where ν is decomposed as $\nu = \alpha - \beta$, with α and β increasing in $[-1, 0]$ (in particular, the decomposition (6) is included); $g(\lambda)$ can then be rewritten as

$$g(\lambda) = \int_{-1}^{0} e^{-\lambda r(\theta)}\, d\alpha(\theta) - \int_{-1}^{0} e^{-\lambda r(\theta)}\, d\beta(\theta) .$$

Theorem 1. *Let* $\theta_0 \in [-1, 0]$ *be such that* $\|r\| = r(\theta_0)$. *If*

$$\max\Big\{\omega_\alpha^+(\theta_0),\, \omega_\alpha^-(\theta_0)\Big\} > \|\beta\| ,$$

then (1) has, at least, a nonoscillatory solution.

Proof: Assume, for example, $\omega_\alpha^+(\theta_0) > \|\beta\|$. For every real λ and $\varepsilon > 0$ sufficiently small, we obtain, by application of a mean value property, that

$$\int_{-1}^{0} e^{-\lambda r(\theta)}\, d\alpha(\theta) \geq \int_{\theta_0}^{\theta_0+\varepsilon} e^{-\lambda r(\theta)}\, d\alpha(\theta) = e^{-\lambda r(\theta_0+\delta\varepsilon)}\Big(\alpha(\theta_0+\varepsilon) - \alpha(\theta_0)\Big) ,$$

for some $0 \leq \delta \leq 1$, depending upon, r, θ_0 and ε. Therefore, making $\varepsilon \to 0^+$, we have

$$\int_{-1}^{0} e^{-\lambda r(\theta)}\, d\alpha(\theta) \geq e^{-\lambda r(\theta_0)}\, \omega_\alpha^+(\theta_0) = e^{-\lambda\|r\|}\, \omega_\alpha^+(\theta_0) .$$

On the other hand, for every real $\lambda < 0$,

$$\left|\int_{-1}^{0} e^{-\lambda r(\theta)}\, d\beta(\theta)\right| \leq e^{-\lambda\|r\|}\, \|\beta\| ,$$

and consequently

$$g(\lambda) \geq e^{-\lambda\|r\|}\Big(\omega_\alpha^+(\theta_0) - \|\beta\|\Big) .$$

Thus $g(\lambda) \to +\infty$, as $\lambda \to -\infty$, and so (1) has, at least, a nonoscillatory solution. ∎

Theorem 2. *Let $r \in C^+$ be such that its absolute maximum is attained at a single point $\theta_0 \in [-1,0]$. If θ_0 is a point of increase of α and a point of constancy of β, then (1) has, at least, a nonoscillatory solution.*

Proof: In fact, assuming, for example, that $\theta_0 \in\,]-1,0[$, we have for some $\varepsilon > 0$,

$$g(\lambda) \;\geq\; \int_{\theta_0-\varepsilon}^{\theta_0+\varepsilon} e^{-\lambda r(\theta)}\, d\alpha(\theta) \;-\; \int_{-1}^{\theta_0-\varepsilon} e^{-\lambda r(\theta)}\, d\beta(\theta) \;-\; \int_{\theta_0+\varepsilon}^{0} e^{-\lambda r(\theta)}\, d\beta(\theta) \;.$$

Take $0 < \delta < \varepsilon$ such that

$$m_0 \;=\; \min_{\theta_0-\delta\leq\theta\leq\theta_0+\delta} r(\theta) \;>\; \begin{cases} M_1 = \displaystyle\max_{-1\leq\theta\leq\theta_0-\varepsilon} r(\theta) \;, \\ M_2 = \displaystyle\max_{\theta_0+\varepsilon\leq\theta\leq 0} r(\theta) \;. \end{cases}$$

Since for every real $\lambda < 0$, it holds that

$$\left| \int_{-1}^{\theta_0-\varepsilon} e^{-\lambda r(\theta)}\, d\beta(\theta) \;+\; \int_{\theta_0+\varepsilon}^{0} e^{-\lambda r(\theta)}\, d\beta(\theta) \right| \;\leq\; \left(e^{-\lambda M_1} + e^{-\lambda M_2}\right) \|\beta\| \;,$$

we obtain, by consequence,

$$\begin{aligned} g(\lambda) \;&\geq\; e^{-\lambda m_0}\Big(\alpha(\theta_0+\delta) - \alpha(\theta_0-\delta)\Big) - \left(e^{-\lambda M_1} + e^{-\lambda M_2}\right) \|\beta\| \\ &\geq\; e^{-\lambda m_0}\left[\Big(\alpha(\theta_0+\delta) - \alpha(\theta_0-\delta)\Big) - \left(e^{\lambda(m_0-M_1)} - e^{\lambda(m_0-M_2)}\right) \|\beta\|\right] \;, \end{aligned}$$

and so, also in this case, $g(\lambda) \to +\infty$, as $\lambda \to -\infty$. ∎

Remark 1. These two results are similar but of different kind. In fact, θ_0 is a point of increase of α if $\omega_\alpha(\theta_0) > 0$, which occurs whenever either $\omega_\alpha^+(\theta_0) > 0$, or $\omega_\alpha^-(\theta_0) > 0$; that is, relatively to θ_0 and α, the assumption in the first theorem implies the assumption in the second. However, the assumptions on β are completely independent. □

