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ON SOME GENERAL NOTIONS OF SUPERIOR LIMIT, INFERIOR LIMIT AND VALUE OF A DISTRIBUTION AT A POINT

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Abstract: For distributions defined on open sets of $\mathbb{R}^n$, we define and study notions of superior limit, inferior limit, and a consequent concept of value at a point, that is more general than that introduced by S. Lojasiewicz and considered in various works of Mikusinski, Sebastião e Silva and the present author.

1 – Introduction

In this work we give definitions of the notions of superior limit and inferior limit of a real distribution of n variables at a point of its domain and study some properties of these notions, showing that they are well connected with the fundamental algebraic operations of distribution theory. For distributions of one variable these questions were studied in [4], the aim of the present work being essentially to extend to the case of several variables some results of that paper. The concepts of superior and inferior limit generate naturally a notion of value of a (real or complex) distribution at a point. This notion, keeping all the essential properties of the homonymous concept considered in [3], [11], [12] and [14], is much more general than this one (as can be seen in [4] and [8]). The extension to the case of distributions defined on open sets of $\mathbb{R}^n$ of other subjects treated in [4] — namely applications to the integration of distributions — and also of some questions studied in other papers ([3], [5] to [9] and [13]) will be the object of future works.

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2 – Superior and inferior limits and value for strictly bounded distributions

Let n be a positive integer, $I_1, I_2, ..., I_n$ open (non empty) intervals of $\mathbb{R}$, $I = I_1 \times \cdots \times I_n$, $a = (a_1, ..., a_n)$ a point in I and $I_a = (I_1 \backslash \{a_1\}) \times \cdots \times (I_n \backslash \{a_n\})$. We shall denote by $BC(I_a)$ the (real) vector space of all bounded continuous real functions defined on I_a. For $F \in BC(I_a)$, $\overline{F(a)}$ and $\underline{F(a)}$ will be, respectively, the upper and the lower limit of the function F at the point a; so, if $\|\cdot\|$ is (for instance) the euclidean norm on $\mathbb{R}^n$, we shall have:

$$\overline{F(a)} = \lim_{\epsilon \to 0^+} \sup\Big\{F(x) \colon \|x-a\| < \epsilon,\ x \in I_a\Big\} ,$$

$$\underline{F(a)} = \lim_{\epsilon \to 0^+} \inf\Big\{F(x) \colon \|x-a\| < \epsilon,\ x \in I_a\Big\} .$$

For each $j \in \{1, ..., n\}$ let ρ_j be the operator such that, for $F \in BC(I_a)$ and $x \in I_a$,

$$(\rho_j\, F)(x) = \frac{1}{x_j - a_j} \int_{a_j}^{x_j} F(x_1, ..., x_{j-1}, \xi, x_{j+1}, ..., x_n)\, d\xi .$$

Then we have:

Proposition 2.1. *ρ_j is a linear injection of $BC(I_a)$ into itself; for each $F \in BC(I_a)$, putting $G = \rho_j\, F$, we have: $\underline{F(a)} \le \underline{G(a)} \le \overline{G(a)} \le \overline{F(a)}$.*

Proof: It is clear that $G \colon I_a \to \mathbb{R}$ is a continuous function and that (since each value $G(x)$ is a certain kind of mean value of the function F) it is also bounded on I_a. To see that $\overline{G(a)} \le \overline{F(a)}$ notice that, if λ is a real number greater than $\overline{F(a)}$, there exists $\epsilon > 0$ such that, for $x \in I_a$ and $\|x-a\| < \epsilon$, we have $F(x) < \lambda$; then, for the same values of x,

$$\frac{1}{x_j - a_j} \int_{a_j}^{x_j} F(x_1, ..., x_{j-1}, \xi, x_{j+1}, ..., x_n)\, d\xi < \lambda$$

and so $\overline{G(a)} < \lambda$; so we have $\overline{G(a)} \le \overline{F(a)}$. The relation $\underline{F(a)} \le \underline{G(a)}$ is obtained in a similar way. Finally, since we have, for each $x \in I_a$

$$F(x) = \frac{\partial}{\partial x_j}\Big[(x_j - a_j)\, G(x)\Big] ,$$

we see that ρ_j is injective. ∎

It is clear that each function $F \in BC(I_a)$ (being defined almost everywhere on I and locally integrable on this interval) can be identified with a (real) distribution defined on I; it is also clear that the distributions that correspond to two distinct functions of the space $BC(I_a)$ are always distinct distributions. So, denoting by $\mathcal{D}'_{\mathbb{R}}(I)$ the space of the real distributions defined on I, we can write $BC(I_a) \subset \mathcal{D}'_{\mathbb{R}}(I)$. Now, let us denote by ∂_j (for $j \in \{1, ..., n\}$) the operator defined on $\mathcal{D}'_{\mathbb{R}}(I)$ and such that

$$\partial_j(g) = f \quad \text{iff} \quad f = D_j\Big[(\widehat{x}_j - a_j)\, g\Big] \ ,$$

where the symbol D_j denotes derivation in order to x_j in the sense of distributions and the accent over a variable means that it is a dummy variable.

It is obvious that, for each $F \in BC(I_a)$ and each j, $\partial_j(\rho_j\, F) = F$ and also that two operators, ∂_i and ∂_j, are always interchangeable. We shall put $\partial = \partial_1 \partial_2 \cdots \partial_n$ and, for each $p \in \mathbb{N}$, $\partial^p = \partial_1^p \partial_2^p \cdots \partial_n^p$. Now we can introduce the following definition: The distribution f is said to be *strictly bounded at the point* a — and we can write $f \in \mathcal{B}^*_a(I)$ — if there exists $F \in BC(I_a)$ and $p \in \mathbb{N}$ such that $f = \partial^p F$. It is clear that, if $p < q$ $(p, q \in \mathbb{N})$ and $f = \partial^p F$ with $F \in BC(I_a)$, there exists $F_1 \in BC(I_a)$ such that $f = \partial^q F_1$: it is sufficient to take $F_1 = \rho^{q-p} F$, where $\rho^{q-p} = \rho_1^{q-p} \cdots \rho_n^{q-p}$. From this it follows easily that $\mathcal{B}^*_a(I)$ is a subspace of the real vector space $\mathcal{D}'_{\mathbb{R}}(I)$.

Obviously $BC(I_a) \subset \mathcal{B}^*_a(I)$. But a distribution that, on the set I_a, is identical to a function that does not belong to $BC(I_a)$ may well be an element of the space $\mathcal{B}^*_a(I)$. For instance — with $n = 1$, $I = \mathbb{R}$ and $a = 0$ — the distribution $\sin \frac{1}{x} - D(x \sin \frac{1}{x})$, that coincides with the function $\frac{1}{x} \cos \frac{1}{x}$ in the set $\mathbb{R}\backslash\{0\}$, is strictly bounded at the origin.

Now, let us introduce a result that is essential in the sequel:

Theorem 2.2. *For each $j \in \{1, ..., n\}$, the operator ∂_j, restricted to the space $\mathcal{B}^*_a(I)$, is an automorphism of this vector space.*

Before the proof of this theorem (and even before introducing the lemma that will precede that proof) it is convenient to remember the definition of the notion of pseudo-polynomial.

Let $J = J_1 \times J_2 \times \cdots \times J_n$ be a (non degenerate) interval of $\mathbb{R}^n$ and $\overline{r} = (r_1, r_2, ..., r_n) \in \mathbb{N}^n$. A pseudo-polynomial defined on J and of degree less than $\overline{r}$ is any function P that can be put in the form:

$$P \;=\; P_1 + P_2 + \cdots + P_n \ ,$$

where, for each $j \in \{1, ..., n\}$, P_j is a "polynomial" in x_j of degree less than r_j and with "coefficients" that are (real or complex) continuous functions defined on J and independent of x_j, that is

$$P_j(x) = P_j(x_1, ..., x_n) = \sum_{k=0}^{r_j-1} a_{jk}(x_1, ..., x_{j-1}, x_{j+1}, ..., x_n)\, x_j^k ,$$

where the functions a_{jk} are continuous on J.

