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ON A CLASS OF SECOND ORDER ODE WITH A TYPICAL DEGENERATE NONLINEARITY

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Abstract: Global solutions of the second order ODE: $u''+u'+f(u)=0$ are studied where f is a C^1 function satisfying $f(0)=0$, $f(u)>0$ for all $u\neq 0$, $f(u)=o(|u|)$ as $u\to 0$; a typical case is $f(u)=c\,u^2$ or more generally $f(u)=c\,|u|^\alpha$ with $c>0$, $\alpha>1$. It is shown that all global solutions u on $[0,+\infty)$ are bounded with $u'+u>0$ and $\lim_{t\to\infty}\{|u(t)|+|u'(t)|+|u''(t)|\}=0$. Moreover if $f(s)=c\,|s|^\alpha$ for some $c>0$, $\alpha>1$, there exists a unique global maximal negative solution $u_-\in C^2(0,+\infty)$ and a unique global maximal solution $u_+\in C^2(0,+\infty)$ such that $\mathrm{Sup}_{t\in(0,+\infty)}\,u_+$ achieves its maximum value. The set of initial data giving rise to global trajectories for $t\geq 0$ is the unbounded closed domain $\mathcal{D}$ enclosed by the union of the two trajectories of u_+ and u_- in the phase plane. Finally it is shown that $\mathrm{meas}(\mathcal{D})<\infty$.

1 – Introduction and main results

In this paper we study the second order ODE

$$u''+u'+f(u)\;=\;0\;, \tag{1.1}$$

where f is a function satisfying the following conditions

$$f\in C^1(\mathbb{R})\,,\qquad f(0)=0\;, \tag{1.2}$$

$$\forall\, u\neq 0\,,\qquad f(u)>0 \tag{1.3}$$

and

$$f(u)=o(|u|)\qquad\text{as}\;\;u\to 0\;.$$

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A typical case is $f(u) = cu^2$ or more generally $f(u) = c\,|u|^\alpha$ with $c > 0$, $\alpha > 1$. We shall first establish the following simple and general result.

Theorem 1.1. *Under the hypotheses (1.2) and (1.3), let $u \not\equiv 0$ be a global bounded solution of (1.1) on $[0, +\infty)$. Then u satisfies the following properties*

i) $\lim_{t\to\infty} \Big\{|u(t)| + |u'(t)| + |u''(t)|\Big\} = 0.$

ii) $u' + u > 0,\ f(u) \in L^1(\mathbb{R}^+)$ *and we have the formula*

$$\forall\, t \geq 0\,, \qquad u' + u = \int_t^\infty f(u(s))\, ds\ . \tag{1.4}$$

iii) *If for some $t_0 \geq 0$ we have $u(t_0) \geq 0$, it follows that $u(t) \geq 0$, $\forall\, t \geq t_0$. Moreover*

$$\forall\, t \geq t_0\,, \qquad u(t) \geq e^{-(t-t_o)}\, u(t_0)\ .$$

iv) *If for some $t_0 \geq 0$ we have $u'(t_0) = 0$, it follows that*

$$u(t) > 0, \quad \forall\, t \geq t_0 \qquad \text{and} \qquad u'(t) < 0, \quad \forall\, t > t_0\ .$$

From Theorem 1.1 it is immediate to deduce the following

Corollary 1.2. *Let $u \not\equiv 0$ be a global bounded solution of (1.1) on $[0, +\infty)$. Then u satisfies either of the following alternatives 1) and 2):*

1) $u' > 0$ *on* $[0, +\infty)$, *and therefore* $u < 0$ *on* $[0, +\infty)$.

2) *There is $t_0 \geq 0$ such that $u' > 0$ on $[0, t_0)$, $u'(t_0) = 0$, $u'(t) < 0$ for all $t > t_0$. In addition $u(t) > 0$ for all $t \geq t_0$.*

If for example $f(s) = s^2$ it is natural to wonder whether all global solutions of (1.1) on $[0, \infty)$ are bounded. Such a property has to do with the growth of f at infinity, for instance if f is sublinear no blow-up can happen. Actually the following simple result is true:

Theorem 1.3. *Assume that f satisfies (1.2), (1.3) and*

$$\exists\, A > 0, \quad \exists\, a > 0, \quad \forall\, s \geq A\,, \qquad f(s) \geq a\ , \tag{1.5}$$

$$\exists\, T > 0, \quad f(T) > 0 \qquad \text{and} \qquad \forall\, s \in (-\infty, -T], \quad f'(s) \leq 0\ , \tag{1.6}$$

$$\int_T^{+\infty} \frac{ds}{\sqrt{-F(-s)}} < \infty \tag{1.7}$$

with

$$F(s):\quad \int_0^s f(x)\,dx \;\le\; 0 \qquad \forall\, s \le 0\ .$$

Then any global solution of (1.1) on $[0,+\infty)$ *is bounded.*

The main result of this paper is the following

Theorem 1.4. *Assume that* $f(s) = c\,|s|^\alpha$ *for some* $c > 0$, $\alpha > 1$. *Then there exists a unique global maximal negative solution* $u_- \in C^2(0,+\infty)$ *of (1.1) and a unique global maximal solution* $u_+ \in C^2(0,+\infty)$ *of (1.1) such that* $\mathrm{Sup}_{t\in(0,+\infty)}\, u_+$ *achieves its maximum value. In addition we have* $u_+ > 0$ *and* $u'_+ < 0$ *for* t *large, while*

$$\lim_{t\to 0} u_\pm(t) = -\infty \qquad \textit{and} \qquad \lim_{t\to 0} u'_\pm(t) = +\infty\ .$$

Finally the set of initial data giving rise to global trajectories of (1.1) for $t \ge 0$ *is the unbounded closed domain* $\mathcal{D}$ *enclosed by the union of the two trajectories of* u_+ *and* u_- *in the phase plane.*

Remark 1.5. **1**) The nonlinearity $f(s) = c\,|s|^\alpha$ is degenerate in the sense that f' vanishes at the equilibrium point 0. For a system without degenerate equilibria, the set of solutions tending to an equilibrium is generally either open or lower dimensional (the "stable" manifold). Here, the attraction basin of $\{0,0\}$ happens to be closed with nonempty interior, and in addition $\{0,0\}$ lies at the *boundary* of the basin. This behavior may be typical in presence of this kind of degeneracy.