Corollary 1. *Let $r \in C^+$ be such that its absolute maximum is attained at a single point $\theta_0 \in [-1,0]$. If θ_0 is a point of increase of ν, then (1) has, at least, one nonoscillatory solution.*

Proof: Notice that in this case, for some $\varepsilon > 0$, the negative variation, $n(\theta)$, of ν, is constant on $[\theta_0-\varepsilon,\, \theta_0+\varepsilon]$ and so the corresponding positive variation $p(\theta) = \nu(\theta) - c$, on that interval, for some real number c. Then applying the Theorem 2 to the decomposition (6) we obtain the statement. ∎

Example 1. Consider the equation

$$x(t) = \int_{-1}^{0} \sin(2\pi\theta)\, x(t - r(\theta))\, d\theta \ ,$$

where the delay function, $r(\theta)$, is any strictly decreasing positive function in $[-1, 0]$. The corresponding function of NBV is $\nu(\theta) = \int_{-1}^{\theta} \sin(2\pi s)\, ds$, which has a point of increase at $\theta = -1$, where $r(\theta)$ attains its absolute maximum. Hence the equation has at least, a nonoscillatory solution. □

The above corollary can also be applied, for example, to equation (3), for the case where $a(\theta)$ is the null function, that is to equation

$$x(t) = \sum_{k=1}^{\infty} a_k\, x(t - r_k) \ , \tag{7}$$

where $0 < \sigma \le r_k \le \tau$ $(k = 1, 2, \ldots)$ and the $a_k \neq 0$ $(k = 1, 2, \ldots)$ are such that

$$\sum_{k=1}^{\infty} |a_k| < \infty \ .$$

Corollary 2. *If $r_1 > \ldots > r_k > \ldots$, then (7) has a nonoscillatory solution provided that $a_1 > 0$.* ∎

Theorem 3. *Let $\theta_0 \in [-1, 0]$ be such that $\|r\| = r(\theta_0)$ and $\omega = \omega_\beta(\theta_0) > 0$. For*

$$k(\theta) = \begin{cases} 1/\omega, & \text{if } r(\theta) = \|r\| \ , \\ \dfrac{1}{\omega^{r(\theta)/\|r\|}} \left(\dfrac{r(\theta)}{\|r\|}\right)^{r(\theta)/\|r\|} \left(1 - \dfrac{r(\theta)}{\|r\|}\right)^{1 - r(\theta)/\|r\|}, & \text{if } r(\theta) \neq \|r\| \ , \end{cases}$$

if

$$\int_{-1}^{0} k(\theta)\, d\alpha(\theta) < 1 \ ,$$

then (1) is totally oscillatory.

Proof: We have, for every real λ, $g(\lambda) < 1$ if and only if

$$\frac{\displaystyle\int_{-1}^{0} e^{-\lambda r(\theta)}\, d\alpha(\theta)}{1 + \displaystyle\int_{-1}^{0} e^{-\lambda r(\theta)}\, d\beta(\theta)} < 1 \ .$$

Since,

$$\int_{-1}^{0} e^{-\lambda r(\theta)}\, d\beta(\theta) \;\geq\; e^{-\lambda\|r\|}\, \omega_\beta(\theta_0) \;,$$

(1) becomes totally oscillatory, provided that

$$\int_{-1}^{0} \frac{d\alpha(\theta)}{e^{\lambda r(\theta)} + \omega\, e^{\lambda(r(\theta)-\|r\|)}} \;<\; 1 \;,$$

for every real λ.

As for each $\theta \in [-1,0]$, the function $e^{\lambda r(\theta)} + \omega\, e^{\lambda(r(\theta)-\|r\|)}$, for λ in the real line, is bounded below, we can analyze the value

$$\sup_{\lambda\in\mathbb{R}} \frac{1}{e^{\lambda r(\theta)} + \omega\, e^{\lambda(r(\theta)-\|r\|)}} \;.$$

