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ON SECOND GRADE FLUIDS WITH VANISHING VISCOSITY

V. Busuioc

Abstract: We consider the equation of a second grade fluid with vanishing viscosity, also known as Camassa–Holm equation, and we prove local existence and uniqueness of solutions for smooth initial data. We also give a blow-up condition which implies that the solution is global in dimension 2. Finally, we prove the convergence of the solutions of second grade fluid equation to the solution of the Camassa–Holm equation as the viscosity tends to zero.

Introduction

This paper is devoted to the study of a family of incompressible, non newtonian fluids of grade two with vanishing viscosity whose flow is given by the equation

$$\partial_t(u-\alpha\,\Delta u)-\nu\,\Delta u+(u-\alpha\,\Delta u)_j\,\nabla u_j+u\,\nabla(u-\alpha\,\Delta u) \;=\; \nabla P+f\;, \tag{1}$$

where u is the velocity field, P is the pressure and the constant α is positive. We suppose that we are in the incompressible case, i.e.,

$$\operatorname{div} u=0\;. \tag{2}$$

The domain under consideration is $\mathbb{R}^n$.

We are here interested in equation (1) when $\nu=0$, that is to say

$$\partial_t(u-\alpha\,\Delta u)+(u-\alpha\,\Delta u)_j\,\nabla u_j+u\,\nabla(u-\alpha\,\Delta u) \;=\; \nabla P+f\;. \tag{3}$$

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In the one-dimensional case, this equation is the shallow water Camassa–Holm equation (see, for instance, Camassa and Holm [4] and Camassa, Holm and Hyman [5]). There is a wide literature on this equation, we refer the reader, for instance, to Constantin [9], Constantin and Escher [10]. Equation (3) can be considered as the generalization to higher dimensions of the space of the shallow water equation. In the sequel, we will refer to (3) as the Camassa–Holm equation.

On the other hand, observe that for $\alpha = 0$, equation (3) is nothing else than the classical Euler equation. It is known that the Euler equation describes the geodesics on the volume-preserving diffeomorphism group for the L^2-norm as shown in Ebin and Marsden [12], see also Chemin [6]. Let us mention that (3) also describes the geodesics on the volume-preserving diffeomorphism group but for the H^1_α-norm defined by

$$\|u\|_{H^1_\alpha} = \left(\|u\|^2_{L^2} + \alpha \,\|\nabla u\|^2_{L^2}\right)^{1/2} .$$

This was proved by Holm, Marsden and Ratiu in [15] and [16]. It is why (3) is also called the α-Euler equation (see for further details Shkoller [20]).

Fluids of second grade (or grade-two fluids) are a particular class of the non newtonian Rivlin–Ericksen fluids of differential type (see Noll and Truesdell [18]). Their general constitutive law is

$$\sigma = -p\,I + \nu A_1 + \alpha_1 A_2 + \alpha_2\, A_1^2 \;, \tag{4}$$

where σ is the stress tensor, the scalar function p represents the pressure and A_1, A_2 are defined by

$$A_1 = L + L^T \,, \qquad L_{ij} = \frac{\partial u_i}{\partial x_j} \;, \tag{5}$$

$$A_2 = \dot{A}_1 + A_1 L + L^T A_1 \;, \tag{6}$$

where the dot denotes the derivative $\partial_t + u\cdot\nabla$. The constant ν is the kinematic viscosity, α_1 and α_2 are normal stress moduli. Hence, the equation of motion of incompressible fluids of second grade is

$$\begin{cases} \operatorname{div}(-p\,I + \nu A_1 + \alpha_1 A_2 + \alpha_2\, A_1^2) + f = \dot{u} \;, \\ \operatorname{div} u = 0 \;, \end{cases} \tag{7}$$

whose unknowns are u and p. One has to add of course, initial conditions and boundary conditions if one has to solve this problem in a bounded domain Ω.

In 1974, Dunn and Fosdick [11] (see also Fosdick and Rajagopal [13]) studied the thermodynamics and stability of this type of special fluids. Their analysis established that ν, α_1 and α_2 have to verify

$$\nu \geq 0\,, \qquad \alpha_1 + \alpha_2 = 0\ , \tag{8}$$

as a consequence of Clausius–Duhem inequality, and

$$\alpha_1 \geq 0\ ,$$

if the Helmholtz free energy is minimum at the equilibrium. We will actually assume that $\alpha_1 > 0$ as if $\alpha_1 = 0$ we obtain the Navier–Stokes equations which are extensively studied. Consequently, using (5), (6), (7) and (8) one can further write $\operatorname{div}\sigma$ in the form

$$\operatorname{div}\sigma \;=\; -\nabla p + \nu\,\Delta u + \alpha\,\partial_t \Delta u + \alpha\ \operatorname{curl}\Delta u \times u\ , \tag{9}$$

where $\alpha = \alpha_1 = -\alpha_2$. Replacing (9) into (7), leads to the system

$$\begin{cases} \partial_t (u - \alpha\,\Delta u) - \nu\,\Delta u + \operatorname{curl}(u - \alpha\,\Delta u) \times u \;=\; \nabla \widetilde{P} + f\ , \\ \operatorname{div} u \;=\; 0\ , \end{cases} \tag{10}$$

where

$$\widetilde{P} \;=\; p - \frac{1}{2}\,|u|^2 - \frac{\alpha}{2}\,|\nabla u|^2 - \alpha\ u \cdot \Delta u\ .$$

An easy computation shows that the equation in (10) is of the form (1) with a modified pressure P (see, for instance, [3]).

