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SECTOR ESTIMATES FOR KLEINIAN GROUPS

R. Sharp

Abstract: We study the number of lattice points in a fixed sector for certain Kleinian groups. We show that they are asymptotically distributed according to the Patterson–Sullivan measure.

0 – Introduction

Let Γ be a (non-elementary) group of isometries of the $(n+1)$-dimensional real hyperbolic space $\mathbb{H}^{n+1}$. Given a point $x \in \mathbb{H}^{n+1}$, we shall be interested in the behaviour of its orbit Γx under the action of Γ. This orbit accumulates only on the boundary of $\mathbb{H}^{n+1}$, which we may regard as S^n. We call the set of accumulation points, which is independent of x, the *limit set* of Γ and denote it by L_Γ. The limit set is a closed perfect subset of S^n; either $L_\Gamma = S^n$ or L_Γ is nowhere dense in S^n. Write $\mathcal{C}(\Gamma) \subset \mathbb{H}^{n+1} \cup S^n$ for the convex hull of L_Γ; if $\Gamma \backslash (\mathcal{C}(\Gamma) \cap \mathbb{H}^{n+1})$ is compact then we say that Γ is *convex co-compact*. (Note that this condition is weaker than requiring that Γ be co-compact, i.e., that $\Gamma \backslash \mathbb{H}^{n+1}$ is compact, since co-compact groups have $L_\Gamma = S^n$.) In this paper we shall be concerned exclusively with convex co-compact groups.

Given points $p, q \in \mathbb{H}^{n+1}$ we can define the orbital counting function $N_\Gamma(p,q,T) = \#\{g \in \Gamma \colon\, d(p,gq) \le T\}$, where $d(\cdot,\cdot)$ denotes distance in $\mathbb{H}^{n+1}$. A lot of effort has gone into understanding the asymptotic behaviour of this function and it is known that, for convex co-compact Γ, $N_\Gamma(p,q,T) \sim \mathcal{C}(p,q,\Gamma) e^{\delta T}$, as $T \to \infty$, where $\mathcal{C}(p,q,\Gamma) > 0$ is a constant and where $0 < \delta \le n$ is the exponent of convergence of Γ [12]. (A more precise estimate is known under the additional hypothesis that $\delta > n/2$ [7].)

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A more delicate question is to understand the asymptotics of the number of orbit points lying in a fixed sector. Fix a (closed) ball $B \subset S^n$ and let $\widehat{B}$ denote the sector in $\mathbb{H}^{n+1}$ formed by the set of geodesic rays emanating from p with end-points in B. Define

$$N_\Gamma^B(p,q,T) \;=\; \#\Big\{g \in \Gamma\colon\ d(p,gq) \le T \ \text{ and } \ gq \in \widehat{B}\Big\}\ .$$

The behaviour of this function is closely related to the so-called Patterson–Sullivan measure $\mu_{p,q}$ on S^n. It is known that there exist constants $0 < C_1 < C_2$ (depending only on Γ) such that if the centre of B lies in the limit set of Γ then, for all sufficiently large T,

$$C_1\,\mu_{p,q}(B)\,N_\Gamma(p,q,T) \;\le\; N_\Gamma^B(p,q,T) \;\le\; C_2\,\mu_{p,q}(B)\,N_\Gamma(p,q,T)$$

[8]. Our object in this paper is to obtain a more precise result for certain classes of Kleinian groups; namely groups satisfying the condition defined below.