It is well known that, if F is a continuous function defined on J, the equality

$$D^{\overline{r}} F = D_1^{r_1} D_2^{r_2} \cdots D_n^{r_n} F = 0$$

(where D_j denotes, as usual, the operator of derivation in order to x_j in the sense of distributions) is verified iff the function F is a pseudopolynomial of degree less than $\overline{r}$ defined on J.

Now we can prove the following lemma:

Lemma 2.3. *Let $J_1, J_2, ..., J_n$ be n intervals of $\mathbb{R}$ unbounded on the left and $J = J_1 \times \cdots \times J_n$. If the pseudo-polynomial $P(x_1, ..., x_n)$ defined on J, has the limit zero when each one of the variables x_j tends to $-\infty$ (the other $n-1$ variables being fixed in arbitrary points of their domains) then $P(x_1, ..., x_n)$ is identically zero.*

Proof (of the lemma)**:** We shall use induction on the number of variables, n; since the result is obvious for $n = 1$, we shall accept its truth in the case where the number of variables is $n-1$ and we shall prove it for a pseudopolynomial $P(x_1, ..., x_n)$ in the conditions of the hypothesis of the lemma. We can suppose that the "polynomials" in $x_1, x_2, ..., x_n$, of which $P(x_1, ..., x_n)$ is the sum, are all of the same "degree" $p-1$, since it would always be possible to reduce ourselves to that case adding, if necessary, some terms with a null coefficient.

So, let us suppose that we have

$$P(x_1, ..., x_n) = \sum_{j=1}^{n} \sum_{k=0}^{p-1} a_{jk}(x_1, ..., x_{j-1}, x_{j+1}, ..., x_n)\, x_j^k \tag{1}$$

and, chosen p distinct points $\xi_0, \xi_1, ..., \xi_{p-1}$ in the interval J_n, let us consider the system (with p equations in the unknowns $a_{n,0}(x_1, ..., x_{n-1})$, ..., $a_{n,p-1}(x_1, ..., x_{n-1})$):

$$\sum_{j=1}^{n-1} \sum_{k=0}^{p-1} a_{jk}(x_1, ..., x_{j-1}, x_{j+1}, ..., x_{n-1}, \xi_l)\, x_j^k + \sum_{k=0}^{p-1} a_{nk}(x_1, ..., x_{n-1})\, \xi_l^k =$$
$$= P(x_1, ..., x_{n-1}, \xi_l) \qquad (l = 0, ..., p-1) ,$$

whose determinant is the Vandermonde determinant of $(\xi_0, ..., \xi_{p-1})$. We can easily see that the solution of that system can be put in the form

$$a_{nk}(x_1, ..., x_{n-1}) = \\ = \sum_{l=0}^{p-1} \beta_{kl} \, P(x_1, ..., x_{n-1}, \xi_l) + \sum_{j=1}^{n-1} \sum_{l=0}^{p-1} b_{jkl}(x_1, ..., x_{j-1}, x_{j+1}, ..., x_{n-1}) \, x_j^l \\ (k = 0, ..., p-1) \; ,$$

where the β_{kl} are constants and the functions b_{jkl} are independent of x_j and of x_n.

If we substitute these values in the equality (1) we obtain a new equality of the form:

$$\sum_{j=1}^{n-1} \sum_{l=0}^{p-1} c_{jl}(x_1, ..., x_{j-1}, x_{j+1}, ..., x_n) \, x_j^l = \\ = P(x_1, ..., x_n) - \sum_{k=0}^{p-1} \sum_{l=0}^{p-1} \beta_{kl} \, P(x_1, ..., x_{n-1}, \xi_l) \, x_n^k \; .$$

According to the hypothesis of the lemma, if the variable x_j tends to $-\infty$, all the other variables being fixed, the second member will tend to zero (for every value of j). So, for any value of x_n in the interval J_n, the first member (that is a pseudo-polynomial in $x_1, ..., x_{n-1}$ if x_n is fixed) tends to zero whenever any of its variables tends to $-\infty$. Then it follows from the induction hypothesis that we must have

$$P(x_1, ..., x_n) = \sum_{k=0}^{p-1} \sum_{l=0}^{p-1} \beta_{kl} \, P(x_1, ..., x_{n-1}, \xi_l) \, x_n^k \; .$$

Now, taking into account that the first member — and then also the second, that is a "polynomial" in x_n — has the limit zero when $x_n \to -\infty$, we can conclude that $P(x_1, ..., x_n)$ is identically zero. ∎

Proof (of Theorem 2.2)**:** It is easily seen that the restriction of ∂_j to the space $\mathcal{B}_a^*(I)$ (restriction that will be denoted by the same symbol, ∂_j) is a linear operator from this space onto itself; it is also clear that, to prove that ∂_j is one-to-one, it will be enough to show that, from one equality of the form $\partial^p F = 0$ (with $F \in BC(I_a)$ and $p \in \mathbb{N}$), it follows necessarily $F = 0$.

Suppose then that we have $\partial^p F = 0$ and denote by K_1 the set of all points $x = (x_1, ..., x_n)$ of the interval I such that $x_j > a_j$ for every value of j; it is clear that K_1 is one of the connected components of the set I_a. Denote also by F_1 the restriction of the function F to the set K_1.

Now, starting from the equality $\partial^p F_1 = 0$ (that follows immediately from the hypothesis $\partial^p F = 0$), let us change the variables $x_1, ..., x_n$ to new variables $u_1, ..., u_n$, by means of the formulas:

$$u_j = \log(x_j - a_j) \quad (j \in \{1, ..., n\}) \ .$$

We shall obtain (on one interval J_1, that is the cartesian product of n intervals unbounded on the left) an equality of the form

$$D^p G_1(u_1, ..., u_n) = 0 \tag{2}$$

where

$$G_1(u_1, ..., u_n) \ = \ e^{u_1+\cdots+u_n} \, F_1(a_1 + e^{u_1}, ..., a_n + e^{u_n}) \ .$$

Taking into account the equality (2) and the fact that G_1 is a continuous function on J_1, we see that G_1 is a pseudo-polynomial defined on this interval; since the function F is bounded we see also that G_1 tends to zero when each one of the variables u_j tends to $-\infty$, the other variables staying fixed. Now the lemma allows us to conclude that $G_1 = 0$ and so that $F = 0$ on K_1.

We should prove analogously that F is equal to zero in each one of the other connected components of I_a (the change of variables to consider would be defined in each case by the system $u_j = \log|x_j - a_j|$, $j \in \{1, ..., n\}$). So we can conclude that F is the null function on I_a. ∎

For each j, let ∂_j^{-1} be the inverse (of the restriction to $\mathcal{B}_a^*(I)$) of the operator ∂_j; for each $p \in \mathbb{N}$, put $\partial^{-p} = \partial_1^{-p} \cdots \partial_n^{-p}$. Moreover, if f is a distribution of the space $\mathcal{B}_a^*(I)$, put $f_p = \partial^{-p} f$. Then it is clear that we have $f_p \in \mathcal{B}_a^*(I)$ for every value of p and also $f_p \in BC(I_a)$ if the value of p is sufficiently large.