If $\theta \in [-1,0]$ is such that $r(\theta) = \|r\|$, this supremum is $1/\omega$. In fact, in that case $e^{\lambda\|r\|} + \omega > \omega$, for every real λ and $e^{\lambda\|r\|} + \omega \to \omega$ as $\lambda \to -\infty$. If $\theta \in [-1,0]$ is such that $r(\theta) \neq \|r\|$, since the only stationary point of $e^{\lambda r(\theta)} + \omega\, e^{\lambda(r(\theta)-\|r\|)}$ is $\lambda = \frac{1}{\|r\|} \log \frac{\omega(\|r\|-r(\theta))}{r(\theta)}$, we have that

$$\min_{\lambda\in\mathbb{R}} \Big(e^{\lambda r(\theta)} + \omega\, e^{\lambda(r(\theta)-\|r\|)}\Big) \;=\; \left(\frac{\omega\Big(\|r\| - r(\theta)\Big)}{r(\theta)}\right)^{r(\theta)/\|r\|} \frac{\|r\|}{\|r\| - r(\theta)} \;,$$

and consequently,

$$\sup_{\lambda\in\mathbb{R}} \frac{1}{e^{\lambda r(\theta)} + \omega\, e^{\lambda(r(\theta)-\|r\|)}} \;=\; \frac{1}{\omega^{r(\theta)/\|r\|}} \left(\frac{r(\theta)}{\|r\|}\right)^{r(\theta)/\|r\|} \left(1 - \frac{r(\theta)}{\|r\|}\right)^{1-r(\theta)/\|r\|} \;.$$

Notice that if $\theta' \in [-1,0]$ is such that $r(\theta') = \|r\|$ then, as $\theta \to \theta'$,

$$\frac{1}{\omega^{r(\theta)/\|r\|}} \;\to\; \frac{1}{\omega} \;,$$

$$\left(\frac{r(\theta)}{\|r\|}\right)^{r(\theta)/\|r\|} \;\to\; 1 \;,$$

$$\left(1 - \frac{r(\theta)}{\|r\|}\right)^{1-r(\theta)/\|r\|} \;\to\; 1 \;.$$

Therefore the function $k(\theta)$ is continuous on $[-1,0]$ and if

$$\int_{-1}^{0} k(\theta)\, d\alpha(\theta) \;<\; 1 \;,$$

equation (1) is totally oscillatory. ∎

By separation of positive and negative terms, the equation (7) can be rewritten as

$$x(t) = \sum_{k=1}^{\infty} c_k\, x(t-\gamma_k) + \sum\nolimits_{k=1}^{\infty} b_k\, x(t-\beta_k) \,, \tag{8}$$

with $c_k \geq 0$, $b_k \leq 0$ $(k = 1, 2, ...)$ and

$$\sum_{k=1}^{\infty} \Big(|c_k| + |b_k|\Big) < \infty \,.$$

Assuming $0 < \gamma < ... < \gamma_k < ... < \gamma_1$, $0 < \beta < ... < \beta_k < ... < \beta_1$ $(k = 2, 3, ...)$, $\gamma_1 < \beta_1$ and $b_1 < 0$, the conditions for applying Theorem 3 are met, which yields as a corollary the following extension of [1, Theorem 3] (see also [2, Chapter 7]).

Corollary 3. *If*

$$\sum_{k=1}^{\infty} \frac{c_k}{\beta_1} \left(\frac{\gamma_k}{|b_1|}\right)^{\gamma_k/\beta_1} (\beta_1 - \gamma_k)^{(\beta_1-\gamma_k)/\beta_1} < 1 \,,$$

then (8) is totally oscillatory. ∎

3 – Oscillations and the delays

For a given $\nu \in NBV$, looking to the real function g, introduced before, as a function of the pair (λ, r) in $\mathbb{R} \times \mathbb{C}^+$,

$$g(\lambda; r) = \int_{-1}^{0} e^{-\lambda r(\theta)}\, d\nu(\theta) \,,$$

let

$$K = \Big\{ r \in C^+ \colon\ g(\lambda; r) < 1, \text{ for every } \lambda \in \mathbb{R} \Big\} \,,$$

be the set of all delays for which the corresponding equation (1) is totally oscillatory.

First notice, that if K is not empty then it has the following cone property:

$$r \in K \ \Rightarrow\ \gamma\, r \in K, \quad \text{for each } \ \gamma > 0 \,,$$

as one easily sees through the relation

$$g(\lambda; \gamma\, r) = g(\gamma\lambda; r) \,,$$

valid for every real λ and γ.

Two extreme cases of K are referred to in the next two examples.