Definition. A Kleinian group Γ is said to satisfy the *even corners condition* if Γ admits a fundamental domain R which is a finite sided polyhedron (possibly with infinite volume) such that $\bigcup_{g\in\Gamma} g\,\partial R$ is a union of hyperplanes. (This definition was introduced by Bowen and Series [3] for the case $n=1$ and studied by Bourdon [2] for $n \ge 2$.) □

Theorem 1. *Let Γ be a convex co-compact group acting on $\mathbb{H}^{n+1}$. If Γ satisfies the even corners condition then for any $p,q \in \mathbb{H}^{n+1}$ and any Borel set $B \subset S^n$ such that $\mu_{p,q}(\partial B) = 0$ we have*

$$\lim_{T\to\infty} \frac{N_\Gamma^B(p,q,T)}{N_\Gamma(p,q,T)} = \mu_{p,q}(B)\ . \ ■$$

This result is known in certain cases. In particular, it is known if Γ is co-compact [9], [10] (in which case $\mu_{p,q}$ is equivalent to n-dimensional Lebesgue measure on S^n) or if Γ is a Schottky group [6].

More generally, Theorem 1 is known to hold if Γ is convex co-compact and the points p and q lie in the convex hull of L_Γ [13]. In this case, the result follows from an approach based on an analysis of the orbit structure of hyperbolic flows. More precisely, writing $M = \Gamma\backslash\mathbb{H}^{n+1}$, consider the projection $\pi\colon\ \mathbb{H}^{n+1} \to M$ and the geodesic flow $\phi_t\colon SM \to SM$ on the unit-tangent bundle of M. The counting function $N_\Gamma^B(p,q,T)$ may be reinterpreted as the number of ϕ-orbits, with length not exceeding T, passing from the fibre $S_{\pi(p)}M$ to the fibre $S_{\pi(q)}M$, such that the initial point lies in $B \subset S_{\pi(p)}M$. (It is a standard procedure to identify the boundary of $\mathbb{H}^{n+1}$ with the fibre $S_{\pi(p)}M$ lying over a fixed base point.) The non-wandering set $\Omega \subset SM$ for ϕ consists of all vectors tangent to the projection $\pi(\mathcal{C}(\Gamma) \cap \mathbb{H}^{n+1})$ and the restriction $\phi_t : \Omega \to \Omega$ is a uniformly hyperbolic flow. If $p, q \in \mathcal{C}(\Gamma)$ then $N_\Gamma^B(p,q,T)$ counts orbits which lie in Ω and the methods of [13] give the required result. (Roughly speaking, $N_\Gamma^B(p,q,T)$ is approximated by functions counting orbits passing from small pieces of unstable manifold to small pieces of stable manifold; these latter quantities admit a symbolic description to which one may apply the techniques of thermodynamic formalism.)

However, if p and q do not lie in $\mathcal{C}(\Gamma)$ then the relevant orbits would lie outside Ω and the above arguments no longer hold. In this paper we impose no restrictions on p and q. Instead of formulating the problem in terms of hyperbolic flows, we shall obtain a symbolic description directly from Γ. The "even corners" property ensures that that this description matches the geometry of the action on $\mathbb{H}^{n+1}$.

We end the introduction by giving two classes of examples of even cornered groups.

Example 1. Let $K_1, ..., K_{2k}$ be $2k$ disjoint n-dimensional spheres in $\mathbb{R}^{n+1}$, each meeting S^n at right angles. For $i = 1, ..., k$, let g_i be the isometry which maps the exterior of K_i onto the interior of K_{k+i}. Then the group Γ generated in this way is called a *Schottky group* and satisfies the even corners condition. Viewed as an abstract group, it is the free group on k generators. In this case, L_Γ is a Cantor set. □

Example 2. Let R be a polyhedron in $\mathbb{H}^{n+1}$ with a finite number of faces and with interior angles all equal to π/k, $k \in \mathbb{N}$, $k \geq 2$. Let Γ be the Kleinian group generated by reflections in the faces of R. Then Γ satisfies the even corners condition. For instance, let R be a regular tetrahedron in $\mathbb{H}^3$ with infinite volume and with dihedral angles $\pi/4$. In this case, L_Γ is a Sierpiński curve [1], [2]. □

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1 – Kleinian groups and Patterson–Sullivan measure

Let $\mathbb{H}^{n+1}$ denote the real hyperbolic space of dimension $n+1$. A convenient model for $\mathbb{H}^{n+1}$ is the open ball $\{x \in \mathbb{R}^{n+1} : ||x||_2 < 1\}$, equipped with the metric

$$ds^2 = \frac{4\,(dx_1^2 + \cdots + dx_{n+1}^2)}{(1 - ||x||_2^2)^2} \,.$$

(In particular, geodesics passing through 0 are just Euclidean straight lines.) We can then naturally identify the ideal boundary of $\mathbb{H}^{n+1}$ with the n-dimensional unit sphere S^n.

