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TOPOLOGICAL PROPERTIES OF SOLUTION SETS FOR FUNCTIONAL DIFFERENTIAL INCLUSIONS GOVERNED BY A FAMILY OF OPERATORS

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Abstract: Let $r > 0$ be a finite delay and $C([-r,t],E)$ be the Banach space of continuous functions from $[-r,0]$ to the Banach space E. In this paper we prove an existence theorem for functional differential inclusions of the form: $\dot{u}(t) \in A(t)\,u(t) + F(t,\tau(t)u)$ a.e. on $[0,T]$ and $u=\psi$ on $[-r,0]$, where $\{A(t)\colon\, t \in [0,T]\}$ is a family of linear operators generating a continuous evolution operator $K(t,s)$, F is a multifunction such that $F(t,\cdot)$ is weakly sequentially hemi-continuous and $\tau(t)\,u(s) = u(t+s)$, for all $t \in [0,T]$ and all $s \in [-r,0]$. Also, we are concerned with the topological properties of solution sets.

1 – Introduction

The existence of solutions for functional differential inclusions (FDI) and the topological properties of solution sets are studied extensively (see, for example, [1], [2], [9], [10], [11], [12], [13]). However, not much study has been done for functional differential inclusions governed by operators. Mainly, recently, Castaing–Marques [3] considered a functional differential inclusions governed by sweeping process while Castaing–Faik–Salvadori [5] considered a functional differential inclusion governed by m-accretive operators which are independent of the time. That is, they proved the existence of integral solutions for the following FDI:

$$\begin{cases} \dot{u}(t) \in A(u(t)) + F(t,\tau(t)u), & \text{a.e. on } [0,T], \\ u = \psi & \text{on } [-r,0]\ , \end{cases}$$

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where $r>0$ is a finite delay, A is m-accretive operator on a separable Banach space E, F is a multifunction, ψ is a continuous function from $[-r,0]$ to E and for each $t\in[0,T]$ $\tau(t)\,u$ is a continuous on $[-r,0]$ such that for each $s\in[-r,0]$, $(\tau(t)\,u)(s)=u(t+s)$.

The purpose of this paper is to obtain conditions on the data that guaranteed the existence of integral solutions and to characterize topological properties of solution sets for a functional differential inclusion (differential inclusion with delay) of the form:

$$(P)\quad \begin{cases} \dot u(t)\in A(u(t))+F(t,\tau(t)\,u), & \text{a.e. on } [0,T],\\ u=\psi & \text{on } [-r,0]\ , \end{cases}$$

where $\{A(t):\, t\in[0,T]\}$ is a family of densely defined, closed, linear operators on a separable Banach space E. Also, we obtain a continuous dependence result that examines the change in the solution set as we vary the initial function.

Our results generalize many previous theorems. In the important case $A(t)=0$, $\forall\, t\in I$, we have that $K(t,s)=Id$ and an integral solution, in fact, a strong solution. Then, as special case, we obtain a generalization of the results of Deimling [7], Kisielewicz [14] and Papageorgiou [16], [17]. In addition, if $A(t)\neq 0$ then many results of this kind are generalized too. For example, Cichon [6], Frankwska [8] and Papageorgiou [18] considered the problem (P) without delay. Moreover, Castaing, Faik and Salvadori [5] investigated the problem (P) in the case when A is an m-accretive multivalued operator and dependent of t. Finally Castaing and Ibrahim [2] considered the problem (P) when $A(t)=0$, $\forall\, t\in I$.

2 – Definitions, notations and preliminaries

We will use the following definitions and notations.

- E is a separable Banach space, E' the topological dual of E and E_w is the vector space E equipped with the $\sigma(E,E')$ topology.
- $c(E)$ (resp. $ck(E)$) is the family of nonempty convex closed (resp. nonempty convex compact) subsets of E.
- If Z is a subset of E, $\delta^*(\cdot,Z)$ is the support function of Z and $|Z|=\{\|z\|:\, z\in Z\}$.
- $r>0$, $T>0$ and $I=[0,T]$.
- $L^1(I,E)$ is the Banach space of Lebesque–Bochner integrable functions $f\colon I\to E$ endowed with the usual norm and $\mathcal{L}(E)$ is the Banach space of all linear continuous operators on E.