The existence and uniqueness of solutions of (1) for a bounded domain with Dirichlet condition on the boundary $\partial\Omega$, was proved by Cioranescu and Ouazar [8]. This solution was obtained as an element of $H^3(\Omega)$. Moreover, in [8] it is also proved that in the two-dimensional case the solution is global in time, and local for small data in the three-dimensional case. This last result was improved by Cioranescu and Girault in [7], which showed that the solution in the three-dimensional case is global under some appropriate assumptions on the data. A fixed point method is used by Galdi and Sequeira [14] to obtain similar results and global existence for small 3D initial data. The proof of a priori estimates in the three-dimensional case relies on the "damping term" $-\nu\,\Delta u$. Consequently, one cannot take directly $\nu = 0$ in (1). The situation is simpler in the two-dimensional case. Indeed, the a priori estimates from [8] are independent of ν. Following the method from [8], one gets without any difficulty the existence and uniqueness of the solution of (3), belonging to $L^\infty(0,\infty; H^3(\Omega))$.

As mentioned before, we are here concerned by equation (3) in the case where Ω is the entire space $\mathbb{R}^n$. We will prove that for smooth enough data, there exists a local in time unique strong solution of (3). This solution is global in the two-dimensional case. Finally, we prove that the solution of Camassa–Holm equation (3) is the limit, when $\nu \to 0$, of the solution of (1). Let us mention that for a two-dimensional bounded domain, a convergence result in $L^\infty(0,\infty;H^3(\Omega))$ is straightforward by using the estimates from [8].

The paper is organized as follows. In Section 1 we prove some *a priori* estimates satisfied by the solutions of problem (3). These estimates imply the local existence of strong solutions; the uniqueness of the solution is also proved.

In Section 2 it is shown that if the solution fails to exist over a certain interval of time, then the supremum of the $\|\mathrm{curl}(u-\alpha\,\Delta u)\|_{L^\infty}$ has to blow up. This result is similar to that proved by Beale, Kato and Majda [1] for the Euler equations (see also [19]) and relies on a logarithmic estimate. In Section 3, we show that in the two-dimensional case the blow-up can never occur in finite time. Hence, in this case, the solution is global in time.

Finally, in Section 4, we prove the strong convergence of the solution u^ν of equation (1) to the solution u of the equation (3) when $\nu \to 0$. The convergence holds on the time interval where the local solution of the Camassa–Holm equation exists and in dimension two, on any bounded interval of time. To do so, we are first led to give existence and uniqueness theorems for equation (1) in the whole space. The existence is global in time in $\mathbb{R}^2$ and local in $\mathbb{R}^n$ for any $n > 2$.

To prove the convergence of u^ν to u in the two-dimensional case, we establish a bound for the H^s-norm of the solution u^ν that is independent of ν.

These results can be summarized in the following theorem:

Theorem. *Let $s > \frac{n}{2} + 1$, $u_0 \in H^{s+2}$, $f \in L^1(0,\infty;H^s)$. Then, there exists a unique solution u of system (3) such that*

$$u \in L^\infty(0,T;H^{s+2})\ ,$$

where

$$T = \frac{C}{\|u_0\|_{s+2} + \|f\|_{L^1(0,\infty;H^s)}}\ , \tag{11}$$

where C is a constant independent of s and the data u_0 and f.

If $T^\star$, the maximal time for which one has the existence of u, is finite, then necessarily

$$\int_0^{T^\star} \|\mathrm{curl}(u-\alpha\,\Delta u)\|_{L^\infty} \;=\; +\infty\ .$$

In the two-dimensional case, the solution is global in time, i.e.,

$$u \in L^\infty(0,+\infty; H^{s+2}) \ .$$

Consider now a family of initial data u_0^ν belonging to H^{s+2}, such that $\lim_{\nu\to 0} u_0^\nu = u_0$ in H^{s+2}. Then, when $\nu \to 0$, the solution u^ν of (1) exists at least on $(0,T)$ with T given in (11) and converges strongly to u in $L^\infty(0,T; H^{s+2-\varepsilon})$, $\forall\, \varepsilon > 0$. In the two-dimensional case, the solutions to both systems are global in time and the result of convergence holds for all $T < \infty$. ∎

1 – A local existence and uniqueness theorem

We place ourselves in $\mathbb{R}^n$, we denote by H^s the usual Sobolev space and by $\|\cdot\|_s$ the corresponding H^s norm. The following classical properties will be frequently used:

- If $s > \frac{n}{2}$ then the following embedding holds: $H^s \subset L^\infty$.
- If $s > \frac{n}{2}$ then H^s is an algebra and we have the following *tame estimates* (see Chemin [6]):

$$\|u \cdot v\|_s \leq C\Big(\|u\|_{L^\infty}\, \|v\|_s + \|u\|_s\, \|v\|_{L^\infty}\Big) \ . \tag{12}$$

- If $s \geq 0$ and D is a partial derivative of order less or equal to s, then we have the following *commutator* type estimate (see Klainermann and Majda [17]):

$$\left|\int D(u\cdot\nabla v)\, Dv\right| \leq C\, \|v\|_s \Big(\|u\|_s\, \|\nabla v\|_{L^\infty} + \|\nabla u\|_{L^\infty}\, \|v\|_s\Big) \ . \tag{13}$$

Let us consider the system

$$\left\{\begin{aligned} \partial_t v + v_j \nabla u_j + u\, \nabla v &= \nabla p + f \ , \\ v &= u - \alpha\, \Delta u \ , \\ \operatorname{div} u &= 0 \ , \\ u(0,x) &= u_0(x) \ . \end{aligned}\right. \tag{14}$$