2) Motivated by the convergence result of [2], in [3] the asymptotic behavior of solutions to the equation

$$u'' + u' + c\,u\,(u^2 - R^2)^2 \;=\; 0$$

has been investigated quite exhaustively. When for instance $|u - R| \ll 1$, this equation is very close to

$$v'' + v' + 4\,c\,R^3\,v^2 \;=\; 0$$

where $v := u - R$. In [3] a set of two closed regions $\mathcal{D}_\pm$ quite similar to the set $\mathcal{D}$ in Theorem 1.4 appears, corresponding to the attraction basins of the two instable solutions $\pm(R,0)$. All solutions starting in $\mathbb{R}^2 \backslash (\mathcal{D}_+ \cup \mathcal{D}_-)$ converge to $(0,0)$. Of course the basic difference with equation (1.1) is that here all solutions are global for $t \ge 0$, and this makes the problem much more complex. □

This paper is organised as follows: in section 2 we prove the simple general properties of Theorem 1.1, Corollary 1.2 and Theorem 1.3. Section 3 is devoted to the construction of small non-trivial solutions under a simple estimate on f'. Sections 4 and 5 are respectively devoted to the global analysis of the negative solutions and general global solutions. With these tools at hand, in section 6 we give the proof of Theorem 1.4. Section 7 contains a few additional facts on the general global solutions. In the final section 8, we show that when $f(u) = c\,|u|^\alpha$ with $c > 0$, $\alpha > 1$ such solutions are exceptional: more precisely with the notation of Theorem 1.4 we have $\operatorname{meas}(\mathcal{D}) < \infty$.

2 – General properties

Proof of Theorem 1.1:

i) If u is a bounded solution of (1.1) we have

$$u'' + u' + u \;=\; u - f(u) \;\in\; L^\infty(\mathbb{R}^+)$$

from which boundedness of u' and u'' follows at once. Then the identity

$$\frac{d}{dt}\Big(\frac{1}{2}\,u'^2 + F(u)\Big) \;=\; -u'^2$$

where F stands for any primitive of f shows that $\Phi(u,v) := \frac{1}{2}\,v^2 + F(u)$ is a Liapunov function for (1.1) in the phase plane. It follows immediately (cf. e.g. [1]) that as $t \to +\infty$, we have $u'(t) \to 0$ and $f(u(t)) \to 0$. Then by our hypotheses on f we also have $u(t) \to 0$ and finally by the equation, $u''(t) \to 0$.

ii) It is clear by integration that

$$\forall\, t \geq 0, \quad \forall\, \theta \geq t\,, \qquad u'(t) + u(t) \;=\; u'(\theta) + u(\theta) + \int_t^\theta f(u(s)\,ds\;.$$

The result then follows easily by letting $t \to +\infty$.

iii) We have

$$\frac{d}{dt}(e^t u(t)) \;=\; e^t\Big(u'(t) + u(t)\Big) \;\geq\; 0\,, \qquad \forall\, t \geq t_0\;.$$

The conclusion follows by integration.

iv) We have

$$\frac{d}{dt}(e^t u'(t)) \;=\; e^t\Big(u''(t) + u'(t)\Big) \;=\; -e^t\, f(u(t)) \;\leq\; 0\,, \qquad \forall\, t \geq t_0\;.$$

Assuming $u \not\equiv 0$, by local uniqueness we have $f(u(t_0)) > 0$ and therefore $u'(t) < 0$, $\forall\, t > t_0$. Finally by (1.4)

$$u(t) \;=\; -u'(t) + \int_t^\infty f(u(s)\,ds \;>\; 0\,, \qquad \forall\, t \geq t_0 \;.$$

Theorem 1.1 is completely proved. ∎

Proof of Theorem 1.3: Let u be an *unbounded* global solution of (1.1) on $[0,+\infty)$. We distinguish 2 cases

1) If $u' > 0$ everywhere on $[0,+\infty)$, we clearly have $u(t) \to +\infty$ as $t \to +\infty$. As a consequence of (1.5) we deduce

$$\exists\, T > 0, \;\; \forall\, t \geq T\,, \qquad u''(t) = -u'(t) - f(u(t)) \leq -a < 0 \;.$$

From this inequality it follows that $u'(t) \to -\infty$ as $t \to +\infty$. A contradiction.

2) Assuming $u'(t_0) \leq 0$, as previously we find $u' < 0$ everywhere in $(t_0,+\infty)$, then clearly $u(t) \to -\infty$ as $t \to +\infty$. If $u'' > 0$ on $[t_0,+\infty)$, then

$$u'(t) = -u''(t) - f(u(t)) \leq -f(u(t)) \qquad \text{on} \;\; [t_0,+\infty) \;.$$

In order to prove that this is impossible, first we establish

Lemma 2.1. *Under the hypotheses of Theorem 1.3 we have*

$$\int_T^{+\infty} \frac{ds}{f(-s)} \;<\; \infty \;. \tag{2.1}$$

Proof: From condition (1.6) it first follows immediately that $F(s) \to -\infty$ as $s \to -\infty$ and with $C := -F(-T)$,

$$\forall\, s \geq T\,, \qquad 0 \leq -F(-s) \leq s\, f(-s) + C \;. \tag{2.2}$$

Now let T_1 be such that for all $t \geq T_1$, we have $F(t) \leq -2\,C$: for any $x \geq 2\,T_1$, we have by the non-decreasing character of $f(-x)$

$$\frac{x}{2\,\sqrt{x}\,\sqrt{\,f(-x)}} \;\leq\; \int_{\frac{x}{2}}^{x} \frac{ds}{\sqrt{s}\,\sqrt{f(-s)}} \;\leq\; \int_{\frac{x}{2}}^{x} \frac{ds}{\sqrt{-F(-s)-C}} \;\leq\; \\ \leq\; \sqrt{2} \int_T^\infty \frac{ds}{\sqrt{-F(-s)}} \;<\; \infty$$

by (1.7). In particular we find

$$\exists\, M > 0, \;\; \forall\, s \geq 2\,T_1\,, \qquad s \leq M f(-s) \;. \tag{2.3}$$

Finally (2.2) and (2.3) give

$$\forall\, s \geq 2T_1\,, \qquad 0 \leq -F(-s) \leq M(f(-s))^2\ . \tag{2.4}$$

Then (2.1) follows immediately from (1.7). ∎

Proof of Theorem 1.3 continued: If $u'' > 0$ on $[t_0, +\infty)$, letting $v = -u$ we find for all t large enough, as a consequence of (2.1)

$$\frac{d}{dt}\int_{v(t)}^{\infty}\frac{ds}{f(-s)} \;=\; \frac{-v'(t)}{f(-v(t))} \;=\; \frac{u'(t)}{f(u(t))} \;\leq\; -1\ ,$$

an obvious contradiction since the integral remains nonnegative. The same contradiction obviously appears if we assume the weaker condition $u'' > 0$ on $[\theta, +\infty)$ for some $\theta > 0$. On the other hand if $u''(t_1) \leq 0$ for some $t_1 \geq \max\{t_0, T\}$, from the hypothesis $f' \leq 0$ on $(-\infty, -T]$ we deduce that