- $C(I,E)$ is the Banach space of continuous functions $f : I \to E$ with the norm of uniform convergence, $C_0 = C([-r,0],E)$, $\psi \in C_0$.
- For any $t > 0$ we denote by $\tau(t)$ the mapping from $C([-r,T],E)$ to $C_0 = C([-r,0],E)$ defined by $\tau(t)\,u(s) = u(s+t)$, $\forall\, s \in [-r,0]$, $\forall\, u \in C([-r,T],E)$.
- A multifunction $G : E \to 2^E - \{\emptyset\}$ with closed values is upper semicontinuous (u.s.c) if and only if $G^-(Z) = \{x \in E : G(x) \cap Z \neq \emptyset\}$ is closed whenever $Z \subset E$ is closed. Taking on E its weak topology, $\sigma(E,E')$, we obtain in a similar way a notion of $w-w$ upper semicontinuous ($w-w$ u.s.c) that is, upper semicontinuous from E_w to E_w. If the set $G^-(Z)$ is weakly sequentially closed whenever Z is weakly closed, we shall say that G is $w-w$ sequentially u.s.c.
- A multifunction $G\colon E \to 2^E - \{\emptyset\}$ with closed values is called upper hemicontinuous (u.h.c) [weakly upper hemicontinuous, w-u.h.c] if and only if for each $x^* \in E'$ and for each $\lambda \in \mathbb{R}$ the set $\{x \in E\colon \delta^*(x^*, G(x)) < \lambda\}$ is open in E (in E_w).
- A multifunction $G\colon E \to 2^E - \{\emptyset\}$ with closed values is called weakly sequentially upper hemicontinuous (w-seq uhc) if and only if for each $x^* \in E'$, $\delta^*(x^*, G(\cdot))\colon E \to \mathbb{R}$ is sequentially upper semicontinuous from E_w to $\mathbb{R}$, see ([6], [14]).
  If $G\colon I \to 2^E - \{\emptyset\}$ is measurable and integrably bounded with weakly compact values, then, the set of all integrable selections of G, S^1_G, is weakly compact in $L^1(I,E)$, see [4].
- μ is either the Kuratowski or the Hausdorff measure of noncompactness on E.

Let $\{A(t)\colon t \in I = [0,T]\}$ be a family of densely defined, closed, linear operators on E. Suppose that for every $s \in I$ and every $x \in E$ the initial value problem problem

$$(*) \qquad \begin{cases} \dot u(t) \in A(t)u(t)\,, & t \in [s,T] \\ u(s) = x \end{cases}$$

has a unique strong solution. Then an operator $K(\cdot,\cdot)$ can be defined from $\Delta = \{(t,s)\colon 0 \le s \le t \le T\}$ to E by $K(t,s)\,x = u(t)$ where u is the unique solution of $(*)$. The operator $K(\cdot,\cdot)$ is called a fundamental solution of $(*)$ or we say the family $\{A(t) : t \in I\}$ is a generator of a fundamental solutions $K(\cdot,\cdot)$ (see [19]). A continuous function $u\colon [-r,T] \to E$ is called an integral solution of the problem (P) if $u=\psi$ on $[-r,0]$ and for every $t\in I$,

$$u(t) \;=\; K(t,0)\,\psi(0)+\int_0^t K(t,s)\,f(s)\;ds\;,$$

where $f\in L^1(I,E)$ and $f(s)\in F(s,\tau(s)\,u)$ a.e..

The following lemmas will be crucial in the proof of our results.

Lemma 2.1 (Lemma 1, [6]). *Let Y be a Banach space. Assume:*

(1) *$G\colon E\to c(Y)$ be w-seq uhc;*

(2) *$\|G(x)\|\le a(t)$ a.e. on I, for every $x\in E$, where $a\in L^1(I,\mathbb{R})$;*

(3) *$x_n\in C(I,E)$, $x_n(t)\to x_0(t)$ (weakly) a.e. on I;*

(4) *$y_n\to y_0$ (weakly), $y_n,y_0\in L^1(I,E)$;*

(5) *$y_n(t)\in G(x_n(t))$ a.e. on I.*

Thus $y_0(t)\in G(x_0(t))$ a.e. on I.

Lemma 2.2 (Theorem 1, [6]). *Let $\{A(t)\colon t\in I\}$ be a family of densely defined, closed, linear operators on E and is a generators of a fundamental solution $K(\cdot,\cdot)\colon \Delta=\{(t,s)\colon 0\le s\le t\le T\}\to\mathcal{L}(E)$ such that*

$(\mathbf{A}_1)$ *$K(s,s)=Id$, $s\in I$ and $K(r,s)\,K(s,t)=K(r,t)$, $r<s<t$;*

$(\mathbf{A}_2)$ *$K\colon\Delta\to\mathcal{L}(E)$ is strongly continuous;*

$(\mathbf{A}_3)$ *$\|K(t,s)\|\le M$, $\forall\,(t,s)\in\Delta$;*

$(\mathbf{A}_4)$ *$K(\cdot,s)\colon I\to\mathcal{L}(E)$ is uniformly continuous.*

Let $S\colon I{\times}E\to c(E)$ such that

$(\mathbf{S}_1)$ *For each $x\in E$, $S(\cdot,x)$ has a measurable selection;*

$(\mathbf{S}_2)$ *For each $t\in I$, $S(t,\cdot)$ is w-seq. u.h.c.;*

$(\mathbf{S}_3)$ *There exists $a\in L^1(I,\mathbb{R})$ such that for each $x\in E$,*

$$\|S(t,x)\|\;\le\;a(t)\,\Big(1+\|x\|\Big)\qquad\text{a.e.}\,;$$

$(\mathbf{S}_4)$ *For each bounded $B\subset E$*

$$\lim_{\delta\to0}\mu\Big(S(I_{t,\delta}\times B)\Big)\;\le\;w(t,\mu(B))\qquad\text{a.e. on }\;I$$

where $I_{t,\delta} = [t-\delta, t] \cap I$ *and* w *is a Kamke function. Then for each* $x_0 \in E$ *there exists at least an integral solution for the problem:*

$$\begin{cases} \dot{u}(t) \in A(t)\, u(t) + S(t, u(t)), & \text{a.e. on } I \\ u(0) = x_0 \ . \end{cases}$$

Moreover, for each $x_0 \in E$ *the set* $S(x_0)$ *of all integral solutions is compact.*

3 – Existence theorem for (P)

In this section we give an existence theorem for (P).