Theorem 1.1. *Let $s > \frac{n}{2} + 1$, $u_0 \in H^{s+2}$, $f \in L^1(0,\infty; H^s)$. Then, there exist a constant C and an unique local solution of system (14) such that*

$$u \in L^\infty(0,T; H^{s+2}) ,$$

where

$$T = \frac{C}{\|u_0\|_{s+2} + \|f\|_{L^1(0,\infty;H^s)}} . \tag{15}$$

Proof of the existence: Let D be a partial derivative of order not greater than s, $D = D^\beta$, $|\beta| \le s$. Applying D to the equation of v and multiplying by Dv yields

$$\begin{aligned} \partial_t \|Dv\|_{L^2}^2 &\le \left| \int D(v_j \nabla u_j)\, Dv \right| + \left| \int D(u\,\nabla v)\, Dv \right| + \left| \int Df\, Dv \right| \\ &\le \left| \int D(u_j \nabla u_j)\, Dv \right| + \left| \alpha \int D(\Delta u_j \nabla u_j)\, Dv \right| \\ &\quad + \left| \int D(u\,\nabla v)\, Dv \right| + \left| \int Df\, Dv \right| . \end{aligned} \tag{16}$$

We now estimate each of the integrals from the right-hand side. An integration by parts shows that the first term vanishes:

$$I_1 = \left| \int D(u_j \nabla u_j)\, Dv \right| = \frac{1}{2} \left| \int D\,\nabla(|u|^2)\, Dv \right| = \frac{1}{2} \left| \int D(|u|^2)\, D\,\mathrm{div}\, v \right| = 0 , \tag{17}$$

since v is divergence free.

The second integral

$$I_2 = \left| \int D(\Delta u_l \nabla u_l)\, Dv \right| ,$$

can be written as a sum of terms of the type

$$\left| \int D(\partial_i u_l\, \partial_j u_l)\, D\partial_k v \right| .$$

Indeed, integrating by parts we have

$$\begin{aligned} I_2 &= \left| \sum_{i,j,l} \int D(\partial_i^2 u_l\, \partial_j u_l)\, Dv_j \right| \\ &= \left| \sum_{i,j,l} \int D(\partial_i u_l\, \partial_j u_l)\, D\partial_i v_j + \sum_{i,j,l} \int D(\partial_i u_l\, \partial_j\, \partial_i u_l)\, Dv_j \right| . \end{aligned}$$

The first term is now of the required form. The second one vanishes, since the equality

$$\partial_i u_l\, \partial_j\, \partial_i u_l \;=\; \frac{1}{2}\, \partial_j (\partial_i u_l)^2$$

implies that

$$\begin{aligned} 2 \sum_{i,j,l} \int D(\partial_i u_l\, \partial_j\, \partial_i u_l)\, D v_j \;&=\; -\sum_{i,j,l} \int D(\partial_i u_l)^2\, D\partial_j v_j \\ &=\; -\sum_{i,l} \int D(\partial_i u_l)^2\, D(\operatorname{div} v) \;=\; 0 \;. \end{aligned}$$

Observe further that, using (12), one has the estimate

$$\begin{aligned} \left| \int D(\partial_i u_l\, \partial_j u_l)\, D\partial_k v \right| \;&=\; \left| \int \partial_k D(\partial_i u_l\, \partial_j u_l)\, D v \right| \\ &\le\; \|\partial_k D(\partial_i u_l\, \partial_j u_l)\|_{L^2}\, \|Dv\|_{L^2} \\ &\le\; \|\partial_i u_l\, \partial_j u_l\|_{s+1}\, \|u\|_{s+2} \\ &\le\; C\, \|u\|_{s+2}^2\, \|\nabla u\|_{L^\infty} \;, \end{aligned} \tag{18}$$

so the same inequality holds for I_2.

The third integral is estimated with the commutator inequality (13). One has

$$\begin{aligned} I_3 \;=\; \left| \int D(u\, \nabla v)\, D v \right| \;&\le\; C\, \|v\|_s \Big(\|u\|_s\, \|\nabla v\|_{L^\infty} + \|\nabla u\|_{L^\infty}\, \|v\|_s \Big) \\ &\le\; C\, \|u\|_{s+2}^2 \Big(\|\nabla v\|_{L^\infty} + \|\nabla u\|_{L^\infty} \Big) \;. \end{aligned} \tag{19}$$

Finally, one can write the following estimate for the last term in (16):

$$\left| \int D f\, D v \right| \;\le\; \|f\|_s\, \|u\|_{s+2} \;. \tag{20}$$

Using now relations (17), (18), (19) and (20) in (16) one obtains

$$\partial_t \|Dv\|_{L^2}^2 \;\le\; C\, \|u\|_{s+2}^2 \Big(\|\nabla v\|_{L^\infty} + \|\nabla u\|_{L^\infty} \Big) + \|f\|_s\, \|u\|_{s+2} \;.$$

Summing over all partial derivatives D yields

$$\partial_t \|u\|_{s+2}^2 \;\le\; C\, \|u\|_{s+2}^2 \Big(\|\nabla v\|_{L^\infty} + \|\nabla u\|_{L^\infty} \Big) + C\, \|f\|_s\, \|u\|_{s+2} \;,$$

which implies the following *a priori* estimate

$$\partial_t \|u\|_{s+2} \;\le\; C\, \|u\|_{s+2} \Big(\|\nabla v\|_{L^\infty} + \|\nabla u\|_{L^\infty} \Big) + C\, \|f\|_s \;. \tag{21}$$