$$u''' + u'' = -f'(u)\,u' \;\leq\; 0 \qquad \text{on} \;\; [t_1, +\infty)$$

and in particular then $u'' \leq 0$ on $[t_1, +\infty)$. As a consequence, u' is nonincreasing and u'^2 is nondecreasing. For any $t \geq t_1$ we consequently obtain

$$\int_{t_1}^{t} u'^2(s)\,ds \;\leq\; (t - t_1)\,u'^2(t)$$

and the energy identity

$$\frac{1}{2}\,u'^2(t) + F(u(t)) + \int_{t_1}^{t} u'^2(s)\,ds \;=\; \frac{1}{2}\,u'^2(t_1) + F(u(t_1))$$

provides the inequality

$$(t - t_1 + 1)\,u'^2(t) \;\geq\; F(u(t_1)) - F(u(t))\ .$$

Because condition (1.6) obviously implies $F(s) \to -\infty$ as $s \to -\infty$, for t large enough we find

$$v' \;\geq\; \frac{1}{2}\,\frac{\sqrt{-F(-v)}}{\sqrt{t}}\ .$$

Letting

$$G(s) \;=\; \int_{s}^{\infty}\frac{d\sigma}{\sqrt{-F(-\sigma)}}$$

we obtain

$$\frac{d}{dt}G(v(t)) \;=\; -\frac{v'(t)}{\sqrt{-F(-v)}} \;\leq\; -\frac{1}{\sqrt{t}}$$

hence $G(v(t))$ becomes negative for t large, a conclusion which contradicts $G \geq 0$. ∎

3 – Special solutions for small initial values

Before we proceed to the proof of Theorem 1.4, we shall establish a sequence of Lemmas valid under various conditions on f. In this section we start by the construction of uniformly small, exponentially decaying solutions on $\mathbb{R}^+$ by means of a fixed point argument.

Lemma 3.1. *Let $f \in C^1(\mathbb{R})$ be such that $f(0) = 0$ and assume*

$$\exists\, \varepsilon > 0,\ \exists\, \alpha > 0,\ \exists\, C > 0 \quad \forall\, s \in \mathbb{R},\ |s| \le \alpha \implies |f'(s)| \le C\,|s|^\varepsilon\ . \tag{3.1}$$

Then for all u_0 with $|u_0|$ small enough, there exists a function $u \in C^2(\mathbb{R}^+)$ satisfying

$$u(0) = u_0\,, \qquad \sup_{t\ge 0}\Big\{e^t|u(t)|\Big\} < \alpha\ , \tag{3.2}$$

$$\forall\, t \ge 0\,, \qquad u' + u \;=\; \int_t^\infty f(u(s)\,ds\ . \tag{3.3}$$

Proof: If u is a solution of (3.2) and (3.3), as a consequence of (3.1) we have the estimate

$$\forall\, s \in \mathbb{R}^+\,, \qquad |f(u(s))| \;\le\; C\,|u(s)|^{1+\varepsilon} \le C\,e^{-(1+\varepsilon)s}\,|e^s u(s)|^{1+\varepsilon}\ . \tag{3.4}$$

Letting $z(t) := e^t\,u(t)$ for all $t \ge 0$, we can write (3.3) in the form

$$\begin{aligned} \forall\, t \ge 0\,, \qquad z(t) \;&=\; z(0) \;+ \int_0^t e^r \int_r^\infty f(u(s)\ ds\,dr \\ &=\; u_0 \;+ \int_0^t e^r \int_r^\infty f(e^{-s}z(s))\ ds\,dr\ . \end{aligned} \tag{3.5}$$

We are therefore led to introduce the functional set

$$X := X_\beta = \Big\{\zeta \in C_b(\mathbb{R}^+),\ \sup_{t\ge 0}|\zeta(t)| \le \beta\Big\}$$

and the nonlinear map $\mathcal{T} \colon X \to C_b(\mathbb{R}^+)$ defined by

$$\forall\, z \in X,\ \forall\, t \ge 0\,, \qquad (\mathcal{T}z)(t) \;=\; u_0 + \int_0^t e^r \int_r^\infty f(e^{-s}z(s))\ ds\,dr\ . \tag{3.6}$$

As a consequence of (3.4) we find immediately the estimate

$$\begin{aligned}
\forall t \geq 0\,, \qquad |(\mathcal{T} z)(t)| \;&\leq\; |u_0| \,+ \int_0^t e^r \int_r^\infty C\, e^{-(1+\varepsilon)s}\, |z(s)|^{1+\varepsilon}\; ds\, dr \\
&\leq\; |u_0| \,+\, C\, \beta^{1+\varepsilon} \int_0^t e^r \int_r^\infty e^{-(1+\varepsilon)s}\; ds\, dr \\
&\leq\; |u_0| \,+\, \frac{C \beta^{1+\varepsilon}}{1+\varepsilon} \int_0^t e^{-\varepsilon r}\, dr \\
&\leq\; |u_0| \,+\, \frac{C}{\varepsilon(1+\varepsilon)}\, \beta^{1+\varepsilon}\;.
\end{aligned}$$

In particular if $|u_0|$ and β satisfy the conditions

(3.7) $$|u_0| \leq \alpha\,, \qquad |u_0| + \frac{C}{\varepsilon(1+\varepsilon)}\, \beta^{1+\varepsilon} \,\leq\, \beta$$

we have

(3.8) $$\mathcal{T}(X) \subset X\;.$$

For instance, we can select $\beta \leq \alpha$ so that $\dfrac{C}{\varepsilon(1+\varepsilon)}\, \beta^{1+\varepsilon} \leq \dfrac{\beta}{2}$, which reduces to

(3.9) $$\beta \;\leq\; \min\left\{\alpha,\; \left[\frac{\varepsilon(1+\varepsilon)}{2\,C}\right]^{\frac{1}{\varepsilon}}\right\}$$

then we choose u_o such that

(3.10) $$|u_0| \;\leq\; \min\left\{\alpha,\; \frac{\beta}{2}\right\}\;.$$

Let us now show that under condition (3.9), $\mathcal{T}$ is in fact a contraction on $X = X_\beta$. Indeed for any $(z, w) \in X$, we have

$$\begin{aligned}
\forall t \geq 0\,, \qquad |\mathcal{T} z(t) - \mathcal{T} w(t)| \;&\leq\; \int_0^t e^r \int_r^\infty C\, e^{-(1+\varepsilon)s}\, \beta^\varepsilon\, |z(s) - w(s)|\; ds\, dr \\
&\leq\; C\, \beta^\varepsilon\, \|z - w\|_\infty \int_0^t e^r \int_r^\infty e^{-(1+\varepsilon)s}\; ds\, dr \\
&\leq\; \frac{C\, \beta^\varepsilon}{1+\varepsilon}\, \|z - w\|_\infty \int_0^t e^{-\varepsilon r}\, dr \\
&\leq\; \frac{C}{\varepsilon\,(1+\varepsilon)}\; \beta^\varepsilon\, \|z - w\|_\infty\;.
\end{aligned}$$

Therefore if β satisfies (3.9), $\mathcal{T}$ is $\frac{1}{2}$-Lipschitz continuous on X_β. By Banach fixed point Theorem we obtain a solution u of (3.2)–(3.3) for any u_0 satisfying (3.10). In addition u is unique. ∎

Lemma 3.2. *The solutions u constructed in Lemma 3.1 are essentially different for positive and negative values of u_0. More precisely*

1) *If* $u_0 > \dfrac{C}{\varepsilon(1+\varepsilon)}\,\beta^{1+\varepsilon}$, *we have* $u > 0$ *and* $u' < 0$ *on* $[T,+\infty)$ *for some* $T \geq 0$.
2) *If* $u_0 < -\dfrac{C}{\varepsilon(1+\varepsilon)}\,\beta^{1+\varepsilon}$, *we have* $u < 0$ *and* $u' > 0$ *on* $[0,+\infty)$.