For $f \in \mathcal{B}_a^*(I)$ we shall call *degree of f at the point a* — $\deg_a f$ — the least value of $p \in \mathbb{N}$ such that $f_p \in BC(I_a)$. Then , for each $f \in \mathcal{B}_a^*(I)$, we can consider two sequences: $\{\overline{f_p(a)}\}$ and $\{\underline{f_p(a)}\}$ (with $p \in \mathbb{N}$, $p \geq \deg_a f$), where $\overline{f_p(a)}$ [resp. $\underline{f_p(a)}$] denotes as before the upper limit [resp. lower limit] of the function f_p at the point a.

From Proposition 2.1 it follows that, for $p \geq \deg_a f$, we have:

$$\underline{f_p(a)} \ \leq \ \underline{f_{p+1}(a)} \ \leq \ \overline{f_{p+1}(a)} \ \leq \ \overline{f_p(a)} \ .$$

So we can introduce the following definitions:

Let $f \in \mathcal{B}_a^*(I)$ and, for each $p \geq \deg_a f$, let $f_p = \partial^{-p} f$; then, the limit of the sequence $\overline{f_p(a)}$ [resp. $\underline{f_p(a)}$] will be called *superior limit* [resp. *inferior limit*] of the distribution f at the point a and will be denoted by $\limsup_a f$ [resp. $\liminf_a f$].

For any distribution $f \in \mathcal{B}_a^*(I)$ we have clearly: $\liminf_a f \leq \limsup_a f$. If the equality is verified, we say that the distribution f is *strictly continuous at the point* a and the common value of the superior and inferior limits, denoted by $f(a)$, is called the *value of* f *at the point* a.

We shall use the symbol $\mathcal{V}_a^*(I)$ to denote the set of all distributions that are strictly continuous at a.

It is convenient to observe that, when f is a function in the space $BC(I_a)$, the equalities

$$\overline{f(a)} = \limsup_a f\,, \qquad \underline{f(a)} = \liminf_a f$$

are not generally satisfied. For instance, with $n = 1$, $I = \mathbb{R}$, $a = 0$ and $h(x) = \sin\frac{1}{x}$, we have, as it is easily verified,

$$\limsup_0 h(x) = \liminf_0 h(x) = 0$$

and $\overline{h(0)} = 1$, $\underline{h(0)} = -1$.

In any case it is clear that, for $f \in BC(I_a)$, we always have:

$$\underline{f(a)} \leq \liminf_a f \leq \limsup_a f \leq \overline{f(a)}\,. \tag{3}$$

So, in a sense, the notions of superior and inferior limit of a distribution at a point do not precisely generalize the usual homonymous notions for functions, although, as we shall see in the next chapter, they share with them a lot of significant properties.

On the other hand, it follows immediately from the inequalities (3) that, if the function f has a limit at the point a in the usual sense, then it belongs to the space $\mathcal{V}_a^*(I)$ and the value $f(a)$ coincides with that limit.

3 – Superior and inferior limits and value: the general case

Let A be an open set of $\mathbb{R}^n$, a a point in A and f a real distribution defined in A.

We shall say that f is *bounded at the point* a, and we shall write $f \in \mathcal{B}_a(A)$, iff there exists an open interval I of $\mathbb{R}^n$ such that

i) $a \in I \subset A$, and

ii) $f_{|I} \in \mathcal{B}_a^*(I)$,

(where $f_{|I}$ denotes the restriction of f to the interval I).

Now, we easily recognize the coherence of the following definitions: If $f \in \mathcal{B}_a(A)$ and if I is an interval such that conditions i) and ii) are satisfied, the real number $\limsup_a f_{|I}$ [resp. $\liminf_a f_{|I}$] will be called *superior limit* [resp. *inferior limit*] of f at the point a, and will be denoted by $\limsup_a f$ [resp. $\liminf_a f$]. If $\limsup_a f = \liminf_a f$, we shall say that f is *continuous at the point* a, with the value $f(a) = \limsup_a f = \liminf_a f$. The set of distributions defined in A and continuous at the point a will be denoted by $\mathcal{V}_a(A)$.

Suppose now that the open set A coincides with an interval I containing the point a: then we have obviously $\mathcal{B}^*_a(I) \subset \mathcal{B}_a(I)$ and $\mathcal{V}^*_a(I) \subset \mathcal{V}_a(I)$, and these inclusions are strict. For instance, if I is an unbounded open interval of $\mathbb{R}^n$, it is easy to see that each one of the coordinate functions $f_j(x) = x_j$ defined on I, although obviously continuous at each point of this interval, is not strictly bounded at any one of these points. It is also clear that the superior and inferior limits of a distribution that is strictly bounded at the point a are the same if the distribution is considered as an element of $\mathcal{B}^*_a(I)$ or as an element of $\mathcal{B}_a(I)$ (and analogously for the value at the point a of an element of the space $\mathcal{V}^*_a(I)$).

We shall state now some general properties of the concepts just defined. In many cases their proofs are so easy that we have decided to omit them.

Proposition 3.1. *Let $a \in \mathbb{R}^n$ and A, A' be two open sets of $\mathbb{R}^n$ such that $a \in A' \subset A$; let also $f \in \mathcal{D}'_{\mathbb{R}}(A)$. Then $f \in \mathcal{B}_a(A)$ [resp. $f \in \mathcal{V}_a(A)$] iff $f_{|A'} \in \mathcal{B}_a(A')$ [resp. $f_{|A'} \in \mathcal{V}_a(A')$] and in that case*

$$\limsup_a f = \limsup_a f_{|A'}, \quad \liminf_a f = \liminf_a f_{|A'} \qquad [\textit{resp. } f(a) = f_{|A'}(a)] \ .$$

In all the following propositions up to Corollary 3.11 A will continue to be an open set of $\mathbb{R}^n$, and a a point in A.

Proposition 3.2. *Let $f \in \mathcal{B}_a(A)$ [resp. $f \in \mathcal{V}_a(A)$], $g \in \mathcal{D}'_{\mathbb{R}}(A)$ and $j \in \{1, 2, ..., n\}$; if there is an interval I such that $a \in I \subset A$ and $g_{|I} = \partial_j f_{|I}$, then $g \in \mathcal{B}_a(A)$ [resp. $g \in \mathcal{V}_a(A)$] and*

$$\limsup_a g = \limsup_a f, \quad \liminf_a g = \liminf_a f \qquad [\textit{resp. } g(a) = f(a)] \ .$$

Proposition 3.3. *Let $f, g \in \mathcal{B}_a(A)$, $h = f + g$ and denote by α and β, respectively, the smallest and the largest of the two numbers:*

$$\liminf_a f + \limsup_a g \quad \textit{and} \quad \liminf_a g + \limsup_a f \ .$$

Then $h \in \mathcal{B}_a(A)$ *and*

$$\liminf_a f + \liminf_a g \le \liminf_a h \le \alpha \ ,$$
$$\beta \le \limsup_a h \le \limsup_a f + \limsup_a g \ .$$

Corollary 3.4. *If* $f \in \mathcal{V}_a(A)$, $g \in \mathcal{B}_a(A)$ *and* $h = f + g$, *then:*

$$\liminf_a h = f(a) + \liminf_a g \,, \qquad \limsup_a h = f(a) + \limsup_a g \ .$$

Corollary 3.5. *If* $f, g \in \mathcal{V}_a(A)$ *and* $h = f + g$, *then* $h \in \mathcal{V}_a(A)$ *and* $h(a) = f(a) + g(a)$.