We now prove that one can remove $\|\nabla u\|_{L^\infty}$ from the above inequality. Since $v = u - \alpha\,\Delta u$, we infer that ∇u can be obtained from ∇v via a Bessel potential:

$$\nabla u \;=\; (I - \alpha\,\Delta)^{-1}\,\nabla v\ .$$

On the other hand, according to Proposition 2, page 132 of Stein [21], the Bessel potential $(I - \alpha\,\Delta)^{-1}$ is an operator of convolution with a L^1 function. Therefore, Young's inequality implies that

$$\|\nabla u\|_{L^\infty} \le C\,\|\nabla v\|_{L^\infty}\ . \tag{22}$$

So, we obtain from (21) that

$$\partial_t \|u\|_{s+2} \;\le\; C\,\|u\|_{s+2}\,\|\nabla v\|_{L^\infty} + C\,\|f\|_s\ . \tag{23}$$

Using that $H^{s-1} \subset L^\infty$ we finally get

$$\partial_t \|u\|_{s+2} \;\le\; C_1\,\|u\|_{s+2}^2 + C_1\,\|f\|_s\ , \tag{24}$$

for some constant C_1.

At this stage of the proof we are going to estimate the maximal time existence of the solution. Let

$$T \;=\; \sup\Big\{t\ \text{ such that }\ \|u(\tau)\|_{s+2} \le K\|u_0\|_{s+2},\ \forall\, 0 \le \tau \le t\Big\}\ ,$$

where

$$K \;=\; 4 + \frac{8\,C_1\,\|f\|_{L^1(0,\infty;H^s)}}{\|u_0\|_{s+2}}\ .$$

We want to show that

$$T \;\ge\; \frac{1}{16\,C_1\,\|u_0\|_{s+2} + 64\,C_1^2\,\|f\|_{L^1(0,\infty;H^s)}}\ .$$

We will show it by contradiction. Assume that

$$T \;<\; \frac{1}{16\,C_1\,\|u_0\|_{s+2} + 64\,C_1^2\,\|f\|_{L^1(0,\infty;H^s)}}\ .$$

Consequently,

$$T \;<\; \frac{1}{8\,C_1\,K\,\|u_0\|_{s+2}}\ .$$

Then, for $t \in [0, T]$, the *a priori* estimate (24) and the definition of T imply

$$\partial_t \|u\|_{s+2} \;\le\; C_1\,K^2\,\|u_0\|_{s+2}^2 + C_1\,\|f\|_s\ .$$

Integrating from 0 to T gives

$$\begin{aligned}\|u(T)\|_{s+2} &\le \|u_0\|_{s+2} + C_1 K^2 \|u_0\|_{s+2}^2\, T + C_1 \|f\|_{L^1(0,\infty;H^s)} \\ &\le \|u_0\|_{s+2} + C_1 K^2 \|u_0\|_{s+2}^2 \frac{1}{8\, C_1 K \|u_0\|_{s+2}} + \frac{K\,\|u_0\|_{s+2}}{8} \\ &= \|u_0\|_{s+2} + \frac{K}{8}\|u_0\|_{s+2} + \frac{K}{8}\|u_0\|_{s+2} \\ &\le \frac{K}{2}\|u_0\|_{s+2}\ ,\end{aligned}$$

which contradicts the definition of T. The existence now follows from a "modified Galerkin method", also known as Friedrichs method, which will be sketched later. ∎

Proof of the uniqueness: Let u^1 and u^2 be two solutions with the same initial data

$$u^1(0) = u^2(0)\ .$$

Subtracting the equations verified by u^1 and u^2 gives

$$\begin{aligned}&\partial_t(w - \alpha\,\Delta w) \;+\; v_j^1\,\nabla w_j \;+\; (w_j - \alpha\,\Delta w_j)\,\nabla u_j^2 \\ &\qquad + u^2\,\nabla(w - \alpha\,\Delta w) \;+\; w\,\nabla v^1 \;=\; \nabla(p^1 - p^2)\ ,\end{aligned}$$

where $w = u^1 - u^2$. Multiplying by $w - \alpha\,\Delta w$ and integrating in space yields, after some classical estimates,

$$\begin{aligned}\partial_t \|w\|_2^2 &\le C\,\|v^1\|_s\,\|w_j\|_1\,\|w\|_2 \;+\; C\,\|w\|_{L^2}\,\|w\|_2\,\|v^1\|_s \;+\; C\,\|w\|_2^2\,\|\nabla u_j^2\|_{s-1} \\ &\le C\,\|w\|_2^2\left(\|u^1\|_{s+2} + \|u^2\|_{s+2}\right)\ ,\end{aligned}$$

from which, by Gronwall's inequality, one has

$$\|w\|_2^2 \;\le\; \|w_0\|_2^2\ \exp\!\left(C\int_0^t \left(\|u^1\|_{s+2} + \|u^2\|_{s+2}\right)\right)\ .$$

This implies the result since $w_0 = 0$. ∎

Sketch of the "Galerkin method": We follow the proof of the short-time existence of strong solutions for quasi-linear symmetric hyperbolic systems given in Taylor [22]. We denote by a Friedrichs mollifier the operator J_ε given by the convolution:

$$J_\varepsilon u \;=\; j_\varepsilon * u\ ,$$

where

$$j_\varepsilon(x) \;=\; \varepsilon^{-n}\, j\,(\varepsilon^{-1}\,x)\ ,$$

and j is a function whose Fourier transform is a compactly supported smooth function equal to 1 in a neighborhood of 0. We now consider the system

$$(S_\varepsilon)\quad \begin{cases} \partial_t v^\varepsilon + J_\varepsilon\Big(J_\varepsilon v_j^\varepsilon\,\nabla J_\varepsilon u_j^\varepsilon + J_\varepsilon u_\varepsilon\,\nabla J_\varepsilon v_\varepsilon\Big) \;=\; \nabla p^\varepsilon + J_\varepsilon f \;, \\ v^\varepsilon \;=\; u^\varepsilon - \alpha\,\Delta u^\varepsilon \;, \\ \operatorname{div} u^\varepsilon \;=\; 0 \;, \\ u^\varepsilon(0,x) \;=\; J_\varepsilon u_0(x) \;. \end{cases}$$

