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GLOBAL EXISTENCE AND UNIFORM STABILIZATION OF A NONLINEAR TIMOSHENKO BEAM

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Abstract: We study the global existence and the large time behavior of the system governing the non-linear vibrations of a Timoshenko beam. For small initial data we prove global existence of strong solutions and exponential decay of the energy.

1 – Introduction

Our purpose in this paper is to prove the global existence and uniform stabilization of solutions to a nonlinear problem governing nonlinear vibrations of a Timoshenko beam (see [6], [11] for furher discussion on the model). Stabilization of linear or nonlinear Timonshenko beams, has been widley studied in literature (see [4], [6] and [10]). In the present paper, we tackle the same problem but with a dissipation distributed in the whole domain. The equations are:

$$\begin{aligned}\partial_t^2\, w(x,t) \;=\; &-c\,d\,\partial_x\Big(\psi(x,t)-\partial_x\, w\Big)+b\bigg(\int_\Omega(\partial_x\, w)^2\,(y,t)\;dy\bigg)\,\partial_x^2\, w(x,t)\\ &-\alpha\,\partial_t\, w\,, \qquad \Omega{\times}(0,\infty)\;,\end{aligned} \tag{1.1}$$

$$\frac{1}{c}\,\partial_t^2\psi(x,t) \;=\; \partial_x^2\psi(x,t)-c\,d\,\Big(\psi(x,t)-\partial_x w(x,t)\Big)-\beta\,\partial_t\psi\,, \quad \Omega{\times}(0,\infty)\;, \tag{1.2}$$

$$\begin{aligned}w(x,0)=w_0(x),\quad \partial_t\, w(x,0)=w_1(x)\,, \qquad \Omega\;,\\ \psi(x,0)=\psi_0(x),\quad \partial_t\,\psi(x,0)=\psi_1(x)\,, \qquad \Omega\;,\end{aligned} \tag{1.3}$$

$$w(0,t)=w(1,t)=0\,, \qquad (0,\infty)\;, \tag{1.4}$$

$$\partial_x\,\psi(0,t)=\partial_x\,\psi(1,t)=0\,, \qquad (0,\infty)\;, \tag{1.5}$$

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where $\Omega =]0,1[$ and b,c,d,α,β are strictly positive constants (α and β depend on the control device, and see [3], [11] for physical significations of b,c,d). We denote by $w(x,t)$ the deflection of the beam from the equilibrium line and by $\psi(x,t)$ the slope of the deflection curve, for the precise meaning of ψ see [9].

Our paper is organized as follows. In the first section, we prove, as in [4], existence and regularity results for the linear problem related to (1.1)–(1.5) and then, by a fixed point approach, we show existence results for the nonlinear problem (1.1)–(1.5). The second and third sections are devoted to prove the main results in this paper: the global existence of strong solutions and the uniform stabilization.

2 – Local existence

We first consider the following linear problem:

$$\partial_t^2 w(x,t) = -c\,d\,\partial_x \psi(x,t) + \lambda(t)\,\partial_x^2 w(x,t) - \alpha\,\partial_t w\,, \qquad \Omega\times(0,\infty)\,, \tag{2.1}$$

$$\frac{1}{c}\,\partial_t^2\psi(x,t) = \partial_x^2\psi(x,t) - c\,d\,\Big(\psi(x,t) - \partial_x w(x,t)\Big) - \beta\,\partial_t\psi\,, \qquad \Omega\times(0,\infty)\,, \tag{2.2}$$

$$\begin{aligned} w(x,0) = w_0(x),\quad \partial_t\, w(x,0) = w_1(x)\,, \qquad \Omega\,,\\ \psi(x,0) = \psi_0(x),\quad \partial_t\,\psi(x,0) = \psi_1(x)\,, \qquad \Omega\,, \end{aligned} \tag{2.3}$$

$$w(0,t) = w(1,t) = 0\,, \qquad (0,\infty)\,, \tag{2.4}$$

$$\partial_x\,\psi(0,t) = \partial_x\,\psi(1,t)\,, \qquad (0,\infty)\,, \tag{2.5}$$

where $\lambda(t)\in\mathcal{C}^1([0,T])$, $\lambda(t)\ge c\,d>0$ and c,d are two strictly positive constants. We state and prove the following theorem:

Theorem 2.1. *Let $(w_0,w_1)\in[H^2(\Omega)\cap H_0^1(\Omega)]\times H_0^1(\Omega)$ and $(\psi_0,\psi_1)\in H\times H^1(\Omega)$, then the problem (2.1)–(2.5) admits a unique solution (w,ψ) in $\Big(\mathcal{C}([0,T],H^2(\Omega)\cap H_0^1(\Omega))\cap\mathcal{C}^1([0,T],H_0^1(\Omega))\Big)\times\Big(\mathcal{C}([0,T],H)\cap\mathcal{C}^1([0,T],H^1(\Omega))\Big)$, where $H=\Big\{\Phi\in H^2(\Omega),\ \partial_x\Phi(0)=\partial_x\Phi(1)=0\Big\}$.*