Theorem 3.1. *Let* $\{A(t)\colon t \in I\}$ *be a family of densely defined, closed, linear operators on* E *and is a generator of a fundamental solution* $K(\cdot,\cdot)$ *satisfying conditions* (A_1)*–*(A_4). *Let* $F\colon I \times C([-r,0], E) \to c(E)$ *be a multifunction such that*

($\mathbf{F}_1$) *For each* $g \in C([-r,0], E)$, $F(\cdot, g)$ *has a measurable selection;*

($\mathbf{F}_2$) *For each* $t \in I$, $F(t,\cdot)$ *is w-seq. uhc;*

($\mathbf{F}_3$) *There exists* $a \in L^1(I, \mathbb{R})$ *such that for every* $g \in C([-r,0], E)$,

$$\|F(t,g)\| \le a(t)\Big(1 + \|g(0)\|\Big) \qquad \text{a.e.}\,;$$

($\mathbf{F}_4$) *There exists* $\gamma \in L^1(I, \mathbb{R}^+)$ *such that for each bounded subset* Z *of* $C([-r,0], E)$,

$$\mu(F(t \times Z)) \le \gamma(t)\, \mu(Z(0)), \qquad \text{a.e.}.$$

Then for each $\psi \in C([-r,0], E)$ *the problem* (P) *has an integral solution.*

Proof: We construct, by induction, a sequence (u_n) in $C([-r,T], E)$ such that it has a subsequence converges uniformly to a function $u \in C([-r,T], E)$ which is an integral solution of (P). For notional convenience we assume without any loss of generality that $T = 1$.

Step 1. Let $n \ge 1$. Set $u_n = \psi$ on $[-r,0]$. Consider the partition of I by the points $t^n_m = \frac{m}{n}$, $m = 0, 1, 2, ..., n$. We define a step function $\theta_n\colon I \to I$

by $\theta_n(0) = 0$, $\theta_n(t) = t^n_{m+1}$ for $t \in (t^n_m, t^n_{m+1}]$. Now we construct two functions $u_n \in C([-r,T],E)$ and $g_n \in L^1(I,E)$ such that for all $t \in [0,T]$,

$$u_n(t) \;=\; K(t,0)\,\psi(0) + \int_0^t K(t,s)\, g_n(s)\; ds \tag{1}$$

$$g_n(t) \;\in\; F\Big(t,\, \tau(\theta_n(t))\, f_{n\theta_n(t)-1}(\cdot, u_n(t))\Big) \quad \text{a.e. on} \;\; I \;, \tag{2}$$

where for every $m = \{0,1,2,...,n-1\}$, $f_m : [-r, t^n_{m+1}] \times E \to E$, defined by

$$f_m(t,x) \;=\; \begin{cases} u_n(t) & \text{if } \; t \in [-r, t^n_m] \\ u_n(t^n_m) + n\,(t - t^n_m)\,(x - u_n(\frac{m}{n})) & \text{if } \; t \in [t^n_m, t^n_{m+1}] \;. \end{cases}$$

Let $f_0\colon\; [-r, t^n_1] \times E \to E$ be defined by

$$f_0(t,x) \;=\; \begin{cases} \psi(t) & \text{if } \; t \in [-r,0] \\ \psi(0) + n\,t\,(x - \psi(0)) & \text{if } \; t \in [0, t^n_1] \end{cases}$$

and $F_0\colon\; [0, t^n_1] \times E \to c(E)$ be defined by

$$F_0(t,x) = F\Big(t,\, \tau(t^n_1)\, f_0(\cdot, x)\Big) \;.$$

We want to show that F_0 satisfies conditions (S_1)–(S_4) of Lemma 2.2. Clearly Condition (S_1) is verified. Next, to show that F_0 satisfies condition (S_2) is suffices to prove that if $x_k \to x$ weakly in E then $\tau(t^n_1)\, f_0(\cdot, x_k) \to \tau(t^n_1)\, f_0(\cdot, x)$ weakly in $C([-r,0],E)$. So, let γ be a bounded regular measure from $[-r,0]$ to E' and is of bounded variation. We have

$$\begin{aligned} &\lim_{k\to\infty} \int_{-r}^0 \Big(\tau(t^n_1)\, f_0(\cdot, x_k) - \tau(t^n_1)\, f_0(\cdot, x)\Big)(t)\; d\gamma(t) \;= \\ &\qquad = \lim_{k\to\infty} \int_{-r}^0 \Big(f_0(t + t^n_1, x_k) - f_0(t + t^n_1, x)\Big)\; d\gamma(t) \\ &\qquad = \lim_{k\to\infty} \int_0^{t^n_1} f_0\Big(s,\, x_k - f_0(s,x)\Big)\; d\gamma(s) \;. \end{aligned}$$

But, for every $x^* \in E'$ and every $s \in [0, t^n_1]$,

$$\lim_{k\to\infty} \Big(x^*,\, f_0(s,x_k) - f_0(s,x)\Big) \;=\; \lim_{k\to\infty} n\,s(x^*, x_k - x) \;=\; \lim_{k\to\infty} (x^*, x_k - x) \;=\; 0 \;.$$