Proposition 3.6. *Let* $f \in \mathcal{B}_a(A)$, $\lambda \in \mathbb{R}$ *and* $g = \lambda f$; *then* $g \in \mathcal{B}_a(A)$ *and*

$$\limsup_a g = \lambda \limsup_a f, \quad \liminf_a g = \lambda \liminf_a f, \qquad \text{if } \lambda \ge 0 \ ,$$
$$\limsup_a g = \lambda \liminf_a f, \quad \liminf_a g = \lambda \limsup_a f, \qquad \text{if } \lambda < 0 \ .$$

Corollary 3.7. *If* $f \in \mathcal{V}_a(A)$, $\lambda \in \mathbb{R}$ *and* $g = \lambda f$, *then* $g \in \mathcal{V}_a(A)$ *and* $g(a) = \lambda f(a)$.

Corollary 3.8. $\mathcal{B}_a(A)$ *is a vector subspace of the space* $\mathcal{D}'_{\mathbb{R}}(A)$ *of all real distributions defined on* A *and* $\mathcal{V}_a(A)$ *is a vector subspace of* $\mathcal{B}_a(A)$. *The map* $f \mapsto f(a)$, *of the space* $\mathcal{V}_a(A)$ *onto* $\mathbb{R}$, *is linear.*

Now we shall prove the following result:

Theorem 3.9. *Let*[1] $f \in \mathcal{B}_a(A)$, $\varphi \in C^\infty_{\mathbb{R}}(A)$ *and suppose that* $\varphi(a) = 0$; *then* $\varphi f \in \mathcal{V}_a(A)$ *and* $(\varphi f)(a) = 0$.

Proof: Denote by K the set of all distributions $h \in \mathcal{B}_a(A)$ such that, for every $\varphi \in C^\infty_{\mathbb{R}}(A)$ with $\varphi(a) = 0$, we have $\varphi h \in \mathcal{V}_a(A)$ and $(\varphi h)(a) = 0$. We must prove that $K = \mathcal{B}_a(A)$.

First note that, if $g \in \mathcal{D}'_{\mathbb{R}}(A)$ and if there exists an interval I such that $a \in I \subset A$ and $g_{|I} \in BC(I_a)$, then clearly $g \in K$. So, to conclude the proof, it will be sufficient to show that, for any $f, g \in \mathcal{D}'_{\mathbb{R}}(A)$ and every $j \in \{1, ..., n\}$, if we

[1] We denote by $C^\infty_{\mathbb{R}}(A)$ [resp. $C^\infty(A)$] the space of all real [resp. complex] infinitely differentiable functions defined on A.

have $g \in K$ and if there is an interval I (with $a \in I \subset A$) such that $f_{|I} = \partial_j(g_{|I})$, then $f \in K$. Suppose then that g, I and j satisfy the conditions just stated and that we have $f_{|I} = \partial_j(g_{|I})$; if $\varphi \in C^\infty_{\mathbb{R}}(A)$ and $\varphi(a) = 0$ we shall have:

$$\varphi\, f_{|I} \;=\; \varphi\, D_j\Big[(\widehat{x}_j - a_j)\, g_{|I}\Big] \;=\; \partial_j(\varphi\, g_{|I}) - (\widehat{x}_j - a_j)\, \frac{\partial\varphi}{\partial x_j}\, g_{|I}\;. \tag{4}$$

As $g \in K$, it follows from the definition of this set that we have $\varphi\, g \in \mathcal{V}_a(A)$ and $(\varphi\, g)(a) = 0$. So, Proposition 3.1 shows the distribution $\varphi\, g_{|I}$ is an element of the space $\mathcal{V}_a(I)$ with value zero at the point a; then by Proposition 3.2 we have also $\partial_j(\varphi\, g_{|I}) \in \mathcal{V}_a(I)$, and $\partial_j(\varphi\, g_{|I})(a) = 0$. On the other hand, as $(\widehat{x}_j - a_j)\, \frac{\partial\varphi}{\partial x_j}$ belongs to the space $C^\infty_{\mathbb{R}}(A)$ with value 0 at the point a, we shall have in a similar way $(\widehat{x}_j - a_j)\, \frac{\partial\varphi}{\partial x_j}\, g_{|I} \in \mathcal{V}_a(I)$, with value zero at the same point. Then, by means of the equality (4), Corollaries 3.5 and 3.7 and Proposition 3.1, we can conclude that $\varphi\, f \in \mathcal{V}_a(A)$ and $(\varphi\, f)(a) = 0$, which means that $f \in K$, finishing the proof. ∎

Proposition 3.10. *Let $f \in \mathcal{B}_a(A)$, $\varphi \in C^\infty_{\mathbb{R}}(A)$. Then $\varphi\, f \in \mathcal{B}_a(A)$ and*

$$\limsup_a(\varphi\, f) = \varphi(a)\, \limsup_a f, \quad \liminf_a(\varphi\, f) = \varphi(a)\, \liminf_a f \qquad \textit{if}\;\; \varphi(a) \geq 0\;,$$

$$\limsup_a(\varphi\, f) = \varphi(a)\, \liminf_a f, \quad \liminf_a(\varphi\, f) = \varphi(a)\, \limsup_a f \qquad \textit{if}\;\; \varphi(a) < 0\;.$$

To verify this result it is sufficient to consider the equality

$$\varphi\, f \;=\; \varphi(a)\, f + (\varphi - \varphi(a))\, f\;,$$

and take into account Proposition 3.6, Theorem 3.9 and Corollary 3.4. From Proposition 3.10 it follows immediately:

Corollary 3.11. *If $f \in \mathcal{V}_a(A)$ and $\varphi \in C^\infty_{\mathbb{R}}(A)$, then $\varphi\, f \in \mathcal{V}_a(A)$ and $(\varphi\, f)(a) = \varphi(a)\, f(a)$.*

The following properties concerning the tensor product, are also very natural:

Theorem 3.12. *Let m and n be two positive integers, A [resp. B] be an open set in $\mathbb{R}^m$ [resp. $\mathbb{R}^n$], $a \in A$, $b \in B$, $f \in \mathcal{B}_a(A)$, $g \in \mathcal{B}_b(B)$ and $h = f \otimes g$. Then $h \in \mathcal{B}_{(a,b)}(A{\times}B)$ and, putting*

$$\alpha_* = \liminf_a f\,, \quad \alpha^* = \limsup_a f\,, \quad \beta_* = \liminf_b g\,, \quad \beta^* = \limsup_b g\;,$$

we have:

$$\liminf_{(a,b)} h \;=\; \min\Big\{\alpha_*\,\beta_*,\;\alpha_*\,\beta^*,\;\alpha^*\beta_*,\;\alpha^*\beta^*\Big\}\;,$$

$$\limsup_{(a,b)} h \;=\; \max\Big\{\alpha_*\,\beta_*,\;\alpha_*\beta^*,\;\alpha^*\beta_*,\;\alpha^*\beta^*\Big\}\;.$$

Proof: From $f \in \mathcal{B}_a(A)$ it follows the existence of an open interval I such that $a \in I \subset A$ and $f_{|I} \in \mathcal{B}^*_a(I)$; then, for $p \geq \deg_a f_{|I}$, we shall have $f_p = \partial_x^{-p} f_{|I} \in BC(I_a)$ (where, for every $h \in \mathcal{B}^*_a(I)$, we put

$$\partial_x h \;=\; D_{x_1}\cdots D_{x_n}\Big[(\widehat{x}_1 - a_1)\cdots(\widehat{x}_n - a_n)\,h\Big]\;,$$

∂_x being an automorphism of $\mathcal{B}^*_a(I)$ according to Theorem 2.2). Analogously, as $g \in \mathcal{B}_b(B)$, there exists one open interval J such that $b \in J \subset B$ and, supposing $p > \deg_b g_{|J}$, $g_p = \partial_y^{-p} g_{|J} \in BC(J_b)$ (where ∂_y has the obvious meaning).