A Kleinian group Γ is a discrete group of isometries of $\mathbb{H}^{n+1}$. (If n=1, we say that Γ is a Fuchsian group.) Its action on $\mathbb{H}^{n+1}$ extends to an action on S^n. We say that Γ is non-elementary if it does not contain a cyclic subgroup of finite index. In this paper we shall only consider non-elementary groups and all statements implicitly assume that Γ is non-elementary. We say that Γ is geometrically finite if there is a fundamental domain for its action on $\mathbb{H}^{n+1}$ which is a polyhedron with finitely many faces; this includes the class of convex co-compact groups. (Note that, for $n \geq 4$, there are other, inequivalent, notions of geometrical finiteness.) If Γ is geometrically finite then it is finitely generated.

One of the most important quantities attached to a Kleinian group is its exponent of convergence. This is the abscissa of convergence of the Dirichlet series $\sum_{g\in\Gamma} e^{-sd(p,gq)}$ (for any $p, q \in \mathbb{H}^{n+1}$) and is denoted by $\delta = \delta(\Gamma)$. We have $0 < \delta \leq n$. If Γ is geometrically finite then δ is also equal to the Hausdorff dimension of L_Γ and, furthermore, if $L_\Gamma \neq S^n$ then $\delta < n$ (so that, in particular, the n-dimensional Lebesgue measure of L_Γ is equal to zero) [16], [17].

The limit set of a Kleinian group supports a natural family of equivalent measures $\mu_{p,q}$ $(p, q \in \mathbb{H}^{n+1})$ called Patterson–Sullivan measures [11], [15]. Roughly speaking, $\mu_{p,q}$ is the weak* limit, as $s \to \delta+$, of

$$\frac{\displaystyle\sum_{g\in\Gamma} e^{-sd(p,gq)}\, D_{gq}}{\displaystyle\sum_{g\in\Gamma} e^{-sd(q,gq)}},$$

regarded as measures on $\mathbb{H}^{n+1} \cup S^n$, where D_{gq} denotes the Dirac measure at gq. If Γ is convex co-compact, they are characterized as the unique non-atomic measures supported on L_Γ satisfying

(i) for $p_1, p_2 \in \mathbb{H}^{n+1}$,
$$\frac{d\mu_{p_2,q}}{d\mu_{p_1,q}}(\xi) = \left(\frac{P(p_2,\xi)}{P(p_1,\xi)}\right)^\delta,$$
where $P(x,\xi) = (1 - ||x||_2^2)/(||x-\xi||_2^2)$ is the Poisson kernel;

(ii) $g^*\mu_{p,q} = \mu_{g^{-1}p,q}$, for $g \in \Gamma$;

(iii) $g^*\mu_{p,q} = |g'|^\delta\, \mu_{p,q}$, for $g \in \Gamma$.

Since p and q are fixed, we shall write $\mu = \mu_{p,q}$. It is a regular Borel measure on S^n.