Thus,

$$\lim_{k\to\infty} \int_{-r}^0 \Big(\tau(t^n_1)\, f_0(\cdot, x_k) - \tau(t^n_1)\, f_0(\cdot, x)\Big)(t)\; d\gamma(t) \;=\; 0 \;.$$

This show that F_0 satisfies condition (S_3) of Lemma 2.2. Furthermore, for every $(t,x) \in [0,t_1^n]\times E$,

$$\begin{aligned}\|F_0(t,x)\| &= \|F(t,\, \tau(t_1^n) f_0(\cdot,x))\| \\ &\le a(t)\Big(1+\|f_0(t_1^n,x))\|\Big) \\ &= a(t)\Big(1+\|x\|\Big)\ .\end{aligned}$$

Then F_0 satisfies condition (S_3) of Lemma 2.2. Now let B be a bounded subset of E. Set $Z=\{\tau(t_1^n)\, f_0(\cdot,x)\colon\, x\in B\}$. We have,

$$\begin{aligned}\mu(F_0(t,B)) &= \mu(F(t,Z)) \\ &\le \gamma(t)\,\mu Z(0) \\ &= \gamma(t)\,\mu(B)\ .\end{aligned}$$

Applying Lemma 2.2 we get a continuous function $v_0\colon\, [0,t_1^n]\to E$ such that

$$v_0(t) \;=\; K(t,o)\,\psi(0) + \int_0^t K(t,s)\,\sigma_0(s)\;ds\ ,$$

$\sigma_0(s)\in F(s,\,\tau(t_1^n)\, f_0(\cdot,v_0(s)))$ a.e. on $[0,t_1^n]$. Now, we define $u_n=v_0$ and $g_n=\sigma_0$ on $[0,t_t^n]$. Then, for all $t\in[0,t_1^n]$

$$u_n(t) \;=\; K(t,0)\,\psi(0) + \int_0^t K(t,s)\,g_n(s)\;ds$$

$g_n(s)\in F(s,\tau(\theta_n(s))\, f_{n\theta_n(s)-1}(\cdot,u_n(s)))$ a.e. on $[0,t_1^n]$. Thus u_n and g_n are well defined on $[0,t_1^n]$ and satisfy the properties (1) and (2).

Suppose u_n and g_n are well defined on $[0,t_m^n]$ such that the properties (1) and (2) are satisfied on $[0,t_m^n]$. Let

$$f_m\colon\, [-r,t_{m+1}^n]\to E\ ,$$

$$f_m(t,x) \;=\; \begin{cases} u_n(t) & \text{if } t\in[-r,t_m^n] \\ u_n(t_m^n)+n\,(t-t_m^n)\,(x-u_n) & \text{if } t\in[t_m^n,t_{m+1}^n]\ .\end{cases}$$

As above we can show that if $x_n\to x$ weakly in E then $\tau(t_{m+1}^n)\, f_m(\cdot,x_n)\to \tau(t_{m+1}^n)\, f_m(\cdot,x)$ weakly in $C([-r,0],E])$. Thus the multifunction

$$F_m\colon\, [t_m^n,t_{m+1}^n]\times E\to c(E)\ ,$$

defined by

$$F_m(t,x)=F\Big(t,\,\tau(t_{m+1}^n)\, f_m(\cdot,x)\Big)\ ,$$

satisfies conditions (S_1)–(S_4) of Lemma 2.2. Then, by Lemma 2.2, there exists a continuous function $v_m\colon [t^n_m, t^n_{m+1}] \to E$ such that

$$v_m(t) \;=\; K(t, t^n_m)\, u_n(t^n_m) + \int_{t^n_m}^{t} K(t,s)\, \sigma_m(s)\; ds\,, \qquad t \in [t^n_m, t^n_{m+1}]\;,$$

where $\sigma_m \in L^1([t^n_m, t^n_{m+1}], E)$, $\sigma_m(s) \in F_m(s, v_m(s)) = F(s, \tau(t^n_{m+1})\, f_m(s, v_m(s)))$ a.e.. Set $u_n(t) = v_m(t)$ for all $t \in [t^n_m, t^n_{m+1}]$ and $g_n(t) = \sigma_m(t)$ for all $t \in (t^n_m, t^n_{m+1}]$. Then, for every $t \in [t^n_m, t^n_{m+1}]$

$$u_n(t) \;=\; K(t, t^n_m)\, u_n(t^n_m) + \int_{t^n_m}^{t} K(t,s)\, g_n(s)\; ds\;,$$

$$g_n(s) \;\in\; F\Big(s,\; \tau(\theta_n(s))\, f_{n\theta_n(s)-1}(\cdot, u_n(s))\Big) \qquad \text{a.e. on} \;\; [t^n_m, t^n_{m+1}]\;.$$

This proves that g_n satisfies relation (2) on $[t^n_m, t^n_{m+1}]$ We claim that u_n verifies relation (1) on $[t^n_m, t^n_{m+1}]$, So, let $t \in [t^n_m, t^n_{m+1}]$. We have

$$u_n(t^n_m) \;=\; K(t^n_m, 0)\, \psi(0) + \int_0^{t^n_m} K(t^n_m, s)\, g_n(s)\; ds\;.$$

Then

$$\begin{aligned} u_n(t) \;&=\; K(t, t^n_m)\, K(t^n_m, 0)\, \psi(0) \;+ \int_0^{t^n_m} K(t, t^n_m)\, K(t^n_m, s)\, g_n(s)\; ds \\ &\quad + \int_{t^n_m}^{t} K(t,s)\, g_n(s)\; ds \\ &=\; K(t,0)\, \psi(0) \;+ \int_0^{t^n_m} K(t,s)\, g_n(s)\; ds \;+ \int_{t^n_m}^{t} K(t,s)\, g_n(s)\; ds \\ &=\; K(t,0)\, \psi(0) \;+ \int_0^{t} K(t,s)\, g_n(s)\; ds\;. \end{aligned}$$

This proves that u_n and g_n satisfy relations (1) and (2).