One can apply the divergence to the equation of v^ε to find Δp^ε in terms of u^ε, that is p^ε in terms of u^ε without time derivatives. In this way, as in [22], or simply by applying the projection on the divergence-free vector fields to (S_ε), system (S_ε) can be regarded as a system of ODEs for u^ε. By Cauchy's theorem, we know that this sytem has a unique smooth solution. The *a priori* estimates previously proved implies that u^ε exists up to the time given in the statement of the theorem and that

$$\|u^\varepsilon\|_{L^\infty(0,T;H^{s+2})}$$

is bounded independently of ε. It is classical that this is enough to pass to the limit (see [22]). ∎

Remark 1.1. It is easy to prove, as in [22], that relation (24) and the equation imply a stronger regularity result for the solution u:

$$u \,\in\, C([0,T];H^{s+2}) \cap C^1([0,T];H^{s+1}) \;,$$

provided that f is regular enough (continous in time). □

2 – A necessary condition for blow-up

Let us first notice that if $T^\star$, the maximal time-existence of the solution given in Theorem 1.1 is finite, then we must necessarily have

$$\lim_{t\to T^\star} \|u(t)\|_{s+2} \;=\; +\infty \;.$$

Indeed, suppose that there exists $t_k \to T^\star$ such that $\|u(t_k)\|_{s+2}$ is bounded independently of k. Theorem 1.1 gives a local solution starting at each t_k whose time existence may be chosed independent of k (see (15)). Since $t_k \to T^\star$, it follows that the solution may be extended over $T^\star$ but this contradicts the maximality of $T^\star$.

In fact, a stronger blow-up condition holds:

Proposition 2.1. *Assume that $T^\star$, the maximal time-existence of the solution given in Theorem 1.1, is finite. Then the following relation:*

$$\int_0^{T^\star} \|\operatorname{curl} v\|_{L^\infty} \;=\; +\infty \tag{25}$$

holds.

Proof: Applying Gronwall's lemma in (23), we get

$$\begin{aligned}\|u(t)\|_{s+2} \;\le\; &\|u_0\|_{s+2}\, \exp\biggl(C\int_0^t \|\nabla v\|_{L^\infty}\biggr)\\ &+ C\,\exp\biggl(C\int_0^t \|\nabla v\|_{L^\infty}\biggr) \int_0^t \|f\|_s\;.\end{aligned} \tag{26}$$

The blow-up condition (25) will be proved by contradiction. To do so, let us introduce the function

$$\phi(t) \;=\; \int_0^t \|\nabla v\|_{L^\infty}\;. \tag{27}$$

The divergence-free condition on v enables us to express v in terms of $\operatorname{curl} v$, so one has (see [6])

$$\phi'(t) \;=\; \|\nabla v(t)\|_{L^\infty} \le \sum_{i,j} \|\partial_i\partial_j\Delta^{-1}\operatorname{curl} v\|_{L^\infty}\;. \tag{28}$$

In the sequel, in order to simplify the computations, we introduce the notation $\omega = \operatorname{curl} v$.

We will now make use of the following logarithmic inequality (see, for instance, Chemin [6]):

$$\|\partial_i\partial_j\Delta^{-1}\omega\|_{L^\infty} \;\le\; C\,\|\omega\|_{L^\infty} \log\biggl(e+\frac{\|\omega\|_r}{\|\omega\|_{L^\infty}}\biggr) + C\,\|\omega\|_{L^2}\;, \tag{29}$$

where $r > \frac{n}{2}$.

It is easy to bound $\|\omega\|_{L^2}$. The equation of ω is

$$\partial_t\omega + u\,\nabla\omega + \omega\,\nabla u \;=\; \operatorname{curl} f\;,$$

if the dimension is 3. In dimension 2, the last term of the left-hand side dissapears (see [7], [8]). In both cases, multiplying by ω, integrating in space and using that div $u = 0$ yields

$$\begin{aligned}\frac{1}{2}\,\partial_t\|\omega\|_{L^2}^2 &\le \|\omega\|_{L^2}\,\|\omega\|_{L^\infty}\,\|\nabla u\|_{L^2} + \|\omega\|_{L^2}\,\|\operatorname{curl} f\|_{L^2}\\ &\le \|\omega\|_{L^2}\,\|\omega\|_{L^\infty}\,\|\nabla v\|_{L^2} + \|\omega\|_{L^2}\,\|\operatorname{curl} f\|_{L^2}\\ &\le C\,\|\omega\|_{L^2}^2\,\|\omega\|_{L^\infty} + \|\omega\|_{L^2}\,\|f\|_s\ ,\end{aligned}$$

where we have used that $\|\nabla v\|_{L^2} \le C\,\|\operatorname{curl} v\|_{L^2}$; this can be immediately deduced by using Plancherel's theorem or simply by using the more general relation for L^a, $\forall\, 1 < a < \infty$, which is proved in [6]. Gronwall's lemma now gives

$$\|\omega(t)\|_{L^2} \le \Big(\|\omega_0\|_{L^2} + \int_0^t \|f\|_s\Big)\,\exp\Big(C\int_0^t \|\omega\|_{L^\infty}\Big)\ . \tag{30}$$