Proof: Let $(w_0,w_1)\in[H^2(\Omega)\cap H_0^1(\Omega)]\times H_0^1(\Omega)$ and $(\psi_0,\psi_1)\in H\times H^1(\Omega)$. The variational formulation of the problem (2.1)–(2.5) is the following:

$$(\partial_t^2\,w,\Phi) = -c\,d\,(\partial_x\,\psi,\Phi) + \lambda(t)\,(\partial_x^2\,w,\Phi)\,, \tag{2.6}$$

$$\frac{1}{c}\,(\partial_t^2\,\psi,\tilde\Phi) = (\partial_x^2\,\psi,\tilde\Phi) - c\,d\,(\psi-\partial_x\,w,\tilde\Phi)\,, \tag{2.7}$$

for any $\Phi\in H_0^1(\Omega)$, $\tilde\Phi$ in H. We denote by $(.,.)$ the scalar product in $L^2(\Omega)$.

Let $\{\xi_i^1\}_{i=1}^{\infty}$ be a basis of $H_0^1(\Omega)$ and $\{\xi_i^2\}_{i=1}^{\infty}$ a basis of the space H. We define the approximate solutions $w_n(x,t), \psi_n(x,t)$ by

$$\left\{\begin{aligned} w_n(x,t) &= \sum_{i=1}^{n} g_{i,n}(t)\, \xi_i^1(x) \ , \\ \psi_n(x,t) &= \sum_{i=1}^{n} \tilde{g}_{i,n}(t)\, \xi_i^2(x) \ , \end{aligned}\right. \tag{2.8}$$

where the functions $g_{i,n}(t), \tilde{g}_{i,n}(t)$ are such that the following equations hold

$$(\partial_t^2 w_n, \Phi) = -c\,d\,(\partial_x \psi_n, \Phi) - \lambda(t)\,(\partial_x w_n, \partial_x \Phi) \ , \tag{2.9}$$

and

$$\frac{1}{c}\,(\partial_t^2 \psi_n, \tilde{\Phi}) = -(\partial_x \psi_n, \partial_x \tilde{\Phi}) - c\,d\,(\psi_n - \partial_x w_n, \tilde{\Phi}) \ , \tag{2.10}$$

for any Φ in $vect\{\xi_i^1\}_{i=1}^{n}$, $\tilde{\Phi}$ in $vect\{\xi_i^2\}_{i=1}^{n}$ and where

$$\left\{\begin{aligned} w_n(x,0) = w_0^n(x) &= \sum_{i=1}^{n}(w_0, \xi_i^1)\, \xi_i^1(x) \ , \\ \partial_t w_n(x,0) = w_1^n(x) &= \sum_{i=1}^{n}(w_1, \xi_i^1)\, \xi_i^1(x) \ , \\ \psi_n(x,0) = \psi_0^n(x) &= \sum_{i=1}^{n}(\psi_0, \xi_i^2)\, \xi_i^2(x) \ , \\ \partial_t \psi_n(x,0) = \psi_1^n(x) &= \sum_{i=1}^{n}(\psi_1, \xi_i^2)\, \xi_i^2(x) \ . \end{aligned}\right. \tag{2.11}$$

Now if we set $\Phi = \partial_t w_n$ and $\tilde{\Phi} = \partial_t \psi_n$ in the variational equations (2.9) and (2.10), we integrate by parts we get:

$$\partial_t |\partial_t w_n|^2 \ = \ -2\,c\,d\,(\partial_x \psi_n, \partial_t w_n) - \lambda(t)\,\partial_t |\partial_x w_n|^2 - \alpha\,|\partial_t w_n|^2 \ , \tag{2.12}$$

and

$$\begin{aligned} \frac{1}{c}\,\partial_t |\partial_t \psi_n|^2 \ = \ & -\partial_t |\partial_x \psi_n|^2 - \beta\,|\partial_t \psi_n|^2 \\ & -2\,c\,d\,(\psi_n, \partial_t \psi_n) + 2\,c\,d\,(\partial_x w_n, \partial_t \psi_n) \ , \end{aligned} \tag{2.13}$$

where $|.|$ denotes the $L^2(\Omega)$ norm. Summing up the two above equations, we obtain:

$$\begin{aligned} \partial_t\Big(\frac{1}{c}\,|\partial_t \psi_n|^2 + |\partial_x \psi_n|^2 + |\partial_t w_n|^2 + \lambda(t)\,|\partial_x w_n|^2\Big) \ \le \\ \le \ c\,d\,\Big(|\partial_x \psi_n|^2 + |\partial_t w_n|^2\Big) + c\,d\,\Big(|\psi_n|^2 + |\partial_t \psi_n|^2\Big) \\ + \ |\lambda'(t)|\,|\partial_x w_n|^2 + c\,d\,\Big(|\partial_t \psi_n|^2 + |\partial_x w_n|^2\Big) \ , \end{aligned} \tag{2.14}$$

and by Gronwall's Lemma we deduce that w_n and ψ_n (respectively $\partial_x w_n$ and $\partial_x \psi_n$) remain in a bounded set in $L^\infty([0,T],H^1(\Omega))$ (respectively $L^\infty([0,T],L^2(\Omega))$).