Step 2. We claim that:

(**a**) There exists a natural number N such that for all $n \geq 1$

(3) $\quad \|u_n(t)\| \leq N \;$ for all $t \in I \quad$ and $\quad \|g_n(t)\| \leq m(t) = a(t)\,(1+N)\;$ a.e..

(**b**) $(u_n) \to u$ uniformly in $C([-r,T], E)$, where $u = \psi$ on $[-r,0]$ and $g_n \to g$ weakly in $L^1(I,E)$.

So, let $n \geq 1$. For almost all $t \in I$,

$$\begin{aligned}
\|g_n(t)\| &\leq \Big\| F\Big(t,\, \tau(\theta_n(t))\, f_{n\theta_n(t)-1}(\cdot, u_n(t))\Big)\Big\| \\
&\leq a(t)\, \Big(1 + f_{n\theta_n(t)-1}(\theta_n(t), u_n(t))\Big) \\
&= a(t)\, \Big(1 + \|u_n(t)\|\Big)\ .
\end{aligned}$$

Then, for all $t \in I$,

$$\begin{aligned}
\|u_n(t)\| &\leq \|K(t,0)\|\, \|\psi(0)\| \ + \int_0^t \|K(t,s)\|\, \|g_n(s)\|\ ds \\
&\leq M\, \|\psi(0)\| \ + \ M \int_0^t a(s)\, \Big(1 + \|u_n(s)\|\Big)\ ds \\
&\leq M\Big(\|\psi(0)\| + \|a\|\Big) \ + \int_0^t M\, a(s)\, \|u_n(s)\|\ ds\ .
\end{aligned}$$

By Gronwall's Lemma, we get

$$\|u_n(t)\| \leq M\Big(\|\psi(0)\| + \|a\|\Big)\ \exp(M\, \|a\|)\ .$$

Denote the right side of the above inequality by N and put $m(t) = a(t)\,(1+N)$, $\forall\, t \in I$. To prove the property (b) let $t_1, t_2 \in I$, $(t_1 < t_2)$ and let n be a fixed natural number.

$$\begin{aligned}
\Big\|u_n(t_2) - u_n(t_1)\Big\| &\leq \Big\|K(t_2,0) - K(t_1,0)\Big\|\, \|\psi(0)\| \\
&\quad + \int_0^{t_1} \Big\|K(t_2,s) - K(t_1,s)\Big\|\, \|g_n(s)\|\ ds \\
&\quad + \int_{t_1}^{t_2} \|K(t_2,s)\|\, \|g_n(s)\|\ ds \\
&\leq \Big\|K(t_2,0) - K(t_1,0)\Big\|\, \|\psi(0)\| \\
&\quad + \int_0^{T} \Big\|K(t_2,s) - K(t_1,s)\Big\|\, |m(s)|\ ds \\
&\quad + M \int_{t_1}^{t_2} |m(s)|\ ds\ .
\end{aligned}$$

Since for each $s \in I$, $K(\cdot, s)$ is uniformly continuous and $u_n \equiv \psi$ on $[-r, 0]$, the sequence (u_n) is equicontinuous in $C([-r,T], E)$. Next, for each $t \in I$, put

$$Z(t) = \{u_n(t) \colon\, n \geq 1\}\,, \qquad \rho(t) = \mu(Z(t))\ .$$

From the properties of μ and Proposition 1.6 of Monch [15] we get

$$\begin{aligned}\rho(t) &= \mu\Big\{\int_0^t K(t,s)\, g_n(s)\; ds \colon n \geq 1\Big\} \\ &\leq M \int_0^t \mu\Big(\{g_n(s)\colon n \geq 1\}\Big)\; ds\ .\end{aligned}$$

But $\mu(\{g_n(s)\colon n \geq 1\}) \leq \mu\, F(s, H(s))$ a.e., where

$$H(s) = \Big\{\tau(\theta_n(s))\, f_{n\theta_n(s)-1}(\cdot, u_n(s))\colon n \geq 1\Big\}\ .$$

Thus, By condition (F_4) we obtain,

$$\begin{aligned}\rho(t) &\leq M \int_0^t \gamma(s)\, \mu(H(s)(0))\; ds \\ &= M \int_0^t \gamma(s)\, \mu\{u_n(s)\colon n \geq 1\}\; ds \\ &= M \int_0^t \gamma(s)\, \rho(s)\; ds\ .\end{aligned}$$