So, for $p \geq \max\{\deg_a f_{|I},\, \deg_b g_{|J}\}$ we shall have also, putting $K = I\times J$ (and then $K_{(a,b)} = I_a\times J_b$), $h_{|K} = f_{|I} \otimes g_{|J} \in \mathcal{B}^*_{(a,b)}(K)$ since $f_p \otimes g_p \in BC(K_{(a,b)})$ and, with an obvious notation,

$$\partial^p_{(x,y)}(f_p \otimes g_p) \;=\; \partial^p_x(f_p) \otimes \partial^p_y(g_p) \;=\; h_{|K}\;.$$

So we see that $h \in B_{(a,b)}(A\times B)$ (and also that $\deg_{(a,b)} h_{|K} \leq p$).

Putting $h_p = f_p \otimes g_p$, we deduce easily that:

$$\underline{h_p(a,b)} \;=\; \min\Big\{\underline{f_p(a)}\,\underline{g_p(b)},\;\underline{f_p(a)}\,\overline{g_p(b)},\;\overline{f_p(a)}\,\underline{g_p(b)},\;\overline{f_p(a)}\,\overline{g_p(b)}\Big\}\;,$$

$$\overline{h_p(a,b)} \;=\; \max\Big\{\underline{f_p(a)}\,\underline{g_p(b)},\;\underline{f_p(a)}\,\overline{g_p(b)},\;\overline{f_p(a)}\,\underline{g_p(b)},\;\overline{f_p(a)}\,\overline{g_p(b)}\Big\}\;.$$

Now, to complete the proof it is sufficient to let $p \to +\infty$. ∎

As immediate consequences we have the following two corollaries:

Corollary 3.13. *With the same notation of Theorem 3.12, if $f \in \mathcal{V}_a(A)$ then:*

$$\liminf_{(a,b)} h = f(a)\,\liminf_{b} g,\quad \limsup_{(a,b)} h = f(a)\,\limsup_{b} g,\qquad \textit{if}\;\; f(a) \geq 0\;,$$

$$\liminf_{(a,b)} h = f(a)\,\limsup_{b} g,\quad \limsup_{(a,b)} h = f(a)\,\liminf_{b} g,\qquad \textit{if}\;\; f(a) < 0\;.$$

Corollary 3.14. *With the same notation of Theorem 3.12, if $f \in \mathcal{V}_a(A)$ and $g \in \mathcal{V}_b(B)$, $h \in \mathcal{V}_{(a,b)}(A\times B)$ and $h(a,b) = f(a)\,g(b)$.*

Now, we are going to analyse some relations between the chief concepts that we are studying and the operation of composition. As we shall see, the changes of variables that are "well related" to those concepts possess some particular properties, which are convenient to consider immediately.

So, let A and B be two open sets in $\mathbb{R}^n$ and μ a map from A to B; for each $x = (x_1, ..., x_n) \in A$ let $\mu(x) = y = (y_1, ..., y_n)$ and suppose that the map μ can be expressed by means of the system

$$y_j = p_j(x) = p_j(x_1, ..., x_n) \qquad (j \in \{1, ..., n\}) ,$$

where $p_j \in C^\infty_{\mathbb{R}}(A)$. Let also $a = (a_1, ..., a_n)$ be a fixed point in A, $b = (b_1, ..., b_n) = \mu(a)$ and suppose that the jacobian $J_\mu = \frac{\partial(p_1,...,p_n)}{\partial(x_1,...,x_n)}$ does not vanish at the point a. Finally, suppose that there exists one open interval I (with $a \in I \subset A$) satisfying the conditions:

i) the restriction of μ to I, $\mu_{|I}$, is a diffeomorphism from I to the set $\mu(I)$;

ii) the jacobian J_μ is different from 0 at each point of I;

iii) for each $j \in \{1, ..., n\}$ and each $x = (x_1, ..., x_n) \in I$ the conditions

$$p_j(x_1, ..., x_n) = b_j \quad \text{and} \quad x_j = a_j$$

are equivalent(2).

From this we deduce easily that, for each j, there exists a function $\varphi_j \in C^\infty_{\mathbb{R}}(I)$, taking on I values that are all strictly positive or all strictly negative, and such that, in each point $x \in I$ we have

$$y_j - b_j = p_j(x) - b_j = (x_j - a_j)\, \varphi_j(x) .$$

In order to get this result it is sufficient to observe that

$$\begin{aligned} y_j - b_j &= p_j(x_1, ..., x_{j-1}, x_j, x_{j+1}, ..., x_n) - p_j(x_1, ..., x_{j-1}, a_j, x_{j+1}, ..., x_n) \\ &= \int_{a_j}^{x_j} \frac{\partial p_j}{\partial x_j}(x_1, ..., x_{j-1}, u_j, x_{j+1}, ..., x_n)\, du_j , \end{aligned}$$

or, putting $u_j - a_j = (x_j - a_j)\, u_j^*$,

$$y_j - b_j = (x_j - a_j) \int_0^1 \frac{\partial p_j}{\partial x_j}\Big(x_1, ..., x_{j-1}, a_j + (x_j - a_j)\, u_j^*, x_{j+1}, ..., x_n\Big)\, du_j^* ,$$

(2) It is easy to see that, to assure the existence of an interval I satisfying i), ii) and iii) it is sufficient to suppose that, in some neighbourhood of a (and for each $j \in \{1, ..., n\}$) we have $p_j(x_1, ..., x_n) = b_j$ if $x_j = a_j$.

where the function defined by the integral, that we shall denote by φ_j, is clearly of class C^∞.

Besides, it is obvious that φ_j cannot be zero at any point $x \in I$ where $p_j(x) \neq b_j$; but φ_j cannot also be zero at any point x where we have $p_j(x) = b_j$ — and so, by iii), $x_j = a_j$ — because then the jacobian J_μ would be zero at the same point, in contradiction with ii). Being different from zero at each point of I, φ_j must have a fixed sign on this interval. So we see that each one of the 2^n connected components of the set $I_a = \{x \in I : (x_1 - a_1)(x_2 - a_2) \cdots (x_n - a_n) \neq 0\}$ is mapped by μ into a connected component of $\mu(I_a)$. For commodity, we shall suppose in the sequel that the functions φ_j are all positive; without any loss of generality we shall also suppose that, at every point $x \in I$, the inequalities

$$\frac{1}{2}\,\varphi_j(a) \;<\; \varphi_j(x) \;<\; 2\,\varphi_j(a) \qquad (j \in \{1, 2, ..., n\})$$

are satisfied.

Before obtaining the chief result relating the superior and inferior limits with the change of variables, we shall state and prove four lemmas.

Lemma 3.15. *Let x be a point in the set I_a such that the open interval J'_y, determined*(3) *by the points b and $y = \mu(x)$ is contained in $\mu(I)$; let also J_x be the interval determined by the points a and x, and let $A_x = \mu^{-1}(J'_y)$. Then for every $\lambda \in\,]0, 1[$ there exists $\epsilon > 0$ such that, putting*

$$J_x^{1-\lambda} = \lambda\, a + (1 - \lambda)\, J_x \quad \text{ and } \quad J_x^{1+\lambda} = -\lambda\, a + (1 + \lambda)\, J_x \ ,$$

we have $J_x^{1-\lambda} \subset A_x \subset J_x^{1+\lambda}$ if $\|x - a\| < \epsilon$.