We now go back to (29) and we use the fact that for all $\alpha > 0$, the function

$$x \longrightarrow x\,\log\Big(e + \frac{\alpha}{x}\Big)$$

is increasing to obtain

$$\begin{aligned}\|\partial_i\partial_j\Delta^{-1}\omega\|_{L^\infty} &\le C\Big(1 + \|\omega\|_{L^\infty}\Big)\,\log\Big(e + \frac{\|\omega\|_r}{1 + \|\omega\|_{L^\infty}}\Big) + C\,\|\omega\|_{L^2}\\ &\le C\Big(1 + \|\omega\|_{L^\infty}\Big)\,\log\Big(e + \|\omega\|_r\Big) + C\,\|\omega\|_{L^2}\ .\end{aligned}$$

Choosing $r = s - 1$ and recalling (28) and (26) yields

$$\begin{aligned}\phi'(t) &\le C\Big(1 + \|\omega\|_{L^\infty}\Big)\,\log\Big(e + \|\omega\|_{s-1}\Big) + C\,\|\omega\|_{L^2}\\ &\le C\Big(1 + \|\omega\|_{L^\infty}\Big)\,\Big(1 + \log_+\|v\|_s\Big) + C\,\|\omega\|_{L^2}\\ &\le C\Big(1 + \|\omega\|_{L^\infty}\Big)\,\Big(1 + \log_+\|u_0\|_{s+2} + \phi(t) + \log_+\Big(\int_0^t \|f\|_s\Big)\Big) + C\|\omega\|_{L^2}\ ,\end{aligned}$$

where $\log_+ = \max(\log, 0)$. Therefore, by using (30), we get

$$\phi'(t) \le C\Big(1 + \|\omega\|_{L^\infty}\Big)\,\Big(\phi(t) + g(t)\Big)\ ,$$

where

$$g(t) = 1 + \log_+\|u_0\|_{s+2} + \log_+\Big(\int_0^t \|f\|_s\Big) + \Big(\|\omega_0\|_{L^2} + \int_0^t \|f\|_s\Big)\,\exp\Big(C\int_0^t \|\omega\|_{L^\infty}\Big)$$

is a function which is bounded as long as $\int_0^t \|\omega\|_{L^\infty}$ is bounded. Gronwall's lemma gives

$$\phi(t) \le C\int_0^t \Big(1 + \|\omega\|_{L^\infty}\Big)\,g\,d\tau\ \exp\Big(C\int_0^t \Big(1 + \|\omega\|_{L^\infty}\Big)\Big)\ . \tag{31}$$

Suppose that (25) does not hold, i.e.,

$$\int_0^{T^\star} \|\omega\|_{L^\infty} < \infty ,$$

This would imply that

$$\phi(t) < \infty , \qquad \forall\, t \le T^\star .$$

Consequently, from (26) and (27) we have

$$\|u(t)\|_{s+2} \le C_1 , \qquad \forall\, t < T^\star ,$$

where C_1 is a constant depending on $\|u_0\|_{s+2}$, $\|f\|_{L^1(0,\infty;H^s)}$ and $\int_0^{T^\star} \|\omega\|_{L^\infty}$. But, as noticed at the begining of the section, this contradicts the maximality of $T^\star$. ∎

3 – The global existence in dimension 2

The equation of the curl in dimension two implies that the blow-up condition proved in the previous section can not occur in finite time.

Theorem 3.1. *In dimension two, the solution given in Theorem 1.1 is global in time.*

Proof: In order to prove the global existence, it is sufficient to prove that

$$\|\omega(t)\|_{L^\infty} \le \|\omega_0\|_{L^\infty} + \int_0^t \|\mathrm{curl}\, f\|_{L^\infty} , \tag{32}$$

since it would imply that

$$\int_0^t \|\omega\|_{L^\infty} < \infty , \qquad \forall\, t < \infty ,$$

which contradicts the blow-up condition (25). To this end, we start by giving the equation satisfied by ω,

$$\partial_t \omega + u\, \nabla\omega = \mathrm{curl}\, f . \tag{33}$$

It is obtained by applying the curl in (14). Let us note that the equation of the curl has this form only in dimension two.

We now define the flow of u.

Definition 3.1. The flow of u, denoted by ψ, is a continuous application from $\mathbb{R}\times\mathbb{R}^2$ to $\mathbb{R}^2$ such that

$$\begin{cases} \partial_t \psi = u(t,\psi) , \\ \psi(0,x) = x . \; \square \end{cases}$$

It is well-known that the divergence-free condition on u implies that, for each t, the flow is a diffeomorphism which preserves the measure (see [6]). The definition of the flow and relation (33) shows that $\operatorname{curl} v$ is transported by the flow

$$\partial_t(\omega(t,\psi)) \;=\; \partial_t\omega + \partial_1\omega\,\partial_t\psi_1 + \partial_2\omega\,\partial_t\psi_2 \;=\; \partial_t\omega + u\,\nabla\omega \;=\; \operatorname{curl} f(t,\psi)\ .$$

Consequently,

$$\omega(t,\psi(t,x)) \;=\; \omega_0(x) + \int_0^t \operatorname{curl} f(\tau,\psi(\tau,x))\,d\tau\ ,$$

and we obtain (32) by taking the L^∞ norm in space. ∎

4 – Limit of second grade fluids as the viscosity tends to zero

In absence of boundaries, it is easy to prove a convergence theorem for the solutions of (1) to the solution of Camassa–Holm equation (3), when the viscosity goes to 0.