Since $\rho(0) = 0$, Gronwall's Lemma tells us $\rho = 0$. So by Ascoli's theorem we may assume that u_n converges uniformly to $u \in C([-r,T],E)$. Obviously $u = \psi$ on $[-r,0]$. Now, let $t \in I$ such that Condition (F_4) is satisfied. Then,

$$\begin{aligned}\mu\{g_n(t)\colon n \geq 1\} &\leq \mu\Big(\Big\{F\Big(t,\, \theta_n(t)\, f_{n\theta_n(t)}(\cdot, u_n(t))\Big)\colon n \geq 1\Big\}\Big) \\ &\leq \gamma(t)\, \mu\Big(\Big\{\theta_n(t)\, f_{n\theta_n(t)}(\cdot, u_n(t))(0)\colon n \geq 1\Big\}\Big) \\ &= \gamma(t)\, \mu\{u_n(t)\}\ .\end{aligned}$$

Then $\mu(\{g_n(t)\colon n \geq 1\}) = 0$ a.e.. By redefining (if necessary) a multifunction H such that its values are in $c(E)$ and $H(t) = \overline{\text{conv}}\{g_n(t)\colon n \geq 1\}$ a.e.. Thus S_H^1 is nonempty, convex and weakly compact in $L^1(I,E)$. By the Eberlein–Smulian Theorem we may assume $g_n \to g \in L^1(I,E)$ weakly.

Step 3. We claim that the function u obtained in the previous step is the desired solution. That is we claim that

$$u(t) = K(t,0)\, \psi(0) + \int_0^t K(t,s)\, g(s)\; ds\,, \qquad \forall\, t \in I\ , \tag{4}$$

$$g(t) \in F(t, \tau(t)\, u), \qquad \text{a.e.} \tag{5}$$

since $g_n \to g$ weakly in $L^1(I,E)$, u_n tends weakly to $K(t,0)\,\psi(0)+\int_0^t K(t,s)\,g(s)\,ds$. Hence we get relation (4). Moreover, from Lemma 2.2 and relation (2), relation (5) will be true if we show

$$\lim_{n\to\infty} \Big\|\tau(\theta_n(t)) - f_{n\theta_n(t)-1}(\cdot,u_n(t))\Big\| = 0\,, \qquad \forall\, t\in I\ . \tag{6}$$

Let $t\in I$ and $n>\frac{1}{r}$. Let $m\in\{0,1,...,n-1\}$ such that $t\in[t^n_m,t^n_{m+1}]$.

$$\begin{aligned}
&\Big\|\tau(\theta_n(t))\, f_{n\theta_n(t)-1}(\cdot,u_n(t)) - \tau(t)\,u\Big\| \;\le \\
&\quad\le \sup_{s\in[-r,-\frac{1}{n}]} \bigg\|f_m\Big(\frac{m+1}{n}+s,\, u_n(t)\Big) - u\Big(\frac{m+1}{n}+s\Big)\bigg\| \\
&\qquad + \sup_{[-\frac{1}{n},-r]} \bigg\|u_n\Big(\frac{m}{n}+n\Big(s+\frac{1}{n}\Big)\Big)\Big(u_n(t)-u_n\Big(\frac{m}{n}\Big)\Big) - u\Big(\frac{m+1}{n}+s\Big)\bigg\| \\
&\qquad + \bigg\|u\Big(\frac{m+1}{n}+s\Big) - u\,(t+s)\bigg\| \\
&\quad\le \sup_{s\in[-r,-\frac{1}{n}]} \bigg\|u_n\Big(\frac{m+1}{n}\Big) - u\Big(\frac{m+1}{n}+s\Big)\bigg\| \\
&\qquad + \bigg\|u_n(t)-u_n\Big(\frac{m}{n}\Big)\bigg\| + \Big\|u_n(t)-u(t)\Big\| \\
&\qquad + \sup_{s\in[-\frac{1}{n},0]} \bigg(\bigg\|u(t)-u\Big(\frac{m+1}{n}+s\Big)\bigg\| + \bigg\|u\Big(\frac{m+1}{n}+s\Big) - u(s+t)\bigg\|\bigg)\,.
\end{aligned}$$

Since u_n converges uniformly to u on each compact subset of $[-r,T]$, u is uniformly continuous on $[-r,0]$ and each u_n is continuous on $[-r,T]$, relation (6) is true. ∎

4 – Some topological properties of solution sets

In the previous section, we obtained conditions on the data that guaranteed that for every $\psi\in C([-r,0],E)$ the solution set of ψ, $S(\psi)$, is nonempty. In this section we examine the topological properties of this solution set.

Theorem 4.1. *If the hypotheses of Theorem 3.1 hold, then for every $\psi\in C([-r,0],E)$, $S(\psi)$ is compact in $C([-r,T],E)$.*

Proof: Arguing in the proof of Theorem 3.1 we can show that $S(\psi)$ is equicontinuous. Furthermore let (u_n) be a sequence in $S(\psi)$ and $t \in I$. Then

$$\begin{aligned}
\mu\Big(\{u_n(t)\colon\, n \geq 1\}\Big) \;&\leq\; \mu\bigg(\bigg\{\int_0^t K(t,s)\, g_n(s)\, ds\colon\, n \geq 1\bigg\}\bigg), \quad g_n \in S^1_{F(\cdot,\tau(\cdot)u_n)} \\
&\leq\; M \int_0^t \mu\Big(\{g_n(s)\colon\, n \geq 1\}\Big)\; ds \\
&\leq\; M \int_0^t \mu\bigg(F\Big(s, \bigcup_{n=1}^{\infty} \tau(s)\, u_n\Big)\bigg)\; ds \\
&\leq\; M \int_0^t \gamma(s)\, \mu\Big(\{(\tau(s)u_n)(0)\colon\, n \geq 1\}\Big)\; ds \\
&=\; M \int_0^t \gamma(s)\, \mu\Big(\{u_n(s)\colon\, n \geq 1\}\Big)\; ds\;.
\end{aligned}$$