Proof: It follows easily from (3.5) that

$$\forall\, t \geq 0\,, \qquad |z(t) - u_0| \;\leq\; \int_0^t e^r \int_r^\infty C\, e^{-(1+\varepsilon)s}\, |z(s)|^{1+\varepsilon}\; ds\, dr \;\leq\; \frac{C}{\varepsilon(1+\varepsilon)}\,\beta^{1+\varepsilon}\;.$$

Since $z(t) = e^t\, u(t)$, the statement on the sign of u follows immediately. In case 2) it follows at once from (3.3) that $u' > 0$ on $[0,+\infty)$. In case 1) we cannot have $u' > 0$ on $[0,+\infty)$, since u tends necessarily to 0 at infinity as a consequence of Theorem 1.1. By Corollary 1.2 the existence of T is clear. ∎

4 – Properties of negative solutions

In this section we show that solutions of (1.1) which are negative on some interval $[T,+\infty)$ have very special properties.

Lemma 4.1. *Let u be a bounded solution of (1.1) on $[0,+\infty)$ such that $u < 0$ on $[T,+\infty)$ for some $T \geq 0$. Then $u < 0$ and $u' > 0$ on $[0,+\infty)$. Moreover we have*

$$\forall\, s \in [0,+\infty),\;\; \forall\, t \geq s\,, \quad e^{s-t}\, u(s) \leq u(t) < 0\;. \tag{4.1}$$

Hence

$$\forall\, s \in [0,+\infty),\;\; \forall\, t \geq s\,, \quad |u(t)| \leq e^{s-t}\, |u(s)|\;. \tag{4.2}$$

Proof: Since $u(t) < 0$ for t large, by Corollary 1.2 we have alternative 1). In addition, since $\dfrac{d}{dt}(e^t u(t)) = e^t(u(t) + u'(t)) \geq 0$, we obtain (4.1). Then (4.2) follows immediately. ∎

Proposition 4.2. *Assume that f satisfies (3.1) and let u, v be two bounded solutions of (1.1) on $[0,+\infty)$ such that $u, v<0$ on $[T,+\infty)$ for some $T \geq 0$ and $|v(0)| \leq |u(0)|$. Then there exists $\tau \geq 0$ such that*

$$\forall\, t \in [0,+\infty)\,, \qquad v(t)=u(t+\tau)\ . \tag{4.3}$$

Proof: By Lemma 4.1 we can take $T=0$. Moreover it is clear that for t large enough we have

$$\max\Bigl\{|u(t)|, |v(t)|\Bigr\} \;\leq\; \min\Bigl\{\alpha, \frac{\beta}{2}\Bigr\}\ . \tag{4.4}$$

Assume first that this is satisfied on $[0,+\infty)$. Then there is $\tau \geq 0$ such that $v(0)=u(\tau)$. By the uniqueness part in the proof of Lemma 3.1, it follows that (4.3) is satisfied. When (4.4) is satisfied only for $t \geq t_0$, it suffices to replace u and v by $u(t_0+\cdot)$ and $v(t_0+\cdot)$. Then (4.3) is obtained on $[t_0,+\infty)$ for some real number τ. By local uniqueness it extends on $[0,+\infty)$. The statement on the sign of τ is easy to recover from the condition $|v(0)| \leq |u(0)|$. ∎

Theorem 4.3. *Let u be a maximal solution of (1.1) in $(T^*,+\infty)$ such that $u<0$. Then assuming (1.7), we have $T^*>-\infty$ and*

$$\lim_{t\to T^*} u(t)=-\infty\,, \qquad \lim_{t\to T^*} u'(t)=+\infty\ . \tag{4.5}$$

Moreover if $\displaystyle\lim_{s\to+\infty} \frac{f(-s)}{s}=+\infty$, we have the additional property

$$\lim_{t\to T^*} \frac{u'(t)}{u(t)}=-\infty\ . \tag{4.6}$$

Proof: By translating t if necessary we may assume $T^*<0$. We set $v(t)=u(-t)$ for all $t \in [0,-T^*)$. Then

$$v''=v'-f(v)=-u'-f(v) \leq -f(v) \leq 0 \quad \text{ on } \ [0,-T^*)\ .$$

On multiplying by $2\,v'$ and integrating, since $v'=-u' \leq 0$ we obtain

$$v'^2(t) \;\geq\; v'^2(0)-2\,F(v(t))+2\,F(v(0))\ .$$

We introduce $w:=-v$. From the inequality $v'' \leq 0$ we deduce that v' is nonincreasing, hence w' is nondecreasing and in particular $w' \geq \gamma>0$ on $[0,-T^*)$. In particular if $T^* = -\infty$, we have $w(t) \to +\infty$ as $t \to +\infty$. In particular for t large enough, we have

$$-2\,F(v(t)) + 2\,F(v(0)) \;\geq\; -F(v(t)) \;.$$

Therefore

$$w' \geq \sqrt{-F(-w(t))}$$

for t large enough. By (1.7) we derive a contradiction, hence $T^* > -\infty$. It is then rather easy, by using the equation, to see that both w and w' tend to $+\infty$ as $t \to -T^*$. In addition, if $\lim\limits_{s\to+\infty} \dfrac{f(-s)}{s} = +\infty$, the above inequality, which is valid for t close to $-T^*$, provides (4.6). ∎

5 – Properties of general global solutions

In this section, under relevant conditions on f we show that all non trivial solutions of (1.1) which exist globally on some interval $[T, +\infty)$ have essentially the same backward behavior. We start with an a priori estimate.