If we differentiate (2.9) and (2.10) with respect to the variable t and write the corresponding variational formulation we obtain that for any $\Phi \in H_0^1(\Omega)$ and $\tilde{\Phi}$ in H, we have:

$$\begin{aligned}(\partial_t^3 w_n, \Phi) \;=\; & -c\,d\,(\partial_{xt}^2 \psi_n, \Phi) + \lambda(t)\,(\partial_{xxt}^3 w_n, \Phi) + \lambda'(t)\,(\partial_x^2 w_n, \Phi) \\ & -\alpha\,(\partial_t^2 w_n, \Phi)\ ,\end{aligned} \tag{2.15}$$

and

$$\frac{1}{c}\,(\partial_t^3 \psi_n, \tilde{\Phi}) \;=\; (\partial_{xxt}^3 \psi_n, \tilde{\Phi}) - c\,d\,(\partial_t \psi_n - \partial_{xt}^2 w_n, \tilde{\Phi}) - \beta\,(\partial_t^2 \psi_n, \tilde{\Phi})\ . \tag{2.16}$$

Next, if we set in the above relation $\Phi = \partial_t^2 w_n$ and $\tilde{\Phi} = \partial_t^2 \psi_n$, we obtain:

$$\begin{aligned}\frac{1}{2}\,\partial_t\,|\partial_t^2 w_n|^2 \;=\; & -c\,d\,(\partial_{xt}^2 \psi_n, \partial_t^2 w_n) - \frac{\lambda(t)}{2}\,\partial_t\,|\partial_{xt}^2 w_n|^2 \\ & + \lambda'(t)\,(\partial_x^2 w_n, \partial_t^2 w_n) - \alpha\,|\partial_t^2 w_n|^2\ ,\end{aligned} \tag{2.17}$$

and

$$\begin{aligned}\frac{1}{2c}\,\partial_t\,|\partial_t^2 \psi_n|^2 \;=\; & -\frac{1}{2}\,\partial_t\,|\partial_{xt}^2 \psi_n|^2 - c\,d\,(\partial_{xt}^2 \psi_n, \partial_t^2 \psi_n) \\ & + c\,d\,(\partial_{xt}^2 w_n, \partial_t^2 \psi_n) - \beta\,|\partial_t^2 \psi_n|^2\ .\end{aligned} \tag{2.18}$$

Summing up the above identities we obtain:

$$\begin{aligned}& \frac{1}{2c}\,\partial_t\,|\partial_t^2 \psi_n|^2 + \frac{1}{2}\,\partial_t\,|\partial_t^2 w_n|^2 + \partial_t\left(\frac{\lambda(t)}{2}\,|\partial_{xt}^2 w_n|^2\right) \;= \\ & \quad = -c\,d\,(\partial_{xt}^2 \psi_n, \partial_t^2 \psi_n) - c\,d\,(\partial_t \psi_n, \partial_t^2 \psi_n) + c\,d\,(\partial_{xt}^2 w_n, \partial_t^2 \psi_n) \\ & \qquad + \lambda'(t)\,(\partial_x^2 w_n, \partial_t^2 w_n) + \frac{\lambda'(t)}{2}\,|\partial_{xt}^2 w_n|^2 - \frac{1}{2}\,\partial_t\,|\partial_{xt}^2 \psi_n|^2 \\ & \qquad - \alpha\,|\partial_t^2 w_n|^2 - \beta\,|\partial_t^2 \psi_n|^2\ .\end{aligned} \tag{2.19}$$

Once again, Gronwall's Lemma allows us to conclude that $\partial_t^2 w_n$ and $\partial_t^2 \psi_n$ (respectively $\partial_t w_n$ and $\partial_t \psi_n$) remain in a bounded set in $L^\infty([0,T],L^2(\Omega))$ (respectively $L^\infty([0,T],H^1(\Omega))$). We extract then from (w_n,ψ_n) a subsequence still denoted by (w_n,ψ_n) such that:

$$\begin{aligned}& \partial_t^2 w_n \to \partial_t^2 w \quad \text{weakly* in} \;\; L^\infty([0,T],L^2(\Omega))\ , \\ & \partial_t w_n \to \partial_t w \quad \text{weakly* in} \;\; L^\infty([0,T],H^1(\Omega))\ , \\ & \partial_t^2 \psi_n \to \partial_t^2 \psi \quad \text{weakly* in} \;\; L^\infty([0,T],L^2(\Omega))\ , \\ & \partial_t \psi_n \to \partial_t \psi \quad \text{weakly* in} \;\; L^\infty([0,T],H^1(\Omega))\ ,\end{aligned}$$