Remark. In the above, we have used a prime to denote differentiation with respect to the metric obtained by radial projection from p. To mke this more precise, let ψ denote a conformal mapping preserving the unit ball such that $\psi(p) = 0$. For $\xi, \eta \in S^n$, we define $d_p(\xi,\eta) = |\cos^{-1}\psi(\xi)\cdot\psi(\eta)|$ and $|g'(\xi)| = \lim_{\eta\to\xi} d_p(g\xi, g\eta)/d_p(\xi,\eta)$. □

2 – Symbolic dynamics

We shall be interested in the action of Γ on S^n. For groups satisfying the even corners condition, it is possible to replace this action with a single piecewise-analytic expanding map of S^n which has the same orbit structure. This, in turn, may be modeled by a symbolic dynamical system, namely a subshift of finite type $\sigma\colon X_A \to X_A$. This is a particular case of the strongly Markov coding introduced by Cannon [4], [5]. However, if Γ satisfies the even corners condition then this construction is more closely related to the action of Γ on $\mathbb{H}^{n+1}$. In [2] and [14] it was shown how to construct a Hölder continuous function $r\colon X_A \to \mathbb{R}$ which encoded the distances $d(p, gq)$. This facilitated an analysis of the Poincaré series $\sum_{g\in\Gamma} e^{-sd(p,gq)}$ via a family of linear operators acting on a space of Hölder continuous functions defined on X_A.

To begin, we recall the notion of word length: given a (symmetric) generating set $\mathcal{S}$, the word length $|g| = |g|_{\mathcal{S}}$ of an element $g \in \Gamma\backslash\{e\}$ is defined by

$$|g| = \inf\Big\{k \geq 1\colon\ g = g_1\cdots g_k,\ g_i \in \mathcal{S},\ i = 1,...,k\Big\}\ .$$

In particular, $|g| = 1$ if and only if $g \in \mathcal{S}$. (By convention, we set $|e| = 0$.)

Let R be a polyhedron as specified by the even corners condition. Label the faces of R by $\{R_1,...,R_m\}$ and let $g_i \in \Gamma$ denote the isometry for which $g_i R \cap R = R_i$. Write $\mathcal{S} = \{g_1,...,g_m\}$; then, by the Poincaré Polyhedron Theorem, $\mathcal{S}$ generates Γ. For each $i = 1,...,m$, R_i extends to a codimension one hyperbolic hyperplane, which divides $\mathbb{H}^{n+1}\cup S^n$ into two half-spaces. Let H_i denote the half-space which does not contain R and let $U_i = H_i \cap S^n$. In general, the U_i's will overlap; to obtain a partition we let $\mathcal{P} = \{P_1,...,P_k\}$ denote the sets formed by taking the closure of all possible intersections of the interiors of the U_i's. Write $\bar{\mathcal{P}} = \bigcup_{i=1}^k P_i$; then $\bigcup_{i=1}^k P_k = S^n$ and $\operatorname{int} P_i \cap \operatorname{int} P_j = \emptyset$ if $i \neq j$.

Choose an arbitrary ordering $\prec$ on $\mathcal{S}$. Let $g \in \Gamma$. If $g = g_{i_0}\cdots g_{i_{n-1}}$ we say that the word $g_{i_0}...g_{i_{n-1}}$ is lexically shortest if $|g| = n$ and if, whenever $g = h_{i_0}\cdots h_{i_{n-1}}$ with $h_{i_0},...,h_{i_{n-1}} \in \mathcal{S}$, then $g_{i_j} \prec h_{i_j}$, where j is the smallest index at which the terms disagree. Clearly every group element is presented by a unique lexically shortest word.

Define a map $f: \bar{\mathcal{P}} \to S^n$ by $f|_{P_i}(x) = a_i^{-1}x$, where $\operatorname{int} P_i = \operatorname{int} U_{j_1} \cap \cdots \cap \operatorname{int} U_{j_l}$ and where a_i is the $\prec$-smallest element of $\{g_{j_1}, ..., g_{j_l}\}$. (Strictly speaking, f is well-defined on the disjoint union $\coprod_{i=1}^k P_i$.) If necessary refining a finite number of times by considering intersections of sets in $\mathcal{P}, f^{-1}\mathcal{P}, ..., f^{-n}\mathcal{P}$, for some $n \geq 0$, f will satisfy the Markov property: if $f(\operatorname{int} P_i) \cap \operatorname{int} P_j \neq \emptyset$ then $f(P_i) \supset P_j$. We shall now define a graph $\mathcal{G} = (V, E)$, where the set of vertices $V = \{1, ..., k\}$ and the set of edges E is defined by