Proposition 5.1. *Assume that f satisfies (1.2)–(1.3) and*

$$\exists\, c > 0,\ \exists\, \varepsilon > 0,\ \ \forall\, u > 0\,, \qquad f(u) \geq c\, u^{1+\varepsilon}\ , \tag{5.1}$$

and let u be any bounded solution of (1.1) on $[0, +\infty)$ such that $u > 0$ on $[T, +\infty)$ for some $T \geq 0$. Then we have

$$\forall\, t \in [0, +\infty)\,, \qquad u(t) \leq M := \left(\frac{1+\varepsilon}{c}\right)^{\frac{1}{\varepsilon}}\ . \tag{5.2}$$

Proof: Without loss of generality we may assume $T = 0$. Indeed if $u(t) \leq 0$, (5.2) is satisfied, and it is therefore sufficient to estimate $u(t)$ after it has become nonnegative for the first time. Setting $z(t) = e^t\, u(t)$, we have

$$\begin{aligned} z'(t) \;&=\; \frac{d}{dt}(e^t\, u(t)) \;=\; e^t\Big(u(t) + u'(t)\Big) \;=\; e^t \int_t^\infty f(e^{-s} z(s))\, ds \;\geq \\ &\geq\; c\, e^t \int_t^\infty e^{-(1+\varepsilon)s}\, z^{1+\varepsilon}(s)\; ds \;\geq\; c\, e^t\, z^{1+\varepsilon}(t) \int_t^\infty e^{-(1+\varepsilon)s}\, ds\ , \end{aligned}$$

$$z'(t) \;\geq\; \frac{c\, e^{-\varepsilon t}}{1+\varepsilon}\; z^{1+\varepsilon}\ .$$

As a consequence

$$\frac{d}{dt}(z^{-\varepsilon}) \;=\; -\varepsilon\, z' z^{-(1+\varepsilon)} \;\le\; -\frac{c\,\varepsilon}{1+\varepsilon}\; e^{-\varepsilon t}\ .$$

By integrating

$$\frac{1}{z^\varepsilon(t)} - \frac{1}{z^\varepsilon(0)} \;\le\; -\frac{c\,\varepsilon}{1+\varepsilon}\int_0^t e^{-\varepsilon s}\,ds \;=\; \frac{c}{1+\varepsilon}\,(e^{-\varepsilon t}-1)\ .$$

In particular

$$\frac{1}{z^\varepsilon(0)} \;\ge\; \frac{c}{1+\varepsilon}\,(1-e^{-\varepsilon t})$$

and by letting $t\to\infty$ we derive $\dfrac{1}{z^\varepsilon(0)} \ge \dfrac{c}{1+\varepsilon}$, which means

$$u(0) \;=\; z(0) \;\le\; \left(\frac{1+\varepsilon}{c}\right)^{\frac{1}{\varepsilon}}\ .$$

Since our time origin $T=0$ can be replaced by any positive value, this estimate is valid with 0 replaced by any $t>0$. This establishes (5.2). ∎

Theorem 5.2. *Let $u\not\equiv 0$ be any maximal solution of (1.1) on $(T^*,+\infty)$. Then assuming (1.7) and (5.1), we have $T^*>-\infty$ and u satisfies (4.5). Moreover*

$$\lim_{t\to T^*}\frac{u'(t)}{\sqrt{-2\,F(u(t))}} \;=\; 1\ . \tag{5.3}$$

In particular if $\displaystyle\lim_{s\to+\infty}\frac{f(-s)}{s}=+\infty$, we have the additional property (4.6).

Proof: We proceed in 4 steps.

Step 1) It is impossible to have

$$u'(t)<0\,,\qquad \forall\, t\in(T^*,+\infty)\ . \tag{5.4}$$

Indeed in such a case we have

$$0\le u(t)\le M\,,\qquad \forall\, t\in(T^*,+\infty)\ , \tag{5.5}$$

and in particular u is bounded. If $T^*>-\infty$ we must have $u'(t)\to-\infty$ as $t\to T^*$. Selecting a fixed number $\tau>-T^*$ we have

$$u(t)-u(\tau) \;=\; -u'(t)+u'(\tau)+\int_t^\tau f(u(s))\,ds\ ,$$

therefore $u(t) \geq -u'(t)-C$ for some constant $C \geq 0$ and consequently $u(t) \to +\infty$ as $t \to T^*$. This contradicts (5.5). Thus we must have $T^* = -\infty$ and since $u(t) \to c > 0$ as $t \to -\infty$, we have $f(u(t)) \geq \frac{1}{2} f(c) > 0$ as $t \to -\infty$. Then we have for instance

$$u(t) \;=\; u(0) - u'(t) + u'(0) + \int_t^0 f(u(s))\, ds \;\geq\; u'(0) + \int_t^0 f(u(s))\, ds \;\longrightarrow\; +\infty$$

as $t \to -\infty$. This again contradicts (5.5).

Step 2) By step 1 there is $t_0 > T^*$ such that $u'(t_0) = 0$ and $u' > 0$ on (T^*, t_0). Since

$$u'' = -u' - f(u) < 0 \quad \text{on} \;\; (T^*, t_0)$$

we have, selecting any $\tau \in (T^*, t_0)$:

$$u' \geq \eta > 0 \quad \text{on} \;\; (T^*, \tau] \;.$$

In particular we cannot have $u \geq 0$ on (T^*, t_0): if $T^* = -\infty$ the previous inequality implies $u(t) \to -\infty$ as $t \to -\infty$; If $T^* > -\infty$, the classical alternative on blowing up implies $u'(t) \to +\infty$ as $t \to T^*$, and then $u(t) \leq -u'(t) + C$ tends to $-\infty$ as $t \to T^*$, a contradiction.

Step 3) By step 1 there is $t_1 \in (T^*, t_0)$ such that $u(t_1) < 0$, and of course $u'(t_1) > 0$. Then, considering the solution $v(t) = u(t_1 - t)$ of the backward equation, the proof of $T^* > -\infty$ and (4.5) becomes identical to the proof of Theorem 4.3.

Step 4) As in the proof of Theorem 4.3, by translating t if necessary we may assume $T^* < 0$ and $u' > 0$ on $(T^*, 0]$. We set $w(t) = -u(-t)$ for all $t \in [0, -T^*)$. Then

$$w'' = w' + f(-w) \quad \text{on} \;\; [0, -T^*) \;.$$

On multipying by $2\,w'$ and integrating, we obtain

$$w'^2(t) + 2\,F(-w(t)) \;=\; w'^2(s) + 2\,F(-w(s)) + \; 2\int_s^t w'^2(r)\, dr \;. \tag{5.6}$$

Therefore choosing first $s = T \in [0, -T^*)$ fixed we obtain for a certain constant $C \geq 0$

$$w' \geq \sqrt{-2\,F(-w(t)) - C}$$

or in other terms

$$u'(t) \geq \sqrt{-2\,F(u(t)) - C} \tag{5.7}$$

valid for t close to T^*. On the other hand w'^2 is nondecreasing on $[0, -T^*)$. In particular (5.6) implies

$$w'^2(t) + 2\,F(-w(t)) \;\le\; w'^2(s) + 2\,F(-w(s)) + (-T^* - s)\,w'^2(t) \;.$$