and (w,ψ) satisfies the problem (2.1)–(2.5). Hence w,ψ are in $L^\infty([0,T],H^2(\Omega))$ and following Strauss [8] (see also Lions and Magenes [7], page 296), we obtain $w,\psi\in\mathcal{C}([0,T],H^2(\Omega))\cap\mathcal{C}^1([0,T],H^1(\Omega))$. To complete the proof of the theorem, we have to show the uniqueness of such solution. This follows from the following energy inequality:

$$E(t)\;\le\;C\,e^{\displaystyle\int_0^t\frac{|\lambda'(s)|}{\lambda(s)}\,ds}\,E(0)\;,$$

where

$$E(t)\;=\;\Big\{|\partial_t w|^2+\frac{1}{c}\,|\partial_t\psi|^2+\lambda(t)\,|\partial_x w|^2+c\,d\,|\psi|^2+|\partial_x\psi|^2\Big\}\;,$$

and C is a positive constant.

The proof of this inequality is obvious. It suffices to multiply (2.1) and (2.2) by $\partial_t w$ and $\partial_t\psi$, respectively and then integrate by parts and apply Gronwall's Lemma.

We note that using similar arguments we may prove that (2.1) and (2.5) have a unique solution in $\Big(\mathcal{C}([0,T],H^3(\Omega)\cap H^1_0(\Omega))\cap\mathcal{C}^1([0,T],H^2(\Omega)\cap H^1_0(\Omega))\Big)\times\Big(\mathcal{C}([0,T],H^3(\Omega)\cap H)\cap\mathcal{C}^1([0,T],H)\Big)$ if the initial data (w_0,w_1) belongs to $[H^3(\Omega)\cap H^1_0(\Omega)]\times H^1_0(\Omega)$ and (ψ_0,ψ_1) belongs to $[H^3(\Omega)\cap H]\times H$.

Next, we prove the existence of a solution to the problem (1.1)–(1.5) by using a fixed point approach. We have:

Theorem 2.2. *Let (w_0,w_1) be in $[H^2(\Omega)\cap H^1_0(\Omega)]\times H^1_0(\Omega)$ and (ψ_0,ψ_1) be in $H\times H^1(\Omega)$, then there exist $T>0$ and a unique couple of functions $(w,\psi)\in\Big(\mathcal{C}([0,T[,H^2(\Omega)\cap H^1_0(\Omega))\cap\mathcal{C}^1([0,T[,H^1_0(\Omega))\Big)\times\Big(\mathcal{C}([0,T[,H)\cap\mathcal{C}^1([0,T[,H^1(\Omega))\Big)$ solution of the problem (1.1)–(1.5). Furthermore, at least one of these two affirmations is true:*

a) $T=+\infty\,,$

b) $\displaystyle\lim_{t\to T^-}\Big\{||w||_{H^2}+||\psi||_{H^2}+||\partial_t w||_{H^1}+||\partial_t\psi||_{H^1}\Big\}=+\infty\,.$

Proof: Let X be a Hilbert space. We denote by $\mathcal{C}^k([0,T],X-w)$ the set of functions k-differentiable from $[0,T]$ into X, equipped with a weak topology. We consider $R,T>0$ and define

$$X_{T,R}=\Big\{\Phi\in\mathcal{C}([0,T],H^2-w)\,\cap\,\mathcal{C}^1([0,T],H^1-w),\;E(\Phi,t)\le R^2,\;\forall\,t\in[0,T]\Big\}\;,$$

where

$$E(\Phi,t)\;=\;||\Phi||^2_{H^2}+||\partial_t\Phi||^2_{H^1}\;.$$

The set $X_{T,R}$ is a complete metric space under the metric defined by

$$d(\Phi,\tilde{\Phi}) \;=\; \sup_{t\in[0,T]}\left\{\frac{c\,d}{2}\,||\Phi-\tilde{\Phi}||^2_{H^1}+\frac{1}{2}\,||\partial_t\,(\Phi-\tilde{\Phi})||^2_{L^2}\right\}^{1/2},$$

and the space $X_{T,R}\times X_{T,R}$ is also a complete metric space under the metric defined by

$$\begin{aligned}\tilde{d}\Big((\Phi_1,\tilde{\Phi}_1),(\Phi_2,\tilde{\Phi}_2)\Big) \;=\; \sup_{t\in[0,T]}&\left\{\frac{c\,d}{2}\Big(||\Phi_1-\Phi_2||^2_{H^1}+||\tilde{\Phi}_1-\tilde{\Phi}_2||^2_{H^1}\Big)\right.\\ &\left.+\frac{1}{2}\,||\partial_t\,(\Phi_1-\Phi_2)||^2_{L^2}+\frac{1}{2}\,||\partial_t\,(\tilde{\Phi}_1-\tilde{\Phi}_2)||^2_{L^2}\right\}^{1/2}.\end{aligned}$$