Proof: First observe that, without loss of generality, we can suppose that the point a is the center of the interval I (since this interval could always be substituted by a subinterval centered at that point); observe also that, to prove the lemma, it is sufficient to consider the case where, for every j, we have $x_j > a_j$ (if some of the values $x_j - a_j$ were negative, we could reduce ourselves to the first case by means of the change of variables $(x_1, ..., x_n) \mapsto (x'_1, ..., x'_n)$ with $x'_j - a_j = |x_j - a_j|$, for every j). In these conditions we shall have clearly:

$$J_x \;=\; \Big\{u = (u_1, ..., u_n) \colon \;\; \forall j \;\;\; 0 < u_j - a_j < x_j - a_j\Big\}$$

and

$$A_x \;=\; \Big\{u = (u_1, ..., u_n) \colon \;\; \forall j \;\;\; 0 < (u_j - a_j)\, \varphi_j(u) < (x_j - a_j)\, \varphi_j(x)\Big\} \ .$$

(3) We say that the interval $K \subset \mathbb{R}^n$ is determined by the points $u = (u_1, u_2, ..., u_n)$ and $v = (v_1, v_2, ..., v_n)$ iff $K = \{w = (w_1, ..., w_n) \colon \forall j, \min\{u_j, v_j\} < w_j < \max\{u_j, v_j\}\}$.

Finally observe that if $u \in J_x$ we have $\|u-a\| < \|x-a\|$ (where $\|\cdot\|$ is still the euclidean norm in $\mathbb{R}^n$) and that, if $u \in A_x$ then $\|u-a\| < 4\,\|x-a\|$ (as we easily see taking into account that, for every $x \in I$ and every j, we have assumed $\frac{1}{2}\,\varphi_j(a) < \varphi_j(x) < 2\,\varphi_j(a)$).

Now, given $\lambda \in \,]0,1[$ we can determine γ in such a way that $0 < \gamma < \min\{\varphi_1(a),...,\varphi_n(a)\}$ and also, for every j,

$$\frac{\varphi_j(a)+\gamma}{\varphi_j(a)-\gamma} < \frac{1}{1-\lambda}$$

and then $\epsilon' > 0$ such that, for $\|x-a\| < \epsilon'$ and $j \in \{1,...,n\}$,

$$\varphi_j(a)-\gamma \;<\; \varphi_j(x) \;<\; \varphi_j(a)+\gamma\;.$$

Then, if $u \in J_x^{1-\lambda}$ and $\|x-a\| < \epsilon'$ we have also, for every j,

$$\varphi_j(a)-\gamma \;<\; \varphi_j(u) \;<\; \varphi_j(a)+\gamma$$

(since $J_x^{1-\lambda} \subset J_x$ and $\|u-a\| < \|x-a\|$ for $u \in J_x$) and therefore:

$$\frac{u_j-a_j}{x_j-a_j}\,\frac{\varphi_j(u)}{\varphi_j(x)} \;<\; \frac{u_j-a_j}{x_j-a_j}\,\frac{\varphi_j(a)+\gamma}{\varphi_j(a)-\gamma} \;<\; \frac{1}{1-\lambda}\,\frac{u_j-a_j}{x_j-a_j} \;<\; 1\;.$$

From this it follows immediately that, for $\|x-a\| < \epsilon'$, we have $J_x^{1-\lambda} \subset A_x$.

To obtain the other inclusion referred in the lemma, let us suppose again that a number $\lambda \in \,]0,1[$ was given and use it to determine $\gamma' > 0$ such that, for every j,

$$\frac{\varphi_j(a)-\gamma'}{\varphi_j(a)+\gamma'} > \frac{1}{1+\lambda}\;.$$

Next, determine $\epsilon'' > 0$ such that, for $\|x-a\| < \epsilon''$, we have

$$\varphi_j(a)-\gamma' \;<\; \varphi_j(x) \;<\; \varphi_j(a)+\gamma' \qquad (\text{for } j \in \{1,...,n\})\;.$$

Now, if $u \in A_x$ and $\|x-a\| < \frac{\epsilon''}{4}$, we shall have (by one of our previous observations) $\|u-a\| < \epsilon''$ and so, for every j:

$$\varphi_j(a)-\gamma' \;<\; \varphi_j(u) \;<\; \varphi_j(a)+\gamma'\;.$$

From this it follows

$$\frac{1}{1+\lambda}\,\frac{u_j-a_j}{x_j-a_j} \;<\; \frac{u_j-a_j}{x_j-a_j}\,\frac{\varphi_j(a)-\gamma'}{\varphi_j(a)+\gamma'} \;<\; \frac{u_j-a_j}{x_j-a_j}\,\frac{\varphi_j(u)}{\varphi_j(x)} \;<\; 1$$

and then $u \in J_x^{1+\lambda}$. So, given $\lambda \in \,]0,1[$, we shall have $J_x^{1-\lambda} \subset A_x \subset J_x^{1+\lambda}$ for every x such that $\|x-a\| < \epsilon = \min\{\epsilon', \frac{\epsilon''}{4}\}$. ∎

Lemma 3.16. *Let x, J_x and A_x be like in the preceding lemma and let $F \in BC(I_a)$ with $\sup_{x\in I_a} F(x) = M$ and $\inf_{x\in I_a} F(x) = m > 0$. Then*

$$\lim_{x\to a} \frac{\displaystyle\int_{A_x} F(u)\,du}{\displaystyle\int_{J_x} F(u)\,du} = 1 \,.$$

Proof: Given $\delta > 0$, determine $\lambda \in \,]0,1[$ such that

$$\frac{M}{m}\Big[1-(1-\lambda)^n\Big] < \delta \quad \text{and} \quad \frac{M}{m}\Big[(1+\lambda)^n - 1\Big] < \delta$$

and then $\epsilon > 0$ such that, for $\|x-a\| < \epsilon$, we have (with the notation used in Lemma 3.15) $J_x^{1-\lambda} \subset A_x \subset J_x^{1+\lambda}$.

Then, if $\|x-a\| < \epsilon$,

$$\int_{J_x^{1-\lambda}} F(u)\,du \;\le\; \int_{A_x} F(u)\,du \;\le\; \int_{J_x^{1+\lambda}} F(u)\,du$$

and so, since we have (denoting by ν, for instance, Jordan measure):

$$\frac{\displaystyle\int_{J_x\setminus J_x^{1-\lambda}} F(u)\,du}{\displaystyle\int_{J_x} F(u)\,du} \;\le\; \frac{M\,\nu(J_x\setminus J_x^{1-\lambda})}{m\,\nu(J_x)} \;=\; \frac{M\,\nu(J_x)\,[1-(1-\lambda)^n]}{m\,\nu(J_x)} \;<\; \delta$$

and analogously

$$\frac{\displaystyle\int_{J_x^{1+\lambda}\setminus J_x} F(u)\,du}{\displaystyle\int_{J_x} F(u)\,du} \;\le\; \frac{M\Big[(1+\lambda)^n - 1\Big]}{m} \;<\; \delta$$

we can conclude that, for $\|x-a\| < \epsilon$,

$$1-\delta \;<\; 1 - \frac{\displaystyle\int_{J_x\setminus J_x^{1-\lambda}} F(u)\,du}{\displaystyle\int_{J_x} F(u)\,du} \;\le\; \frac{\displaystyle\int_{A_x} F(u)\,du}{\displaystyle\int_{J_x} F(u)\,du} \;\le\; 1 + \frac{\displaystyle\int_{J_x^{1+\lambda}\setminus J_x} F(u)\,du}{\displaystyle\int_{J_x} F(u)\,du} \;<\; 1+\delta \,. \;\blacksquare$$

Now, let us recall and complete some of the notation that will be used in the following results. A and B will still be two open sets of $\mathbb{R}^n$, μ a map from A to B, $a \in A$, $b = \mu(a)$, J will be an interval such that $b \in J \subset B$, I an interval and $\varphi_j \colon I \to \mathbb{R}$ $(j \in \{1,...,n\})$ n functions satisfying the conditions referred to before Lemma 3.15. Without loss of generality we can suppose that $I \subset \mu^{-1}(J)$.