Choosing now $s = -T^* - \eta$ we deduce

$$(1-\eta)\,w'^2(t) \;\le\; -2\,F(-w(t)) + w'^2(-T^* - \eta) + 2\,F(-w(-T^* - \eta)) \;,$$

hence

$$(1-\eta)\,w'^2(t) \;\le\; -2\,F(-w(t)) + D(\eta)\,, \qquad \forall\, t \in [-T^* - \eta,\, -T^*) \;,$$

or in other terms

$$(5.8) \qquad (1-\eta)\,u'^2(t) \;\le\; -2\,F(u(t)) + D(\eta)\,, \qquad \forall\, t \in (T^*,\, T^* + \eta] \;.$$

It is perfectly clear that (5.7) and (5.8) imply (5.3). ∎

6 – Proof of the main result

Before we go to the proof of the main result, we need two lemmata. The first lemma concerns the shape in the phase plane of the trajectories associated to a maximal global solution of (1.1) on $(0, +\infty)$.

Lemma 6.1. *Let u be a maximal global solution of (1.1) on $(0, +\infty)$. Then any straight line*

$$\Delta_a = \Big\{(u, v) \in \mathbb{R}^2,\; u + v = a\Big\}$$

with $a > 0$ intersects the curve $\Gamma(u) = \bigcup_{t>0} \{(u(t), u'(t)\}$ at exactly one point.

Proof: This is an immediate consequence of the fact that $u(t) + u'(t)$ is decreasing, since $u(t)$ vanishes at most once and $[u(t) + u'(t)]' = -f(u(t)) < 0$ whenever $u(t) \ne 0$. ∎

Lemma 6.2. *Assuming (1.2), (1.3) and (5.1), let $M_0 := \operatorname{Sup}\{C \ge 0,\ \exists\, u$ solution of (1.1) on $[0, +\infty)$ such that $u(0) = C$ and $u'(0) = 0\}$. The unique local solution u_+ of (1.1) such that $u_+(0) = M_0$ and $u'_+(0) = 0$ is global on $[0, +\infty)$ and we have*

$$\forall\, t \in [0, +\infty)\,, \qquad 0 \le u_+(t) \le M_0 \;.$$

Proof: Let $C_n \in [0, M_0)$ be a sequence tending to M_0 as $n \to \infty$ and let u_n be the solution of (1.1) on $[0, +\infty)$ such that $u_n(0) = C_n$ and $u_n'(0) = 0$. It is clear that

$$\forall t \in [0, +\infty), \qquad 0 \le u_n(t) \le M_0 \ .$$

Then by the equation

$$(e^t u_n')' = e^t(u_n'' + u_n') = -e^t f(u_n)$$

is uniformly locally bounded on $[0, +\infty)$. Since $u_n'(0) = 0$ it follows that u_n', and then also u_n'' is uniformly locally bounded on $[0, +\infty)$. Then the sequence $\{u_n\}$ has at least a convergent subsequence in $C^1([0, +\infty))$. This is enough to conclude the proof since by the equation the convergence actually takes place in $C^2([0, +\infty))$ and the limiting function u_+ satisfies all the conditions. ∎

Proof of Theorem 1.4: The curve $\mathcal{C}_- = \Gamma(u_-)$ cuts the halfplane

$$\Pi = \Big\{(u, v) \in \mathbb{R}^2, \ u + v > 0\Big\}$$

into two connected open regions Ω_-^- and Ω_-^+, the region Ω_-^+ containing the first quadrant $(\mathbb{R}^+)^2$. If a solution u of (1.1) starts in Ω_-^-, it cannot become ≥ 0 unless it meets $\mathcal{C}_-$ in finite time, by uniqueness this would mean that u is a time-translate of u_-, which is impossible because then the initial state would lie on $\mathcal{C}_-$. Similarly, the curve $\mathcal{C}_+ = \Gamma(u_+)$ cuts Π into two connected open regions Ω_+^- and Ω_+^+, the region Ω_+^- containing the halfline $\{(-s, s),\ s > 0\}$. If a solution u of (1.1) starts in Ω_+^+, in order to be global for $t \ge 0$ it has to take some positive values. Actually the corresponding maximal trajectory, being distinct from the trajectory of u_-, must take is maximum on the axis $\{v = 0\}$ and therefore, since the maximum is less than the maximum of u_+, it has to cross $\mathcal{C}_+$ in finite time; by uniqueness this would mean that u is a time-translate of u_+, which is impossible because then the initial state would lie on $\mathcal{C}_+$. Therefore we have the following inclusion

$$u \text{ global solution of (1.1) on } (0, +\infty) \implies \Gamma(u) \subset \overline{\Omega_+^-} \cap \overline{\Omega_-^+} =: \mathcal{D} \ .$$

Conversely, if a solution of u (1.1) starts in $\mathcal{D}$ at $t = 0$, either it is a time-translate of u_+ or u_-, or $(u(0), u'(0)) \in \Omega_+^- \cap \Omega_-^+$. In the second case we notice that $u(t) + u'(t)$ is non increasing and on the other hand for any constant C the region

$$\Big\{(u, v) \in \mathcal{D}, \ u + v \le C\Big\}$$

is clearly bounded. The solution, being trapped in a bounded region, is global for $t \ge 0$. The proof of Theorem 1.4 is completed. ∎

7 – Additional properties of global solutions

Proposition 7.1. *Let $f \in C^1(\mathbb{R})$ be such that $f(0) = 0$ and assume (3.1). Then for all u_0 with $|u_0|$ small enough, the unique solution $u \in C^2(\mathbb{R}^+)$ of (3.2) and (3.3) is such that*

$$\lim_{t\to\infty} \frac{u'(t)}{u(t)} = -1 \; . \tag{7.1}$$

Proof: As a consequence of (3.1) we have the estimate

$$\forall\, s \in \mathbb{R}^+\,, \qquad |f(u(s))| \le C\,|u(s)|^{1+\varepsilon} \le C_1\, e^{-\gamma s}\, |u(s)|^{1+\gamma} \; . \tag{7.2}$$

with $\gamma := \frac{\varepsilon}{2} > 0$. In particular we have

$$\begin{aligned}
\forall\, t \ge 0\,, \quad 0 \,<\, u'(t) + u(t) \,=\, \int_t^\infty f(u(s))\, ds \;&\le\; C_1 \int_t^\infty e^{-\gamma s}\, |u(s)|^{1+\gamma}\, ds \\
&\le\; C_1\, |u(t)|^{1+\gamma} \int_t^\infty e^{-\gamma s}\, ds \\
&\le\; C_2\, |u(t)|^{1+\gamma} \quad \text{with } \; C_2 := \frac{C_1}{\gamma} \; .
\end{aligned}$$