Since $\mu(\{u_n(0)\colon\, n \geq 1\}) = 0$, by Gronwall's Lemma we get $\mu(\{u_n(t)\colon\, n \geq 1\}) = 0$. For all $t \in I$. Thus (u_n) has a convergent subsequence in $C([-r,T],E)$. ∎

Theorem 4.2. *The multifunction $S : C([-r,0],E) \to C([-r,T],E)$ is upper semicontinuous.*

Proof: Let B be a closed set in $C([-r,T],E)$ and $Z = \{\psi \in C([-r,0],E)\colon S(\psi) \cap B \neq \emptyset\}$. We shall show that Z is closed. So, let $\psi_n \in Z$, $\psi_n \to \psi$ in $C([-r,0],E)$. For each $n \geq 1$, let $u_n \in S(\psi_n) \cap Z$. Then, for every $n \geq 1$, $u_n = \psi_n$ on $[-r,0]$ and for all $t \in I$,

$$u_n(t) \;=\; K(t,0)\, \psi_n(0) \;+ \int_0^t K(t,s)\, g_n(s)\; ds\,, \qquad g_n \in S^1_{F(\cdot,\tau(\cdot)u_n)}\;.$$

Then, for every $t \in I$,

$$\mu\Big(\{u_n(t)\colon\, n \geq 1\}\Big) \;\leq\; M\, \mu\Big(\{\psi_n(0)\colon\, n \geq 1\}\Big) + M\, \mu\bigg(\bigg\{\int_0^t g_n(s)\, ds\colon\, n \geq 1\bigg\}\bigg)$$

since $\psi_n(0) \to \psi(0)$ as $n \to \infty$, we get

$$\mu\Big(\{u_n(t)\colon\, n \geq 1\}\Big) \;\leq\; M\, \mu\bigg(\int_0^t g_n(s)\, ds\colon\, n \geq 1\bigg)\;.$$

As in the proof of Theorem 4.1 we can claim that $\mu(\{u_n(t)\colon\, n \geq 1\}) = 0$. Invoking the Arzela–Ascoli theorem there exists a subsequence $u_{nk} \to u \in Z$ in $C([-r,T],E)$. Clearly $u = \psi$ on $[-r,0]$. Now

$$\begin{aligned}
\mu\Big(\{g_{n_k}(t)\colon\, n \geq 1\}\Big) \;&\leq\; \mu\Big(\{F(t,\tau(t)\, u_{n_k})\colon\, n \geq 1)\}\Big); \quad t \in I \\
&\leq\; \gamma(t)\, \mu\Big(\{(\tau(u_n))(0)\colon\, n \geq 1\}\Big); \quad t \in I \\
&=\; 0\;.
\end{aligned}$$

As in the proof of Theorem 3.1, $g_{n_k} \to g$ weakly in $L^1(I,E)$. Invoking Lemma 2.1, $g(t) \in F(t,\tau(t)\,u)$ a.e.. Thus

$$u(t) \;=\; K(t,0)\,\psi(0) + \int_0^t K(t,s)\,g(s)\;ds\,, \qquad g \in S^1_{F(\cdot,\tau(\cdot)u)}\;.$$

This prove that Z is closed and hence $\psi \to S(\psi)$ is upper semicontinuous. ∎

Corollary 4.1. *For every $\psi \in C([-r,0],E)$ and every $t \in I$ the attainable set $P_t(\psi) = \{u(t)\colon\, u \in S(\psi)\}$ is compact, the multifunction $(\psi,t) \to P_t(\psi)$ is jointely upper semicontinuous.*

Theorem 4.3. *Let Z be a compact subset of $C([-r,0],E)$ and let $\varphi\colon E \to \mathbb{R}$ be lower semicontinuous then the problem*

$$\begin{cases} \dot{u}(t) \;\in\; A(t)\,u(t) + F(t,\tau(t)\,u), & \text{a.e. on } [0,T] \\ u = \psi \in Z \\ \text{minimise } \varphi(u(T)) \end{cases}$$

has an optimal solution, that is, there exists $\psi_0 \in Z$ and $u \in S(\psi_0)$ such that

$$\varphi(u(T)) = \inf\Big\{\varphi(v(T))\colon\; v \in S(\psi),\; \psi \in Z\Big\}\;.$$

Proof: Consider the multifunction

$$\begin{aligned} &P_T\colon\; Z \to 2^E \\ &P_T(\psi) = \{v(T)\colon\; v \in S(\psi)\}\;. \end{aligned}$$

By Corollary 4.1, P_T is upper semicontinuous. Then the set $P_T(Z) = \bigcup_{\psi\in Z} P_T(\psi)$ is compact in E. Since φ is lower semicontinuous on E, there exists $\psi_0 \in Z$ such that $\varphi(\psi_0(T)) = \inf\{\varphi(v(T))\colon\; v \in \bigcup_{\psi\in Z} S(\psi)\}$. ∎