We define for $\Phi\in X_{T,R}$, $S(\Phi,\Phi)$ by

$$S(\Phi,\Phi)=(w,\psi)\ ,$$

where (w,ψ) is the solution of the following problem:

$$\begin{aligned}\partial_t^2\,w \;&=\; -\,c\,d\,\partial_x(\psi-\partial_x\,w)+b\,|\partial_x\,\Phi|^2\,\partial_x^2\,w-\alpha\,\partial_t\,w\,, \qquad \Omega\times(0,\infty)\ ,\\ \frac{1}{c}\,\partial_t^2\,\psi \;&=\; \partial_x^2\,\psi-c\,d\,(\psi-\partial_x\,w)-\beta\,\partial_t\,\psi\,, \qquad \Omega\times(0,\infty)\ ,\\ &w(x,0)=w_0(x),\quad \partial_t\,w(x,0)=w_1(x)\,, \qquad \Omega\ ,\\ &\psi(x,0)=\psi_0(x),\quad \partial_t\,\psi(x,0)=\psi_1(x)\,, \qquad \Omega\ ,\\ &w(0,t)=w(1,t)=0\,, \qquad (0,\infty)\ ,\\ &\partial_x\,\psi(0,t)=\partial_x\,\psi(1,t)=0\,, \qquad (0,\infty)\ .\end{aligned}$$

Let $\Delta=\{(\Phi,\Phi),\ \Phi\in X_{T,R}\}$. It is easy to see for T small enough that S is a contraction under the metric $\tilde{d}$ and $S(\Delta)\subset X_{T,R}\times X_{T,R}$ (see for instance [1]). Let now the mapping $\tilde{S}$ be defined from:

$$X_{T,R}\;\mapsto\;\Delta\;\mapsto\;X_{T,R}\times X_{T,R}\;\mapsto\;X_{T,R}\ ,$$

by

$$\tilde{S}\;=\;\alpha_1\circ S\circ\alpha_2\ ,$$

where α_1 and α_2 are two contractions defined by:

$$\alpha_2(\Phi)=(\Phi,\Phi)\,, \qquad \Phi\in X_{T,R}\ ,$$

and

$$\alpha_1(\Phi,\tilde{\Phi})=\Phi,(\Phi,\tilde{\Phi})\;\in\;X_{T,R}\times X_{T,R}\ .$$

This implies that $\tilde{S}$ is a contraction. Then, using the Banach fixed point theorem, it follows that $\tilde{S}$ has a fixed point and then the problem (1.1)–(1.5) admits a unique solution. The proof of the local existence result is now complete.

The purpose of the next section is to prove global existence of solutions to the problem (1.1)–(1.5) for small initial data.

3 – Global existence

The first main result of this paper is formulated in the following theorem.

Theorem 3.1. *Let (w_0, w_1) be in $[H^2(\Omega) \cap H_0^1(\Omega)] \times H_0^1(\Omega)$ and (ψ_0, ψ_1) be in $H \times H^1(\Omega)$, such that*

$$\begin{aligned}
&4 \sup\Big(\frac{1}{c\,d}, 1\Big) \sqrt{\frac{3\,b}{2\,c\,d}} \Big(c\,d + 2\sqrt{b\,E(0)}\Big) \Big(\frac{1}{2}\,|\partial_x w_1|^2 + \frac{1}{2\,c}\,|\partial_x \psi_1|^2 \\
&\qquad + \frac{c\,d}{2}\,|\partial_x^2 w_0 - \partial_x \psi_0|^2 + \frac{b}{2}\,|\partial_x w_0|^2\,|\partial_x^2 w_0|^2 + |\partial_x^2 \psi_0|^2\Big)^{1/2} < \qquad (3.1)\\
&\quad < \min\left(\min\Big(\frac{\alpha}{2}, \frac{\beta\,c}{2}\Big),\; \frac{1}{\sup\Big(2\,\alpha + \beta + \frac{1}{c} + 2,\; (2 + 2\,\alpha)/c\,d\Big)}\right),
\end{aligned}$$

then there exists a unique couple of functions $(w, \psi) \in \Big(\mathcal{C}([0, +\infty[, H^2(\Omega) \cap H_0^1(\Omega)) \cap \mathcal{C}^1([0, +\infty[, H_0^1(\Omega))\Big) \times \Big(\mathcal{C}([0, +\infty[, H) \cap \mathcal{C}^1([0, +\infty[, H^1(\Omega))\Big)$ solution of equations (1.1)–(1.5).

Proof: From the local existence result, there exists a maximal solution (w, ψ) to the problem (1.1)–(1.5) and we know that $w \in \mathcal{C}([0, T[, H^2(\Omega) \cap H_0^1(\Omega)) \cap \mathcal{C}^1([0, T[, H_0^1(\Omega))$ and $\psi \in \mathcal{C}([0, T[, H) \cap \mathcal{C}^1([0, T[, H^1(\Omega))$. Let

$$\lambda(t) \;=\; c\,d \,+\, b \int_\Omega (\partial_x\, w)^2(y, t)\; dy \;.$$

We introduce

$$F(t) \;=\; \frac{1}{2}\Big\{|\partial_{xt}^2\, w|^2 + \frac{1}{c}\,|\partial_{xt}^2\, \psi|^2 + c\,d\,|\partial_x\, \psi - \partial_x^2 w|^2 + b\,|\partial_x\, w|^2\,|\partial_x^2 w|^2 + |\partial_x^2\, \psi|^2\Big\} \;.$$