$$E = \Big\{(i,j) \in V \times V : \, f(P_i) \supset P_j\Big\} \; .$$

If P_i is contained in only one U_j then we call i a pure vertex; there are precisely $\#\mathcal{S}$ pure vertices. The map $(i_1, ..., i_n) \mapsto a_{i_1} \cdots a_{i_n}$ gives a bijections between the set of paths in $\mathcal{G}$ starting at a pure vertex and Γ. In order that these paths can be written as infinite paths, we shall augment $\mathcal{G}$ by adding an extra vertex 0 and edges $(v, 0)$ for all $v \in V$ to form a new graph $\mathcal{G}'$. Let A and B denote the incidence matrices of $\mathcal{G}'$ and $\mathcal{G}$, respectively.

Define the shift space X_A by

$$X_A = \Big\{x \in (V \cup \{0\})^{\mathbb{Z}^+} : \, A(x_n, x_{n+1}) = 1 \;\; \forall n \geq 0\Big\}$$

and define X_B in a similar way. On each of these spaces, define the shift map σ by $(\sigma x)_n = x_{n+1}$. For notational convenience, we shall use $\dot{0}$ to denote the element of X_A consisting of an infinite sequence of 0's.

For a path $(i_0, i_1, ..., i_n)$ in $\mathcal{G} = (V, E)$, we write

$$P(i_0, i_1, ..., i_n) = \bigcap_{j=0}^{n} f^{-j} P(i_j) \; .$$

We call such a set a geometric n-cylinder. We shall denote the collection of all geometric n-cylinders by $\mathcal{P}_n$ and write $\bar{\mathcal{P}}_n = \bigcup_{P \in \mathcal{P}_n} P$. We have $L_\Gamma = \bigcap_{n=1}^\infty \bar{\mathcal{P}}_n$.

The map f restricts to a map $f: L_\Gamma \to L_\Gamma$ which models the action of Γ on L_Γ. It is an expanding map in the sense that there exists $n \geq 0$ and $\beta > 1$ such that $|(f^n)'(x)| \geq \beta$ for all $x \in \bar{\mathcal{P}}_n$ and it is locally eventually onto. If $(i_0, i_1, ...)$ is an infinite path in (V, E) then $\operatorname{diam} P(i_0, ..., i_n)$ and $\mu(P(i_0, ..., i_n))$ both converge to zero as $n \to \infty$; the latter statement following from the fact that μ is non-atomic and regular. In particular, $\bigcap_{n=0}^\infty P(i_0, ..., i_n)$ consists of a single point $x_{i_0, i_1, ...}$, say.

There is a natural Hölder continuous semi-conjugacy $\Pi: X_B \to L_\Gamma$ between $\sigma: X_B \to X_B$ and $f: L_\Gamma \to L_\Gamma$, defined by $\Pi(i_0, i_1, ...) = x_{i_0, i_1, ...}$ which is bounded-to-one and one-to-one on a residual set. A particular consequence is that the matrix B is aperiodic.

For each $i = 1, ..., k$, define $C(i) = \bigcap_{a \in \mathcal{S}(i)} H_a$ and, for $P(i_0, ..., i_n)$, define

$$C(i_0, ..., i_n) = \bigcap_{j=0}^{n} a_{i_0} \cdots a_{i_{j-1}} C(i_j) \ .$$

We refer to $C(i_0, ..., i_n)$ as the "cap" of $P(i_0, ..., i_n)$. We shall also denote the cap of $P \in \mathcal{P}_n$ by C_P. The following result is immediate from the construction.