We shall also put $I_a = \{x \in I \colon\ (x_1 - a_1) \cdots (x_n - a_n) \neq 0\}$, $J_b = \{y \in J \colon (y_1 - b_1) \cdots (y_n - b_n) \neq 0\}$ and, for each $\Phi \in BC(I_a)$ [resp. $\Psi \in BC(J_b)$] and each $x \in I_a$ [resp. $y \in J_b$]:

$$\begin{aligned}(\rho_{a,x}\Phi)(x) &= \frac{1}{(x_1 - a_1) \cdots (x_n - a_n)} \int_{a_1}^{x_1} \cdots \int_{a_n}^{x_n} \Phi(u_1,...,u_n)\, du_1 \cdots du_n \\ &= \frac{1}{|x_1 - a_1| \cdots |x_n - a_n|} \int_{J_x} \Phi(u)\, du\end{aligned}$$

$$\left[\text{resp.}\quad \begin{aligned}(\rho_{b,y}\Psi)(y) &= \frac{1}{(y_1 - b_1) \cdots (y_n - b_n)} \int_{b_1}^{y_1} \cdots \int_{b_n}^{y_n} \Psi(v_1,...,v_n)\, dv_1 \cdots dv_n \\ &= \frac{1}{|y_1 - b_1| \cdots |y_n - b_n|} \int_{J'_y} \Psi(v)\, dv \end{aligned}\right].$$

Now we can state:

Lemma 3.17. *Denoting by $\mu_{|I_a}$ the restriction of μ to the set I_a, let G be a function in the space $BC(J_b)$ with positive infimum and $F = G \circ \mu_{|I_a}$. Then $F \in BC(I_a)$ and putting*

$$F_p = \rho^p_{a,x} F\,, \qquad G_p = \rho^p_{b,y} G \quad \text{ and } \quad G^*_p = G_p \circ \mu_{|I_a}\,,$$

for each p there exists a continuous function $\lambda_p \colon I_a \to \mathbb{R}$ such that $G^*_p = \lambda_p F_p$ and $\lim_{x \to a} \lambda_p(x) = 1$.

Proof: It is obvious $F \in BC(I_a)$ (and that $\overline{F(a)} = \overline{G(a)}$, $\underline{F(a)} = \underline{G(a)} > 0$). To prove the lemma we shall use induction on p. As the case $p = 0$ (where we can take $\lambda_0 = 1$) is trivial, let us suppose that there exists a function λ_p in the required conditions and consider the equality (where we denote as before by J'_y the interval determined by the points b and $y \in J_b$):

$$G_{p+1}(y) = \frac{1}{|y_1 - b_1| \cdots |y_n - b_n|} \int_{J'_y} G_p(v)\, dv\ .$$

Changing variables by means of $\mu_{|I_a}$ we obtain

$$G^*_{p+1}(x) = \frac{1}{|x_1 - a_1| \cdots |x_n - a_n|\, \varphi_1(x) \cdots \varphi_n(x)} \int_{A_x} G^*_p(u)\, J_\mu(u)\ du\ ,$$

where $A_x = \mu^{-1}(J'_y)$. Now, by the induction hypothesis

$$G^*_{p+1}(x) \;=\; \frac{1}{|x_1 - a_1| \cdots |x_n - a_n|\, \varphi_1(x) \cdots \varphi_n(x)} \int_{A_x} \lambda_p(u)\, F_p(u)\, J_\mu(u)\; du$$

or, since A_x is connected and F_p positive in A_x,

$$G^*_{p+1}(x) \;=\; \frac{1}{|x_1 - a_1| \cdots |x_n - a_n|\, \varphi_1(x) \cdots \varphi_n(x)}\, \lambda_p(\overline{x})\, J_\mu(\overline{x}) \int_{A_x} F_p(u)\, du \;,$$

where $\overline{x}$ is a point in A_x that tends to a when x does. So, if we put

$$\tau_p(x) = \frac{\displaystyle\int_{A_x} F_p(u)\, du}{\displaystyle\int_{J_x} F_p(u)\, du} \qquad \text{and} \qquad \lambda_{p+1}(x) \;=\; \frac{1}{\varphi_1(x) \cdots \varphi_n(x)}\, \lambda_p(\overline{x})\, J_\mu(\overline{x})\, \tau_p(x) \;,$$

we get finally $G^*_{p+1} = \lambda_{p+1}\, F_{p+1}$, where we easily see, taking into account Lemma 3.16, that $\lim_{x\to a} \lambda_{p+1}(x) = 1$. ∎

Lemma 3.18. *If $G \in BC(J_b)$ and $F = G \circ \mu_{|I_a}$, then*

$$\liminf_a F = \liminf_b G\,, \qquad \limsup_a F = \limsup_b G\ .$$

Proof: If the infimum of G is strictly positive, Lemma 3.17 implies immediately that, for each p, $\underline{G_p(b)} = \underline{F_p(a)}$ and $\overline{G_p(b)} = \overline{F_p(a)}$. Then:

$$\limsup_b G \;=\; \lim_{p\to\infty} \overline{G_p(b)} \;=\; \lim_{p\to\infty} \overline{F_p(a)} \;=\; \limsup_a F$$

and the same for the inferior limits. If $m = \inf_{y\in J_b} G(y) \le 0$, let us take a constant c such that $c > |m|$ and put $\Psi(y) = G(y) + c$ (for $y \in J_b$) and $\Phi = \Psi \circ \mu_{|I_a}$. Then Ψ will be a function in the space $BC(J_b)$ with infimum strictly greater than zero and so, from what we have just seen, we shall have $\Phi \in BC(I_a)$ and

$$\liminf_a \Phi = \liminf_b \Psi\,, \qquad \limsup_a \Phi = \limsup_b \Psi\ .$$

Since $\Phi(x) = F(x) + c$ for each $x \in I_a$, it follows, taking into account Corollary 3.4,

$$\limsup_a F \;=\; \limsup_a \Phi - c \;=\; \limsup_b \Psi - c \;=\; \limsup_b G$$

and analogously for the inferior limits. ∎

Theorem 3.19. *Let A, B, μ, a and b be like in the preceding lemmas, $g \in \mathcal{B}_b(B)$ and $f = g \circ \mu$. Then $f \in \mathcal{B}_a(A)$ and*

$$\liminf_a f = \liminf_b g\,, \qquad \limsup_a f = \limsup_b g\ .$$

Proof: Denote by L the set of all distributions $l \in \mathcal{B}_b(B)$ such that $l \circ \mu \in \mathcal{B}_a(A)$ and

$$\limsup_a (l \circ \mu) = \limsup_b l\,, \qquad \liminf_a (l \circ \mu) = \liminf_b l\ .$$

We have to prove that $L = \mathcal{B}_b(B)$.

Let $g \in \mathcal{B}_b(B)$. If, for some interval J, with $b \in J \subset B$, we have $g_{|J} \in BC(J_b)$ then, denoting by I an interval such that $a \in I \subset A$ and $I \subset \mu^{-1}(J)$, we shall clearly have $f_{|I} = g_{|J} \circ \mu_{|I} \in BC(I_a)$ and, by Lemma 3.18,

$$\limsup_a f_{|I} = \limsup_b g_{|J}\,, \qquad \liminf_a f_{|I} = \liminf_b g_{|J}\ ;$$

from this it follows immediately $f \in \mathcal{B}_a(A)$ and

$$\limsup_a f = \limsup_b g\,, \qquad \liminf_a f = \liminf_b g$$

that is, $g \in L$.