For

$$(w_0, w_1) \in [H^3(\Omega) \cap H_0^1(\Omega)] \times [H^2(\Omega) \cap H_0^1(\Omega)]\,, \quad (\psi_0, \psi_1) \in [H^3(\Omega) \cap H] \times H \;,$$

we have

$$(w,\psi) \in \Big(\mathcal{C}\Big([0,T[, H^3(\Omega)\cap H_0^1(\Omega)\Big) \cap \mathcal{C}^1\Big([0,T[, H^2(\Omega)\cap H_0^1(\Omega)\Big)\Big) \times \Big(\mathcal{C}\Big([0,T[, H^3(\Omega)\cap H\Big) \cap \mathcal{C}^1([0,T[, H)\Big) .$$

Since $F(t)\in \mathcal{C}^1([0,T[)$, then if we take its derivative, we obtain

$$\frac{dF}{dt}(t) = -\alpha \int_\Omega (\partial_{xt}^2 w)^2\, dx - \beta \int_\Omega (\partial_{xt}^2 \psi)^2\, dx + \frac{b}{2}\,|\partial_x^2 w(t)|^2\, \frac{d}{dt}\Big(|\partial_x w(t)|^2\Big) .$$

This gives

$$\frac{d}{dt}F(t) \le \sup\Big(\frac{2}{c\,d}, 2\Big)\,|\lambda'(t)|\,F(t)\,, \qquad \forall\, t\in [0,T[.$$

Hence the following estimate holds

$$\begin{aligned} F(t) &\le F(0)\; e^{\int_0^t \sup\left(\frac{2}{c\,d},2\right)|\lambda'(s)|\,ds}, \\ &\le F(0)\; e^{\int_0^T \sup\left(\frac{2}{c\,d},2\right)|\lambda'(s)|\,ds}. \end{aligned}$$

In what follows we show for $T<\infty$ that $|\lambda'(t)|$ remains bounded on $[0,T]$. Let

$$e(t) = \int_\Omega \partial_x w\,\partial_{xt}^2 w\, dx + \frac{1}{c}\int_\Omega \partial_x\psi\,\partial_{xt}^2\psi\, dx + \frac{\alpha}{2}\int_\Omega (\partial_x w)^2\, dx + \frac{\beta}{2}\int_\Omega (\partial_x\psi)^2\, dx .$$

It follows from (3.1) that there exists $\varepsilon>0$ such that

$$\varepsilon < \min\left(\min\Big(\frac{\alpha}{2},\frac{\beta\,c}{2}\Big), \frac{1}{2\sup\Big(2\,\alpha+\beta+\frac{1}{c}+2,\ (2+2\,\alpha)/c\,d\Big)}\right),$$

and

$$\varepsilon > 4\sup\Big(\frac{1}{c\,d},1\Big)\sqrt{\frac{3\,b}{2\,c\,d}}\Big(c\,d+2\sqrt{b\,E(0)}\Big)\Big(\frac{1}{2}\,|\partial_x w_1|^2 + \frac{1}{2\,c}\,|\partial_x\psi_1|^2 + \frac{c\,d}{2}\,|\partial_x^2 w_0 - \partial_x\psi_0|^2 + \frac{b}{2}\,|\partial_x w_0|^2\,|\partial_x^2 w_0|^2 + |\partial_x^2\psi_0|^2\Big)^{1/2} .$$

From

$$|e(t)| \le \sup\Big(2\,\alpha+\beta+\frac{1}{c}+2,\ (2+2\,\alpha)/c\,d\Big)\,F(t) ,$$

we deduce that

$$\varepsilon\,|e(t)| \leq \frac{1}{2}\,F(t)\,, \qquad \forall\, t \in [0,T[\ .$$

This gives

$$\frac{1}{2}\,F(t) \leq F_\varepsilon(t) \leq \frac{3}{2}\,F(t)\,, \qquad \forall\, t \in [0,T[\ , \tag{3.2}$$

where

$$F_\varepsilon(t) = F(t) + \varepsilon\, e(t) \ .$$

On the other hand, from

$$\lambda'(t) \ = \ 2\,b \int_0^1 \partial_x w\, \partial^2_{xt} w \ dx \ ,$$

it follows that the following inequality holds

$$|\lambda'(t)| \ \leq \ 2\sqrt{b\,\lambda(t)}\ |\partial^2_{xt}\, w|\,, \qquad \forall\, t \in [0,T[\ .$$

Since $\lambda(t) \geq c\,d > 0$, we obtain

$$\frac{|\lambda'(t)|}{\lambda(t)} \ \leq \ 2\sqrt{\frac{b}{c\,d}}\ |\partial^2_{xt}\, w|\,, \qquad \forall\, t \in [0,T[\ .$$