Lemma 1. *Suppose that $q \in R$. If $gq \in C(i_0, ..., i_n)$ then $g = a_{i_0} \cdots a_{i_n}$ and this is the lexically shortest representation of g.* ∎

Remark 1. If $q \notin R$ then the above lemma can be simply amended. However, for simplicity, we shall restrict ourselves to the case $q \in R$. □

3 – Approximation

For $g \in \Gamma$, write $\xi(g) \in S^n$ for the (positive) endpoint of the geodesic from p to gq. Then, for any set $F \subset S^n$, we have $N_\Gamma^F(p, q, T) = \#\{g \in \Gamma \colon d(p, q) \le T$ and $\xi(g) \in F\}$.

Let $\epsilon > 0$ be given. Then, since $\mu(\partial B) = 0$, we can find n sufficiently large and collections $\mathcal{Q} \subset \mathcal{Q}'$ of geometric n-cylinders such that

$$\bigcup_{P \in \mathcal{Q}} P \subset B \subset \bigcup_{P \in \mathcal{Q}'} P \cup (S^n \backslash \bar{\mathcal{P}}_n)$$

and

$$\mu(B) - \epsilon \ \le \sum_{P \in \mathcal{Q}} \mu(P) \ \le \sum_{P \in \mathcal{Q}'} \mu(P) \ \le \ \mu(B) + \epsilon \ .$$

Since we then have

$$\sum_{P \in \mathcal{Q}} N_\Gamma^P(p, q, T) \ \le \ N_\Gamma^B(p, q, T) \ \le \sum_{P \in \mathcal{Q}'} N_\Gamma^P(p, q, T) + O(1) \ ,$$

it suffices to show that

$$\lim_{T \to \infty} \frac{N_\Gamma^P(p, q, T)}{N_\Gamma(p, q, T)} \ = \ \mu(P) \ ,$$

whenever P is a geometric n-cylinder.

To do this we need to make a second approximation. First we introduce some notation. Let $\widehat{P}$ denote the sector formed by geodesic rays emanating from p with endpoints in P and let C denote the cap of P.

Choose $\epsilon > 0$ and let $\mathcal{N}_\epsilon(\partial P)$ denote the ϵ-neighbourhood of ∂P in S^n. Since C and $\widehat{P}$ are tangent at ∂P, provided T_0 is sufficiently large and $d(p,gq) > T_0$, if $gq \in C \triangle \widehat{P}$ then $\xi(g) \in \mathcal{N}_\epsilon(\partial P)$. Thus, for $T \geq T_0$,

$$\left| N_\Gamma^P(p,q,T) \;-\; \#\Big\{g \in \Gamma\colon\; d(p,gq) \leq T,\;\, gq \in C\Big\}\right| \;\leq\; N_\Gamma^{\mathcal{N}_\epsilon(\partial P)}(p,q,T)\;.$$

Since $\mu(\mathcal{N}_{2\epsilon}(\partial P)) \to 0$, as $\epsilon \to 0$, the proof of Theorem 1 will be complete once we have shown the following two results. The proof of Proposition 1 will be given in the next section.

Proposition 1.

$$\lim_{T\to\infty} \frac{1}{N_\Gamma(p,q,T)}\; \#\Big\{g \in \Gamma\colon\; d(p,gq) \leq T,\;\, gq \in C\Big\} \;=\; \mu(P)\;. \blacksquare$$

Lemma 2.

$$\limsup_{T\to\infty} \frac{N_\Gamma^{\mathcal{N}_\epsilon(\partial P)}(p,q,T)}{N_\Gamma(p,q,T)} \;\leq\; C_2\, \mu(\mathcal{N}_{2\epsilon}(\partial P))\;.$$