Theorem 4.4. *Let E be a separable Hilbert space and $G(t,\cdot)$ is w-seq uhc and $G(\cdot,g)$ has a measurable selection. Moreover, suppose that there exists a sequence $(G_n)\colon\, I \times C([-r,0],E) \to c(E)$ satisfying the following properties:*

(1) *For all $n \geq 1$, G_n verifies conditions (F_1), (F_2) and (F_4) of Theorem 3.1.*

(2) *For all $(t,g) \in I \times C([-r,0],E)$ we have*

- **(a)** $\|G_n(t,g)\| < L$, $\forall\, n \geq 1$, *for some constant $L > 0$;*
- **(b)** $\lim_{n\to\infty} h(G_n(t,g), G(t,g)) = 0$, *where h is the Hausdorff distance;*
- **(c)** $G_{n+1}(t,g) \subset G_n(t,g)$, $\;\forall\, n \geq 1$;

(**d**) $G(t,g) = \bigcap_{n=1}^{\infty} G_n(t,g)$.

Then for each $\psi \in C([-r,0],E)$, $S_G(\psi) = \bigcap_{n=1}^{\infty} S_{G_n}(\psi)$.

Proof: From the assumptions each G_n satisfies all conditions of Theorem 3.1. Thus $S_G(\psi) \neq \emptyset$. Also from condition (2)(d) we get $S_G(\psi) \subseteq S_{G_n}(\psi)$, $\forall\, n \geq 1$. Now let $u \in \bigcap_{n=1}^{\infty} S_{G_n}(\psi)$. Then for every $n \geq 1$, there exists $g_n \in L^1(I,E)$ such that

$$u(t) \;=\; K(t,0)\,\psi(0) + \int_0^t K(t,s)\,g_n(s)\;ds\,, \qquad \forall\, t \in I\;,$$

$$g_n(t) \;\in\; G_n(t,\tau(t)\,u) \quad \text{a.e.,} \quad \forall\, n \geq 1\;.$$

Thus, by condition 2(b), we obtain

$$g_n(t) \;\in\; G(t,\tau(t)\,u) + \delta_n(t)\,\overline{B_E} \quad \text{a.e.}\,,$$

where, for all $t \in I$, $\delta_n(t) = \lim_{n\to\infty} h(G_n(t,\tau(t)\,u), G(t,\tau(t)u)) \to 0$ and $\overline{B_E}$ is the closed unit ball in E. Invoking condition (2)(a), the sequence (g_n) is uniformly bounded. By extracting a subsequence, denoted again by g_n, we can passing to convex combination of $g_n(t)$, denoted by $\tilde{g}_n(t)$, we have $\tilde{g}_n(t) \to g(t)$ a.e. in E and

$$\tilde{g}_n(t) \;\in\; \sum_{m\geq n} \alpha_m(t)\,\Big(G(t,\tau(t)\,u) + \delta_m(t)\,\overline{B_E}\Big) \quad \text{a.e.}\,,$$

where $\sum_{m\geq n} = 1$, $\alpha_m(t) \geq 0$. Since the values of G are convex, we get

$$\tilde{g}_n(t) \;\in\; G(t,\tau(t)\,u) + (\sup_{m\geq n} \delta_m(t))\,\overline{B_E}\;.$$

Taking the limit as $n \to \infty$ we obtain $g(t) \in G(t,\tau(t)\,u)$ a.e.. Thus $u \in S_G(\psi)$. ∎

5 – Remarks

1. Let for every $t \in I$, $A(t)$ be a bounded linear operator on E such that the function $t \to A(t)$ is continuous in the uniform operator topology. Then for every $x \in E$ and every $s \in [0,T]$, the initial value problem

$$\begin{cases} \dot{u}(t) \in A(t)\,u(t)\,, & t \in [0,T] \\ u(s) = x \end{cases}$$

has a unique strong solution. Thus the operator $K(\cdot,\cdot)$ can be defined and satisfies all conditions (A_1)–(A_4) (see, Ch. 5 [19]). □

2. If we replace condition (F_4) by the condition:

$(\mathbf{F}_4)^*$ There exists an integrably bounded multifunction $\Gamma\colon I \to c\,k(E)$ such that

$$F(t,u) \subset \Big(1 + \|u(0)\|\Big)\,\Gamma(t)\,, \qquad \forall\,(t,u) \in I \times C([-r,0],E)\ ,$$

then the convergence of approximated solutions (u_n) constructed in the proof of Theorem 3.1 is directly ensured.

Indeed, for all $n \ge 1$ and all $t \in I$,

$$\begin{aligned} u_n(t) &\in K(t,0)\,\psi(0) + \int_0^t K(t,s)\,F(t,\tau(\theta_n(s)))\,f_{n\theta_n(s)-1}(\cdot,u_n(s))\;ds \\ &\subseteq K(t,0)\,\psi(0) + M\int_0^t \Big(1 + \|u_n(s)\|\Big)\,\Gamma(s)\;ds\ . \end{aligned}$$

since for each $n \ge 1$, $\|u_n(s)\| \le N$, $\forall\,t \in I$, Theorem v-15 of [4] implies that, $\int_0^t (1 + \|u_n(s)\|)\,\Gamma(s)\,ds$ is in $c\,k(E)$. Thus for all $t \in I$ the set $\{u_n(t)\colon\ n \ge 1\}$ is relatively compact in E. ◻

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