Thus,

$$\frac{|\lambda'(t)|}{\lambda(t)} \ \leq \ 2\sqrt{2}\ \sqrt{\frac{b}{c\,d}}\ \sqrt{F(t)}\,, \qquad \forall\, t \in [0,T[\ . \tag{3.3}$$

From (3.2), we have

$$\frac{|\lambda'(t)|}{\lambda(t)} \ \leq \ 4\sqrt{\frac{b}{c\,d}}\ \sqrt{F_\varepsilon(t)}\,, \qquad \forall\, t \in [0,T[\ .$$

Let now

$$E(t) \ = \ \frac{1}{2}\left\{|\partial_t\, w|^2 + \frac{1}{c}\,|\partial_t\,\psi|^2 + c\,d\,|\psi -\ \partial_x w|^2 + \frac{b}{2}\,|\partial_x\, w|^4 + |\partial_x\,\psi|^2\right\} \ .$$

We can easily verify that

$$\frac{d}{dt}E(t) = -\alpha\,|\partial_t w|^2 - \beta\,|\partial_t \psi|^2 \ .$$

This implies

$$E(t) \leq E(0) \ .$$

Thus,

$$\lambda(t) \;=\; c\,d + b\,|\partial_x w|^2 \;\leq\; c\,d + 2\sqrt{b}\,\sqrt{E(0)}\,, \qquad \forall\, t \in [0,T]\ .$$

The identity (3.3) gives

$$|\lambda'(t)| \;\leq\; 4\sqrt{\frac{b}{c\,d}}\,\left(c\,d + 2\sqrt{b}\sqrt{E(0)}\right)\sqrt{F_\varepsilon(t)}\,, \qquad \forall\, t \in [0,T[\ , \tag{3.4}$$

and

$$|\lambda'(t)| \;\leq\; 2\sqrt{\frac{2\,b}{c\,d}}\,\left(c\,d + 2\sqrt{b}\sqrt{E(0)}\right)\sqrt{F(t)}\,, \qquad \forall\, t \in [0,T[\ . \tag{3.5}$$

Next, if we set $t = 0$ in the inequality (3.4), we obtain

$$|\lambda'(0)| \;\leq\; 4\sqrt{\frac{3\,b}{c\,d}}\,\left(c\,d + 2\sqrt{b}\sqrt{E(0)}\right)\sqrt{F(0)} \;<\; \frac{\varepsilon}{k}\ ,$$

where

$$k \;=\; \sup\left(\frac{1}{c\,d},\, 1\right)\ .$$

We shall prove by contradiction that

$$|\lambda'(t)| < \frac{\varepsilon}{k}\,, \qquad \forall\, t \in [0,T[\ .$$

Assume that there exists $t^* \in [0,T]$ such that

$$|\lambda'(t)| < \frac{\varepsilon}{k}\,, \qquad \forall\, t \in [0,t^*[\ ,$$

and

$$|\lambda'(t^*)| = \frac{\varepsilon}{k}\ ,$$

then, if we take the derivative of $F_\varepsilon(t)$, we find

$$\begin{aligned}
F'_\varepsilon(t) \;&=\; F'(t) + \varepsilon\, e'(t)\ ,\\
&=\; -\,\alpha\int_\Omega (\partial^2_{xt} w)^2\, dx \;-\; \beta\int_\Omega (\partial^2_{xt}\psi)^2\, dx \;+\; \frac{b}{2}\,\frac{d}{dt}\Big(|\partial_x w(t)|^2\Big)\,|\partial^2_x w(t)|^2\\
&\quad +\; \frac{\varepsilon}{c}\int_\Omega (\partial^2_{xt}\psi)^2\, dx \;+\; \varepsilon\int_\Omega (\partial_{xt} w)^2\, dx\\
&\quad +\; \varepsilon\, c\, d\int_\Omega \partial_x(\psi - \partial_x w)\,\partial^2_x w\, dx \;-\; \varepsilon\, b\,|\partial_x w(t)|^2\int_\Omega (\partial^2_x w)^2\, dx\\
&\quad -\; \varepsilon\int_\Omega (\partial^2_x \psi)^2\, dx \;-\; \varepsilon\, c\, d\int_\Omega \partial_x(\psi - \partial_x w)\,\partial_x\psi\, dx\ .
\end{aligned}$$

This gives

$$\begin{aligned}
F_\varepsilon'(t) \;\le\; & \sup\left(\frac{2}{c\,d}, 2\right) |\lambda'(t)|\, F(t) \\
& -\beta \int_\Omega (\partial^2_{xt}\psi)^2 \; dx \;-\; \alpha \int_\Omega (\partial^2_{xt} w)^2 \; dx \\
& +\frac{\varepsilon}{c} \int_\Omega (\partial^2_{xt}\psi)^2 \; dx \;+\; \varepsilon \int_\Omega (\partial^2_{xt} w)^2 \; dx \\
& -\varepsilon\, c\, d \int_\Omega (\partial_x\psi - \partial^2_x w)^2 \; dx \;-\; \varepsilon\, b\, |\partial_x w(t)|^2 \int_\Omega (\partial^2_x w)^2 \; dx \\
& -\varepsilon \int_\Omega (\partial^2_x \psi)^2 \; dx \; .
\end{aligned}$$