Proof: Choose m sufficiently large that if $R \in \mathcal{P}_m$ and $R \cap \mathcal{N}_\epsilon(\partial P) \neq \emptyset$ then $R \subset \mathcal{N}_{2\epsilon}(\partial P)$. Set $\mathcal{R} = \{R \in \mathcal{P}_m\colon\; R \cap \mathcal{N}_\epsilon(\partial P) \neq \emptyset\}$. If $d(p,gq) > T_0$ and $\xi(g) \in \mathcal{N}_\epsilon(\partial P)$ then $gq \in C_R$ for some $R \in \mathcal{R}$. Thus

$$\begin{aligned}
\limsup_{T\to\infty} \frac{N_\Gamma^{\mathcal{N}_\epsilon(\partial P)}(p,q,T)}{N(T)} \;\leq&\\
&\leq \lim_{T\to\infty} \frac{1}{N(T)} \sum_{R\in\mathcal{R}} \#\Big\{g \in \Gamma\colon\; T_0 < d(p,gq) \leq T,\;\, gq \in C_R\Big\}\\
&= \sum_{R\in\mathcal{R}} \mu(R) \;=\; \mu\left(\bigcup_{R\in\mathcal{R}} R\right) \;\leq\; \mu(\mathcal{N}_{2\epsilon}(\partial P))\;,
\end{aligned}$$

where we have used Proposition 1. $\blacksquare$

4 – Poincaré series

In this section we will prove Proposition 1 by considering the analytic domain of a certain function of a complex variable. Before we do this, we need to consider a family of linear operators defined as follows. Note that $X_A \backslash X_B$ consists of all sequences in X_A ending in an infinite string of 0's. Define $r\colon\, X_A \backslash X_B \to \mathbb{R}$ by

$$r(i_0, i_1, ..., i_n, \dot{0}) \;=\; d(p, a_{i_0} \cdots a_{i_n} q) - d(p, a_{i_1}, \cdots a_{i_n} q)\;,$$

so that

$$\sum_{k=0}^{n} r(i_0, ..., i_k, \dot{0}) \;=\; d(p, a_{i_0} \cdots a_{i_n} q) \;.$$

This extends to a Hölder continuous function $r\colon X_A \to \mathbb{R}$ [2], [6], [14].

For $s \in \mathbb{C}$, define $\mathcal{L}_s\colon C^\alpha(X_A) \to C^\alpha(X_A)$ by

$$\mathcal{L}_s \phi(x) \;=\; \sum_{\substack{\sigma y = x \\ y \neq \dot{0}}} e^{-sr(y)}\, \phi(y) \;.$$

(Note that this agrees with the usual definition of the Ruelle transfer operator for $x \in X_A \backslash \{\dot{0}\}$.) The following result is well-known.

Proposition 2.

(i) *The restricted operator $\mathcal{L}_\delta\colon C^\alpha(X_B) \to C^\alpha(X_B)$ has 1 as a simple maximal eigenvalue with a strictly positive associated eigenfunction ψ. The corresponding eigenmeasure ν for $\mathcal{L}_\delta^*$ satisfies $\Pi_* \nu = \mu$, where μ is the Patterson–Sullivan measure.*

(ii) *For s in a neighbourhood of δ, $\mathcal{L}_s$ has a simple eigenvalue $\rho(s)$ which is maximal in modulus such that $s \mapsto \rho(s)$ is analytic and $\rho(\delta) = 1$.*

(iii) *For $\Re s = \delta$, $s \neq \delta$, $\mathcal{L}_s$ does not have 1 as an eigenvalue.*

Proof: Part (i) follows by a standard calculation because $\log|f'|\colon L_\Gamma \to \mathbb{R}$ pulls back under Π to a function cohomologous to $r\colon X_B \to \mathbb{R}$ (i.e. $\log|f' \circ \Pi| = r + u \circ \sigma - u$ for some $u \in C(X_B)$). To see this note that it suffices to show that $r^n(x) := \sum_{k=0}^{n-1} r(\sigma^k x) = \sum_{k=0}^{n-1} \log|f'(\Pi(\sigma^k x))|$, whenever $\sigma^n x = x$ is a periodic point for $\sigma\colon X_B \to X_B$. To every such periodic point, we can associate a conjugacy class in Γ and hence a closed geodesic on $\Gamma \backslash \mathbb{H}^{n+1}$ with length equal to $r^n(x)$. The result now follows as in, for example, Theorem 8 of [6]. Part (ii) is standard. Part (iii) follows from the fact that $N_\Gamma(p, q, T) \sim \mathcal{C}(p, q, \Gamma)\, e^{\delta T}$. ∎

It is easy to see that $\mathcal{L}_s\colon C^\alpha(X_A) \to C^\alpha(X_A)$ and $\mathcal{L}_s\colon C^\alpha(X_B) \to C^\alpha(X_B)$ have the same isolated eigenvalues of finite multiplicity [14]. In particular, we have $\mathcal{L}_\delta \psi = \psi$ for some $\psi \in C^\alpha(X_A)$ with $\psi|_{X_B} > 0$.