So, to conclude the proof, it will be sufficient to show that, if a distribution h belongs to L and if $k \in \mathcal{D}'_{\mathbb{R}}(B)$ is such that, for some $j \in \{1, ..., n\}$ and on some interval J (with $b \in J \subset B$) we have $k_{|J} = \partial_{y_j}(h_{|J}) = D_{y_j}[(\widehat{y}_j - b_j)\, h_{|J}]$, then $k \in L$.

Suppose then that h, k, j and J satisfy the conditions just stated and let I be an interval such that $a \in I \subset A$ and $I \subset \mu^{-1}(J)$; suppose also, as usually, that the restriction of μ to I, $\mu_{|I}$, can be expressed by means of the system

$$y_i - b_i \;=\; (x_i - a_i)\, \varphi_i(x)\ ,$$

with the φ_i strictly positive and of class C^∞. Then, putting $h^* = h \circ \mu$ and $k^* = k \circ \mu$ we shall have:

$$k^*_{|I} \;=\; k \circ \mu_{|I} \;=\; \Big\{ D_{y_j} \Big[(\widehat{y}_j - b_j)\, h_{|J} \Big] \Big\} \circ \mu_{|I} \;=\; \left(\sum_{i=1}^{n} \frac{\partial_{x_i}}{\partial_{y_j}}\, D_{x_i} \right) \Big[(\widehat{x}_j - a_j)\, \varphi_j\, h^*_{|I} \Big]\ .$$

But we easily see that, if $i \neq j$, we have in I:

$$\frac{\partial_{x_i}}{\partial_{y_j}} \;=\; (\widehat{x}_i - a_i)\, \omega_{ij}\ ,$$

where the functions $\omega_{ij} \colon I \to \mathbb{R}$ are of class C^∞; and also that,

$$\frac{\partial x_j}{\partial y_j}(a) = \frac{1}{\varphi_j(a)} \; . \tag{5}$$

So, we have:

$$k^*_{|I} \;=\; \sum_{\substack{i=1 \\ i\neq j}}^{n} (\widehat{x}_j - a_j)\, \omega(ij)(\widehat{x}_i - a_i)\, D_{x_i}(\varphi_j\, h^*_{|I}) + \frac{\partial x_j}{\partial y_j} D_{x_j}\Big[(\widehat{x}_j - a_j)\, \varphi_j\, h^*_{|I}\Big] \; . \tag{6}$$

Now, as we have by hypothesis $h^* \in \mathcal{B}_a(A)$, we see (by Propositions 3.1, 3.10, 3.2 and Corollary 3.4) that, for $i \neq j$, the distribution

$$(\widehat{x}_i - a_i)\, D_{x_i}(\varphi_j\, h^*_{|I}) \;=\; \partial_{x_i}(\varphi_j\, h^*_{|I}) - \varphi_j\, h^*_{|I}$$

belongs to the space $\mathcal{B}_a(I)$ and then (by Theorem 3.9 and Corollary 3.5) we can conclude that the first term of the second member of (6) is continuous and has value zero at the point a. On the other hand, taking into account equality (5) and Propositions 3.2 and 3.10, we see that the distribution $\frac{\partial x_j}{\partial y_j}\, \partial_{x_j}(\varphi_j\, h^*_{|I})$ has at the point a the same superior and the same inferior limits as h^*. From this it follows easily that $k^* \in \mathcal{B}_a(A)$ and that

$$\limsup_a k^* \;=\; \limsup_a k^*_{|I} \;=\; \limsup_a h^* \;=\; \limsup_b h \;=\; \limsup_b k$$

and analogously for the inferior limits. This means that $k \in L$, concluding the proof. ∎

As an immediate consequence we have:

Corollary 3.20. *With the same notation of Theorem 3.19 suppose now that $g \in \mathcal{V}_b(B)$. Then $f = g \circ \mu \in \mathcal{V}_a(A)$ and $f(a) = g(b)$.*

As we saw, all the preceding definitions and results stated in this work concern only real distributions; but it is quite clear that some of them are immediately extensible to (complex) distributions. To prepare the obvious definitions we recall that, as it is well known, if f is a (complex) distribution defined in an open set A of $\mathbb{R}^n$, there exist two real distributions $f_1, f_2 \in \mathcal{D}'_{\mathbb{R}}(A)$, uniquely determined, such that $f = f_1 + i\, f_2$. Then we shall say that f is *bounded* [resp. *continuous*] *at the point* $a \in A$ — and we shall write $f \in \mathcal{B}_a(A)$ [resp. $f \in \mathcal{V}_a(A)$] — iff we have $f_1, f_2 \in \mathcal{B}_a(A)$ [resp. $f_1, f_2 \in \mathcal{V}_a(A)$]. If we have $f \in \mathcal{V}_a(A)$, we call *value of f at the point a* the number $f(a) = f_1(a) + i\, f_2(a)$.

Now it is very easy to verify that many propositions previously stated — namely Corollaries 3.5, 3.7 (with the hypothesis $\lambda \in \mathbb{R}$ changed to $\lambda \in \mathbb{C}$), 3.8 (with $\mathcal{D}'_{\mathbb{R}}(A)$ changed to $\mathcal{D}'(A)$ and $\mathbb{R}$ to $\mathbb{C}$), Theorem 3.9 and Corollary 3.11 (with $C^\infty_{\mathbb{R}}(A)$ changed to $C^\infty(A)$) and Corollaries 3.14 and 3.20 — are still valid in this new context.

REFERENCES

[1] Antosik, P.; Mikusinski, J. and Sikorsky, R. – *Theory of Distributions, The Sequential Approach*, Elsevier, 1973.

[2] Campos Ferreira, J. – Sur la notion de limite d'une distribution à l'infini, *Rend. Acc. Naz. Lincei,* 38(6) (1965).

[3] Campos Ferreira, J. – *Alguns Problemas de Prolongamento de Distribuições, com Aplicações à Integração*, Diss. conc. Prof. Cat., IST, Lisboa, 1967.

[4] Campos Ferreira, J. – Sur une notion de limite d'une distribution, plus générale que celle de M.S. Lojasiewicz, *Rev. Fac. Ciências, Lisboa,* (1973).

[5] Campos Ferreira, J. – Sur l'oscillation et la valeur d'une distribution vectorielle en un point, *CMAF, Lisboa, Textos e Notas,* 6 (1977).

[6] Campos Ferreira, J. – A diferenciabilidade no sentido de Denjoy e a teoria das distribuições, *Mem. Acad. Ciências de Lisboa,* (1986).

[7] Campos Ferreira, J. – Sur une définition de la valeur d'une distribution en un point et son application au problème de Cauchy, *Port. Math.,* 44(3) (1987).

[8] Campos Ferreira, J. – *Introduction to the theory of distributions*, "Monographs and Surveys in Pure and Applied Mathematics", vol. 87, Pitman, 1997.

[9] Campos Ferreira, J. and Silva Oliveira, J. – Problèmes aux conditions initiales dans la théorie des distributions, *Rev. Fac. Ciências, Lisboa,* (1964).

[10] Hoskins, R. and Sousa Pinto, J. – *Distributions, Ultradistributions and Other Generalised Functions*, Ellis Horwood, 1994.

[11] Lojasiewicz, S. – Sur la valeur et la limite d'une distribution en un point, *Studia Math.,* 16 (1957).

[12] Lojasiewicz, S. – Sur la fixation de variables dans une distribution, *Studia Math.,* 17 (1958).

[13] Schwartz, L. – *Théorie des Distributions*, Hermann, 1966.

[14] Sebastião e Silva, J. – *Orders of growth and integrals of distributions (International Course on Distribution Theory)*, Instituto Gulbenkian de Ciência, 1967.

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