Example 2. If ν is such that

$$\nu(0) = \int_{-1}^{0} d\nu(\theta) = 1 ,$$

we have $K = \emptyset$, since then $g(0;r) = 1$, for every $r \in C^+$. □

Example 3. For the equation

$$x(t) = c\int_{-1}^{0} x(t - r(\theta))\, d\theta ,$$

with $c \neq 0$, it holds that,

$$g(\lambda;r) = c\int_{-1}^{0} e^{-\lambda r(\theta)}\, d\theta .$$

Then either $K = \emptyset$ or $K = C^+$, according to whether $c > 0$ or $c < 0$, respectively. In fact, if $c > 0$ then $K = \emptyset$, since for each $r \in C^+$, $g(\lambda;r) \to +\infty$, as $\lambda \to -\infty$; if $c < 0$, the range of g is contained in $]-\infty, 0[$ and so $K = C^+$. □

When $K = C^+$, equation (1) will be said *totally oscillatory globally in the delays*. The opposite case, $K = \emptyset$, means that for every delay function $r \in C^+$ the corresponding equation (1) has at least one nonoscillatory solution. Another example of these two extreme situations is given in the following theorem.

Theorem 4.

(i) *If ν is decreasing, then (1) is totally oscillatory globally in the delays.*

(ii) *If ν is increasing and $\nu(0) > 0$, then for each $r \in C^+$ the corresponding equation (1) has at least one nonoscillatory solution.*

Proof: **(i)** If ν is decreasing, then for each $r \in C^+$ and every λ, real, we have $g(\lambda;r) \leq 0$ and consequently $K = C^+$.

(ii) Notice that if ν has a point of increase at some $\theta_0 \in [-1,0]$, then taking $r \in C^+$ such that $r(\theta_0) = \|r\|$ and $r(\theta) \neq \|r\|$ for every $\theta \neq \theta_0$, by the Corollary 1, the corresponding equation has at least one nonoscillatory solution and (1) is not totally oscillatory globally in the delays. Furthermore, if ν is increasing, then for each $r \in C^+$, we have

$$g(\lambda;r) \geq e^{-\lambda m(r)}\, \nu(0) ,$$

for every real $\lambda \leq 0$, and $\nu(0) > 0$ implies that $g(\lambda;r) \to +\infty$, as $\lambda \to -\infty$, and so $K = \emptyset$. ■

In the following theorem the case $K = C^+$ is characterized for an important class of functions ν.

Theorem 5. *Let ν be a right or left hand continuous step function. Then (1) is totally oscillatory globally in the delays if and only if ν is decreasing.*

Proof: Let $\{\theta_0, ..., \theta_q\}$ be a partition of $[-1, 0]$ ($\theta_0 = -1$, $\theta_q = 0$) such that, $\nu(\theta) = \nu_j$ for $\theta \in]\theta_{j-1}, \theta_j[$, $j = 1, ..., q$. Suppose that (1) is totally oscillatory globally in the delays. Since $\nu(-1) = 0$, if ν is right hand continuous then $\nu(\theta) = 0$ on $[-1, \theta_1[$ and so $\nu_1 = 0$. Otherwise, if $\nu_1 > 0$, then ν is increasing on $[-1, \theta_1[$ and -1 is a point of increase of ν which cannot occur. So, necessarily $\nu_1 \leq 0$, and ν is decreasing on $[-1, \theta_1[$.

If ν is not decreasing on $[-1, 0[$, then for some $j \in \{1, ..., q-1\}$, we have $\nu_j < \nu_{j+1}$. Therefore, with ν either left or right continuous at θ_j, ν is increasing on $]\theta_{j-1}, \theta_{j+1}[$ and θ_j is a point of increase of ν, which is contradictory. Hence ν is necessarily decreasing on $[-1, 0[$.

Finally, if ν is left hand continuous then $\nu(0) = \nu_q$, and ν becomes decreasing on $[-1, 0]$. Otherwise, if $\nu(0) > \nu_q$, ν is increasing on $]\theta_{q-1}, 0]$ and zero is again a point of increase of ν, which contradicts the hypothesis of (1) be totally oscillatory globally in the delays. ∎

As a consequence, the Remark 2 of [1] is obtained.

Corollary 4. *Equation (4) is totally oscillatory globally in the delays if and only if $a_j < 0$ for $j = 1, ..., q$.*

Proof: In this case ν can be taken as a step function of the form

$$\nu(\theta) = \sum_{j=1}^{q} H(\theta - \theta_j)\, a_j \ ,$$

where H denotes the Heaviside function, and $-1 < \theta_1 < ... < \theta_q < 0$; ν is right hand continuous at each point θ_j, $j = 1, ..., q$. Since any of the a_j cannot be zero, ν is decreasing if and only if $a_j < 0$ for every $j = 1, ..., q$. ∎

Remark 2. This situation can be extended to equation (7). Considering a strictly increasing sequence, θ_k, in $]-1, 0]$, converging to zero, and

$$\nu(\theta) = \sum_{k=1}^{\infty} H(\theta - \theta_k)\, a_k \ ,$$

the arguments used in Theorem 5 can be repeated to show that (7) is totally oscillatory globally in the delays if and only if ν is decreasing, that is, if and only if $a_k < 0$ for $k = 1, 2, \ldots$. □

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