Lemma 3 ([6]). *The extension of ψ to X_A is strictly positive.* ∎

A simple argument then shows that the corresponding eigenmeasure can be identified with ν by setting $\nu(X_A \backslash X_B) = 0$.

In view of the above we may write, for s close to δ, $\mathcal{L}_s = \rho(s)\,\pi_s + Q_s$, where π_s is the projection onto the eigenspace associated to $\rho(s)$ and where the spectral radius of Q_s is bounded away from 1 from above. In particular, $\pi_\delta(\phi) = (\int \phi \, d\nu)\,\psi$.

Let $C = C(i_0, ..., i_m)$. To prove Proposition 1, we shall consider the "restricted Poincaré series"

$$\eta_C(s) \; = \sum_{\substack{g\in\Gamma \\ gq\in C}} e^{-sd(p,gq)} \; .$$

This converges to an analytic function for $\Re s > \delta$.

It is easy to see that we may rewrite $\eta_C(s)$ in the form

$$\eta_C(s) \; = \; \sum_{n=0}^{\infty} \mathcal{L}_s^n \chi(\dot{0}) \; ,$$

where χ is the characteristic function of $[i_0, ..., i_m] := \{x \in X_A \colon x_j = i_j, \; j = 0, ..., m\}$.

Combining these observations, we see that $\eta_C(s)$ has a meromorphic extension to a neighbourhood of $\Re s \geq \delta$, has no poles on $\Re s = \delta$ apart from $s = \delta$, and, for s close to δ, satisfies

$$\eta_C(s) \; = \; \frac{\displaystyle\int \chi \, d\nu \;\; \psi(\dot{0})}{\displaystyle\delta \int r \, d\nu \;\; (s-\delta)} + \omega(s) \; ,$$

where $\omega(s)$ is analytic. Noting that $\int \chi \, d\nu = \mu(C)$ and comparing with the Dirichlet series $\sum_{g\in\Gamma} e^{-sd(p,gq)} = \sum_{n=0}^{\infty} \mathcal{L}_s 1(\dot{0})$, allows us to rewrite this last expression as

$$\eta_C(s) \; = \; \frac{\mathcal{C}(p,q,\Gamma)\,\mu(C)}{s-\delta} + \omega(s) \; .$$

Applying the Ikehara Tauberian Theorem, we obtain that

$$\#\Big\{g \in \Gamma \colon \; d(p,gq) \leq T, \;\; gq \in C\Big\} \; \sim \; \mathcal{C}(p,q,\Gamma)\,\mu(C)\,e^{\delta T} \; ,$$

from which Proposition 1 follows.

Remark. It is straightforward to extend the above analysis to cover the case of a subgroup $\overline{\Gamma} \triangleleft \Gamma$ (of an even cornered Kleinian group Γ) satisfying $\Gamma/\overline{\Gamma} \cong \mathbb{Z}^k$, and obtain

$$\lim_{T\to\infty} \frac{N_{\overline{\Gamma}}^B(p,q,T)}{N_{\overline{\Gamma}}(p,q,T)} = \mu(B) \; .$$

(In this case $N_{\overline{\Gamma}}(p,q,T) \sim$ const. $e^{\delta T}/T^{k/2}$, as $T \to \infty$.) $\square$

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Richard Sharp,
Department of Mathematics, University of Manchester,
Oxford Road, Manchester M13 9PL – U.K.