By Lemma 3.2 we know that u does not change sign for t large. If $u > 0$, dividing through by $u(t)$, we obtain

$$\forall\, t \ge 0\,, \qquad 0 < \frac{u'(t)}{u(t)} + 1 \;\le\; C_2\, |u(t)|^\gamma \; . \tag{7.3}$$

Then (7.1) follows by letting $t \to +\infty$ in (7.3). On the other hand if $u < 0$, dividing through by $u(t)$, we obtain

$$\forall\, t \ge 0\,, \qquad 0 > \frac{u'(t)}{u(t)} + 1 \;\ge\; -C_2\, |u(t)|^\gamma \; . \tag{7.3$'$}$$

Then (7.1) follows by letting $t \to +\infty$ in (7.3'). ∎

Remark 7.2. It follows in particular from Lemma 7.1 and the results of section 3 that the negative global solutions on $[T, +\infty)$ satisfy (7.1). The question naturally arises whether (7.1) is also true for the global solutions which are positive for t large. As we shall see next this is only valid for those solutions which are time-translates of the "largest" solution u_+. □

Theorem 7.3. *Let $f \in C^1(\mathbb{R})$ be such that $f(0) = 0$ and assume (3.1). Then actually the restriction of u_+ on some interval $[T, \infty)$ is one of the positive solutions given by Lemmata 3.1–3.2 and we have*

$$\lim_{t\to\infty} \frac{u_-'(t)}{u_-(t)} = \lim_{t\to\infty} \frac{u_+'(t)}{u_+(t)} = -1 \ . \tag{7.4}$$

On the other hand, under the conditions of Theorem 1.3 if u is any solution of (1.1) on $[T, +\infty)$ which is not a time translate of either u_+ or u_-, then we have

$$\lim_{t\to\infty} \frac{u'(t)}{u(t)} = 0 \ . \tag{7.5}$$

Proof: For $u < 0$ on $[T, +\infty)$ this is just Proposition 7.1. Otherwise we proceed in 3 steps.

Step 1) For any global solution u of (1.1) on $[T, +\infty)$ we have

$$\frac{u'(t)}{u(t)} > -1 \,, \qquad \forall\, t \in [T, +\infty) \ , \tag{7.6}$$

and either

$$\lim_{t\to+\infty} \frac{u'(t)}{u(t)} = -1 \tag{7.7}$$

or

$$\lim_{t\to+\infty} \frac{u'(t)}{u(t)} = 0 \ . \tag{7.8}$$

Indeed (7.6) follows from Theorem 1.1, ii). In addition, setting

$$\forall\, t \in [T, +\infty) \,, \qquad p(t) := \frac{u'(t)}{u(t)} \ ,$$

we have

$$\forall\, t \in [T, +\infty) \,, \qquad p'(t) \ = \ \frac{u''(t)}{u(t)} - \frac{u'^2(t)}{u^2(t)} \ = \ -p(t) - p^2(t) - \frac{f(u(t))}{u(t)} \ .$$

By our hypothesis on f we have

$$\lim_{t\to+\infty} \frac{f(u(t))}{u(t)} = 0 \ .$$

Given $\varepsilon \in (0,1)$, we may assume

$$\forall\, t \in [T(\varepsilon), +\infty)\,, \qquad \frac{f(u(t))}{u(t)} < \varepsilon - \varepsilon^2\ . \tag{7.9}$$

Now if (7.7) is not satisfied, there is some $\varepsilon \in (0,1)$ and a sequence $t_n \to \infty$ such that $p(t_n) > -\varepsilon$, in particular

$$\exists\, \tau \in [T(\varepsilon), +\infty)\,, \qquad p(\tau) > -\varepsilon\ . \tag{7.10}$$

We show that

$$\forall\, t \in [\tau, +\infty)\,, \qquad p(t) > -\varepsilon\ . \tag{7.11}$$

Indeed (7.11) is true for t sufficiently close to τ. Assume, by contradiction, that

$$T := \operatorname{Sup}\Big\{t \in [\tau, +\infty),\ \forall\, s \in [\tau, t],\ p(s) > -\varepsilon\Big\} < \infty\ . \tag{7.12}$$

Then of course $p(T) = -\varepsilon$ and as a consequence of (7.9) we derive $p'(\tau) > 0$. This contradiction establishes (7.11). Next by choosing for $\varepsilon \in (0,1)$ any number such that $-\varepsilon < \limsup\limits_{t\to+\infty} p(t)$ we find by the above argument

$$\forall\, t \in [\tau(\varepsilon), +\infty)\,, \qquad p(t) > -\varepsilon\ . \tag{7.13}$$

In particular there exists

$$\lim_{t\to+\infty} \frac{u'(t)}{u(t)} = l > -1\ .$$

Since

$$\forall\, t \in [T, +\infty)\,, \qquad p'(t) = -p(t) - p^2(t) - \frac{f(u(t)}{u(t)}\ ,$$

we have

$$l + l^2 = -\lim_{t\to+\infty} p'(t) = 0\ .$$

We conclude that $l = 0$ and step 1 is achieved.

Step 2) Let u be any solution given by Lemma 3.1 with $u > 0$ for t large. By Theorem 5.2, $u > 0$ can be continued backwards and in particular we can, by a suitable time translation if necessary, assume that u is defined on $[0, +\infty)$ and

$$0 < u(0) = \max_{t\ge 0} u(t) \le u_+(0) = M_0\ .$$

By Lemma 7.1 we know that u satisfies (7.1). On the other hand if for any $t \geq 0$ we define $\theta(t)$ by

$$u(t) = u_+(\theta(t)) \ ,$$

clearly since $u'_+ < 0$ on $(0, +\infty)$, $\theta(t)$ is unique and $\theta(t) \to +\infty$ as $t \to +\infty$. Now since

$$u'_+(\theta(0)) \leq 0 = u'(0) \ ,$$

it follows that either u is a time-translate of u_+, or

$$\forall\, t \in [0, +\infty) \,, \qquad u'_+(\theta(t)) \leq u'(t) \ .$$

As a consequence

$$\forall\, t \in [0, +\infty) \,, \qquad \frac{u'_+(\theta(t))}{u_+(\theta(t))} \leq \frac{u'(t)}{u(t)} \ ,$$

and consequently u_+ satisfies (7.1). Since By Lemma 7.1 we already know that u_- satisfies (7.1), property (7.4) follows.