Theorem 4.1. *Consider a family of initial data u_0^ν belonging to H^{s+2}, $s > \frac{n}{2}+1$, such that*

$$\lim_{\nu\to 0} u_0^\nu = u_0 \quad \text{strongly in}\ \ H^{s+2}\ .$$

Let u^ν be the solution of the second grade equation

$$\begin{cases} \partial_t(u-\alpha\,\Delta u) - \nu\,\Delta u + (u-\alpha\,\Delta u)_j\nabla u_j + u\,\nabla(u-\alpha\,\Delta u) - \nabla P \;=\; f\ , \\ \operatorname{div} u = 0\ , \\ u(x,0) = u_0\ . \end{cases} \tag{34}$$

Then, when $\nu\to 0$, u^ν exists at least on the time interval given in Theorem 1.1 and moreover,

$$u^\nu \to u \quad \text{strongly in}\ \ L^\infty(0,T;H^{s+2-\varepsilon}),\quad \forall\,\varepsilon>0\ ,$$

where u is the solution of system (13), given in Theorem 1.1 and T is given in (15).

In $\mathbb{R}^2$, the solutions of both systems are global in time and the convergence result holds for all $T<\infty$.

Proof: The assertion on the short-time existence of u^ν follows trivially from the convergence of the initial data and from the remark that when making energy estimates, the viscosity term has the good sign, so it can be neglected to obtain the same estimates on the short-time existence as in the zero-viscosity case.

In dimension two, further uniform estimate for $\operatorname{curl} v$, namely (32), has been used to deduce the global existence of the solution of the Camassa–Holm equation. Lemma 4.1 below proves that such an estimate holds also for the second grade fluid. This *a priori* estimate will imply the global existence of the solution of system (34), since they can be used in the same way in the Galerkin method as we did in the proof of Theorem 2.1. Note that the additional viscosity term is linear, so it does not count in the limiting process. The uniqueness also holds true since it was proved via energy estimates. Therefore, we obtain a global existence and uniqueness theorem for solutions of system (34) in $\mathbb{R}^2$.

From the estimates in the proof of Theorem 1.1 one can deduce that the solution u^ν is bounded in $L^\infty(0,T;H^{s+2})$ independently of ν, where T is given in Theorem 1.1. In dimension two, Lemma 4.1 as well as relations (26), (27) and (31) show that the same holds for all $T<\infty$.

We now prove that if T is such that u^ν is bounded in $L^\infty(0,T;H^{s+2})$, then u^ν converges to u, strongly in $L^\infty(0,T;H^{s+2-\varepsilon})$, $\forall\, \varepsilon>0$. To do so, it is sufficient to prove that u^ν converges to u in $L^\infty(0,T;H^2)$. The result will follow from the following well-known interpolation inequality:

$$\|u\|_{s+2-\varepsilon} \;\le\; \|u\|_2{}^{\frac{\varepsilon}{s}}\, \|u\|_{s+2}{}^{1-\frac{\varepsilon}{s}} \;.$$

Therefore, it is sufficient to prove that $v^\nu\to v$ in $L^\infty([0,T];L^2)$. In order to estimate $v^\nu-v$ we subtract the equations satisfied by v^ν and v to obtain:

$$\begin{aligned}\partial_t(v^\nu-v)-\nu\,\Delta u^\nu+({v_j}^\nu-v_j)\,\nabla u_j^\nu+v_j\nabla(u_j^\nu-u_j)+u^\nu\,\nabla(v^\nu-v)+(u^\nu-u)\,\nabla v \;&=\\ =\;\nabla(p^\nu-p)\;.&\end{aligned}$$

Multiplying by $v^\nu-v$ and integrating in space gives

$$\begin{aligned}\partial_t\|v^\nu-v\|_{L^2}^2 \;\le\; \nu\left|\int \Delta u^\nu (v^\nu-v)\right| \;+\; \left|\int (v_j^\nu-v_j)\,\nabla u_j^\nu\,(v_j^\nu-v_j)\right|&\\ +\;\left|\int v_j\,\nabla(u_j^\nu-u_j)\,(v^\nu-v)\right| \;+\; \left|\int (u^\nu-u)\,\nabla v\,(v^\nu-v)\right|&\;.\end{aligned}$$

Let us bound the right-hand side. One has

$$\left|\int \Delta u^\nu\,(v^\nu-v)\right| \;\le\; \|u^\nu\|_{s+2}\,\|v^\nu-v\|_{L^2}\;.$$

Clearly

$$\left|\int (v_j^\nu - v_j)\, \nabla u_j^\nu\, (v_j^\nu - v_j)\right| \le C\, \|u^\nu\|_{s+2}\, \|v^\nu - v\|_{L^2}^2 \ .$$

We also have that

$$\left|\int v_j\, \nabla (u_j^\nu - u_j)\, (v^\nu - v)\right| \le C\, \|u\|_{s+2}\, \|v^\nu - v\|_{L^2}^2 \ .$$

and that

$$\left|\int (u^\nu - u)\, \nabla v\, (v^\nu - v)\right| \le C\, \|u\|_{s+2}\, \|v^\nu - v\|_{L^2}^2 \ .$$

Putting together the above inequalities yields

$$\partial_t \|v^\nu - v\|_{L^2}^2 \le C\,\nu\, \|u^\nu\|_{s+2}\, \|v^\nu - v\|_{L^2} + C\, \|u\|_{s+2}\, \|v^\nu - v\|_{L^2}^2 \ . \tag{35}$$