Step 3) In order to establish (7.6), first we notice that if u is not a time-translate of u_-, then $u > 0$ for t large and by a time translation we may assume $u > 0$ for $t \geq 0$. Now by the same method as in the proof of Proposition 7.1 we find that if u satisfies (7.1), first $u(t) \leq K e^{-\frac{t}{2}}$ since $u'(t) + \frac{1}{2}\, u(t) < 0$ for t large, and then

$$0 \,<\, u'(t) + u(t) \,=\, \int_t^\infty f(u(s))\, ds \,\leq\, C\, |u(t)|^{1+\gamma} \quad \text{with} \ \ \gamma := \frac{\varepsilon}{2} \ . \tag{7.14}$$

Letting $z(t) := e^t\, u(t)$ we find successively

$$z'(t) \,\leq\, C\, e^{-\gamma t}\, z(t)^{1+\gamma} \ ,$$

$$[z^{-\gamma}]'(t) \,=\, -\gamma\, z(t)^{-(1+\gamma)} \,\geq\, -C\, \gamma\, e^{-\gamma t} \ ,$$

$$[z^{-\gamma}](t) - [z^{-\gamma}](0) \,\geq\, -C \int_0^t \gamma\, e^{-\gamma s}\, ds \,\geq\, -C \int_0^\infty \gamma\, e^{-\gamma s}\, ds \,=\, -C$$

and finally

$$e^t u(t) \,=\, z(t) \,\leq\, \left(\frac{1}{[z^{-\gamma}](0) - C} \right)^{\frac{1}{\gamma}}$$

assuming $[z^{-\gamma}](0) - C > 0$. Because the inequality (7.14) is preserved under time-translation for a fixed C and we have $u(t) \to 0$ as $t \to +\infty$, the above condition, reducing to $u(0) < C$, can be achieved by some additional change of the time origin. Finally we obtain

$$\forall\, t \in [T, +\infty) \,, \qquad u(t) \leq K e^{-t} \ ,$$

for some positive constants T, K. By considering large values of t and a final time-translation we conclude that u is the time-translate of one of the small positive solutions of Lemmas 3.1–3.2. In particular this argument applies to u_+ as a consequence of step 2. Therefore any solution of (1.1) on a halfline $[T,+\infty)$ which satisfies (7.1) and is not a time-translate of u_- has to be a time-translate of u_+. By step 1 the proof of Theorem 7.3 is now complete. ∎

8 – Global solutions are exceptional

In this section, under some conditions on f we show that global solutions of (1.1) on $[0,+\infty)$ are in fact exceptional. This will come from the structure of the set $\mathcal{D}$ and the fact that the geometrical supports of the extreme solutions become very close to each other near blowing-up. More precisely we have

Theorem 8.1. *Assume that f satisfies (1.2), (1.3) and (1.5) and*

$$\exists\, C>0,\ \ \exists\, \varepsilon>0\,, \qquad \lim_{s\to-\infty}\left(\frac{f'(s)}{|s|^\varepsilon}\right) = -C\ . \tag{8.1}$$

Then we have

$$\operatorname{meas}(\mathcal{D}) < \infty\ . \tag{8.2}$$

Proof: We proceed in 3 steps.

Step 1) First by introducing $c := \dfrac{C}{1+\varepsilon}$, as $s\to-\infty$ we have

$$f(s) \sim c\,|s|^{1+\varepsilon}\ ; \qquad F(s) \sim -\frac{c}{2+\varepsilon}\,|s|^{2+\varepsilon}\ ; \qquad f'(s) \sim -c\,(1+\varepsilon)\,|s|^\varepsilon\ .$$

For any global solution u of (1.1) on $[0,+\infty)$ we have as t tends to the blow-up time $T^*(u)$

$$v(t) \;=\; u'(t) \;\sim\; \sqrt{-2\,F(u(t)}\;\sim\; \left(\frac{2\,c}{2+\varepsilon}\right)^{\frac{1}{2}} |u(t)|^{1+\frac{\varepsilon}{2}}\ . \tag{8.3}$$

It follows in particular from (8.3) that

$$\lim_{t\to T^*}\frac{f(u(t))}{v(t)} \;=\; +\infty\ . \tag{8.4}$$

Step 2) Let u be any solution of (1.1); for any $v>0$ let us call $u(v)$ the value $u(t)$ where t is the only value in $[0,+\infty)$ such that $u'(t)=v$. We have

$$\frac{du}{dv}=\frac{u'}{u''}=\frac{v}{-v-f(u)}=-1+\frac{f(u)}{v+f(u)}\ .$$

In particular if $u, \hat{u}$ are two different solutions we have

$$\forall\, v>0\,,\quad \frac{du}{dv}-\frac{d\hat{u}}{dv}=\frac{f(u)}{v+f(u)}-\frac{f(\hat{u})}{v+f(\hat{u})}=\frac{f(u)-f(\hat{u})}{v\Big(1+\frac{f(u)}{v}\Big)\Big(1+\frac{f(\hat{u})}{v}\Big)}\ . \tag{8.5}$$

As a consequence of (8.4) and (8.6) we now find

$$\text{As}\ \ v\to+\infty\,,\quad \frac{du}{dv}-\frac{d\hat{u}}{dv}\sim\frac{v(f(u)-f(\hat{u}))}{f(u)f(\hat{u})}\ . \tag{8.6}$$

Finally by (8.3) it is clear that $u(v)\sim\hat{u}(v)$ and then by the property (8.1) of f'

$$\text{As}\ \ v\to+\infty\,,\quad \frac{du}{dv}-\frac{d\hat{u}}{dv}\sim\frac{v\,f'(u)\,(u-\hat{u})}{f^2(u)}\ . \tag{8.7}$$

Step 3) We introduce

$$G(v)\ =\ u(v)-\hat{u}(v)\ . \tag{8.8}$$

By (8.7) we have

$$\text{As}\ \ v\to+\infty\,,\quad v\,G'(v)\sim\frac{v^2 f'(u)}{f^2(u)}\,G(v)\ . \tag{8.9}$$

On the other hand as $v\to+\infty$,

$$v^2\sim\frac{2\,c}{2+\varepsilon}\,|u|^{2+\varepsilon}\,;\qquad f'(u)\sim-c\,(1+\varepsilon)\,|u|^{\varepsilon}\,;\qquad f^2(u)\sim c^2\,|u|^{2+2\varepsilon}\ . \tag{8.10}$$

Combining (8.9) and (8.10) we obtain

$$\text{As}\ \ v\to+\infty\,,\quad v\,G'(v)\sim-\frac{2+2\,\varepsilon}{2+\varepsilon}\,G(v)\ . \tag{8.11}$$

Selecting $u=u_+,\ \hat{u}=u_-$ we have

$$\forall\, v>0\,,\quad G(v)=u_+(v)-u_-(v)>0\ .$$

By (8.11) it is immediate to check that

$$\forall\,\alpha\in\left(0,\frac{\varepsilon}{2+\varepsilon}\right),\ \ \exists\, M(\alpha),\ \ \forall\, v\geq1\,,\qquad G(v)\leq M(\alpha)\,v^{-(1+\alpha)}\ . \tag{8.12}$$

It is clear that (8.12) implies (8.2). The proof of Theorem 8.1 is now complete. ∎

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