Let K be such that

$$\|u^\nu\|_{L^\infty(0,T;H^{s+2})} + \|u\|_{L^\infty(0,T;H^{s+2})} \le K \ .$$

It follows from (35) that

$$\partial_t \|v^\nu - v\|_{L^2} \le C\, K\Big(\nu + \|v^\nu - v\|_{L^2}\Big) \ ,$$

or, equivalently,

$$\partial_t \Big(\log\Big(\nu + \|v^\nu - v\|_{L^2}\Big)\Big) \le CK \ .$$

Integrating in time yields

$$\log \frac{\nu + \|v^\nu - v\|_{L^2}}{\nu} \le CK\, t \ ,$$

so

$$\|v^\nu - v\|_{L^2} \le \nu\Big(\exp(CK\, t) - 1\Big) \le \nu\Big(\exp(CKT) - 1\Big) \ .$$

Taking the upper bound in t implies

$$\|v^\nu - v\|_{L^\infty(0,T;L^2)} \le \nu\Big(\exp(CKT) - 1\Big) \ ,$$

which gives

$$v^\nu \to v \quad \text{in} \ \ L^\infty(0,T;L^2) \ ,$$

and this ends the proof of Theorem 4.1. ∎

It remains to prove *a priori* estimates in $\mathbb{R}^2$ for the solutions of (1). These are given by the following lemma:

Lemma 4.1. *Consider a two-dimensional solution of the second grade equation (34) with $u_0 \in H^{s+2}$. There exists a constant C independent of the viscosity ν such that*

$$\|\operatorname{curl} v(t)\|_{L^\infty} \le \left(\|\operatorname{curl} v(0)\|_{L^\infty} + \int_0^t \|\operatorname{curl} f\|_{L^\infty}\right) e^{Ct\nu/\alpha} .$$

Proof: Applying the curl to relation (34), one finds the following equation for $\operatorname{curl} v$:

$$\partial_t \operatorname{curl} v + u \nabla \operatorname{curl} v - \nu \Delta \operatorname{curl} u = \operatorname{curl} f .$$

As in (32), we deduce that

$$\|\operatorname{curl} v(t)\|_{L^\infty} \le \|\operatorname{curl} v(0)\|_{L^\infty} + \int_0^t \|\nu \Delta \operatorname{curl} u + \operatorname{curl} f\|_{L^\infty} . \tag{36}$$

We know that $v = u - \alpha \Delta u$. Taking the curl we get

$$\operatorname{curl} v = \operatorname{curl} u - \alpha \Delta \operatorname{curl} u ,$$

which implies

$$\Delta \operatorname{curl} u = \frac{1}{\alpha} (\operatorname{curl} u - \operatorname{curl} v) .$$

Using this in (36), one obtains

$$\begin{aligned}\|\operatorname{curl} v(t)\|_{L^\infty} \le\; & \|\operatorname{curl} v(0)\|_{L^\infty} \\ & + \frac{\nu}{\alpha} \int_0^t \Big(\|\operatorname{curl} v\|_{L^\infty} + \|\operatorname{curl} u\|_{L^\infty}\Big) + \int_0^t \|\operatorname{curl} f\|_{L^\infty} .\end{aligned}$$

Since $\operatorname{curl} u$ is obtained from $\operatorname{curl} v$ via a Bessel potential,

$$\operatorname{curl} u = (I - \alpha \Delta)^{-1} \operatorname{curl} v ,$$

we have as in (22) that

$$\|\operatorname{curl} u\|_{L^\infty} \le C \|\operatorname{curl} v\|_{L^\infty} .$$

Therefore

$$\|\operatorname{curl} v(t)\|_{L^\infty} \le \|\operatorname{curl} v(0)\|_{L^\infty} + C \frac{\nu}{\alpha} \int_0^t \|\operatorname{curl} v\|_{L^\infty} + \int_0^t \|\operatorname{curl} f\|_{L^\infty} . \tag{37}$$

Let

$$h(t) = \|\operatorname{curl} v(0)\|_{L^\infty} + \int_0^t \|\operatorname{curl} f\|_{L^\infty} .$$

We have from (37) that

$$\partial_t\Big(e^{-Ct\nu/\alpha}\int_0^t \|\operatorname{curl} v\|_{L^\infty}\Big) \;=\; \Big(\|\operatorname{curl} v\|_{L^\infty} - C\,\frac{\nu}{\alpha}\int_0^t \|\operatorname{curl} v\|_{L^\infty}\Big)\,e^{-Ct\nu/\alpha}$$
$$\le\; h(t)\,e^{-Ct\nu/\alpha}\;.$$

Integrating in time yields

$$\int_0^t \|\operatorname{curl} v\|_{L^\infty} \;\le\; \int_0^t h(\tau)\,e^{C(t-\tau)\nu/\alpha}\,d\tau \;\le\; h(t)\,(e^{Ct\nu/\alpha}-1)\;\frac{\alpha}{\nu\,C}\;.$$

Using this in (37) gives

$$\|\operatorname{curl} v(t)\|_{L^\infty} \;\le\; \Big(\|\operatorname{curl} v(0)\|_{L^\infty} + \int_0^t \|\operatorname{curl} f\|_{L^\infty}\Big)\,e^{Ct\nu/\alpha}\;,$$

which is the desired inequality. ∎

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Valentina Busuioc,
Laboratoire d'Analyse Numérique, Université Pierre et Marie Curie, BC 187,
175, rue Chevaleret, 75252 Paris – FRANCE
E-mail: `busuioc@ann.jussieu.fr`