Thus,

$$F_\varepsilon'(t) \;\le\; \sup\left(\frac{2}{c\,d}, 2\right) |\lambda'(t)|\, F(t) \;-\; 2\, \min(\alpha-\varepsilon,\, \beta\, c - \varepsilon,\, \varepsilon)\, F(t) \; .$$

Since ε is such that

$$\varepsilon \;\le\; \min\left(\frac{\alpha}{2}, \frac{\beta\, c}{2}\right) \; ,$$

then from

$$F_\varepsilon'(t) \;\le\; -\left(2\, \min(\alpha-\varepsilon,\, \beta\, c-\varepsilon,\, \varepsilon) \;-\; \sup\left(\frac{2}{c\,d}, 2\right) |\lambda'(t)|\right) F(t) \; ,$$

and since we have

$$\min(\alpha-\varepsilon,\, \beta\, c-\varepsilon,\, \varepsilon) \;=\; \varepsilon$$

and

$$k \;=\; \sup\left(\frac{1}{c\,d}, 1\right) \; ,$$

we obtain

$$F_\varepsilon(t^*) \le F_\varepsilon(0) \; .$$

By a density argument we can show that the above inequality remains true for (w_0, w_1) and $(\psi_0, \psi_1) \in H^2(\Omega) \times H^1(\Omega)$. But from (3.4), we have

$$|\lambda'(t^*)| < \frac{\varepsilon}{k} \; ,$$

which contradicts the hypothesis. We conclude that $|\lambda'(t)|$ is bounded for any $t \in [0,T]$. This shows that the quantity $(||w||_{H^2} + ||\psi||_{H^2} + ||\partial_t w||_{H^1} + ||\partial_t \psi||_{H^1})$ is uniformaly bounded for $t \in [0,T]$. Hence the global existence of a solution to (1.1)–(1.5) with small initial data holds according to Theorem 2.2. ∎

4 – Stabilization

We define the energy of the system ((1.1)–(1.5)) by:

$$E(t) \;=\; E(w,\psi,t) \;=\; \frac{1}{2}\left\{|\partial_t w|^2 + \frac{1}{c}\,|\partial_t \psi|^2 + c\,d\,|\psi - \partial_x w|^2 + \frac{b}{2}\,|\partial_x w|^4 + |\partial_x \psi|^2\right\} .$$

By simple computation we have

$$E'(w,\psi,t) \;=\; -\,\alpha \int_\Omega (\partial_t w)^2\; dx \;-\; \beta \int_\Omega (\partial_t \psi)^2\; dx \;\le\; 0 \;. \tag{4.1}$$

This shows that the energy is decreasing.

The second important result in this paper is:

Theorem 4.1. *If (w,ψ) is a global strong solution of the problem (1.1)–(1.5) then the energy satisfies the following estimate:*

$$E(t) \;\le\; k\,E(0)\,e^{-\omega\,t} \;, \tag{4.2}$$

where k,ω are two strictly positive constants independent of the initial data.

In order to prove this theorem, we need the following technical result (see [5], Theorem 8.1, page 103, for a proof):

Lemma 4.2. *Let $F:\,\mathbb{R}^+ \mapsto \mathbb{R}^+$ be a decreasing function. If we assume that there exists $A>0$ such that:*

$$\int_t^{+\infty} F(s)\; ds \;\le\; A\,F(t)\,, \qquad \forall\, t \in \mathbb{R}^+ \;,$$

then we have

$$F(t) \le F(0)\,e^{1-t/A}\,, \qquad \forall\, t \in \mathbb{R}^+ \;. \;\blacksquare$$

We now go back to the proof of Theorem 4.1. For $0 \le S < T < \infty$, arbitrary fixed, we have from (4.1):

$$E(S) - E(T) \;=\; \alpha \int_S^T \int_\Omega (\partial_t w)^2\, dx\;dt \;+\; \beta \int_S^T \int_\Omega (\partial_t \psi)^2\, dx\;dt \;.$$

Lemma 4.3. *We have:*

$$2\int_S^T E(t)\;dt \;\le\; c\,E(S) \;-\; \alpha \int_S^T \int_\Omega w\,\partial_t w\;dx\;dt \;-\; \beta \int_S^T \int_\Omega \psi\,\partial_t \psi\;dx\;dt \;.$$

Proof: We multiply (1.1) by $-w$ and (1.2) by $-\psi$, integrating the sum of these two equations on $[S,T]\times\Omega$, we obtain

$$\begin{aligned}0 = &-\Big[\int_\Omega w\,\partial_t w\;dx\Big]_S^T - \frac{1}{c}\Big[\int_\Omega \psi\,\partial_t\psi\;dx\Big]_S^T \\ &- \int_S^T\Big(-|\partial_t w|^2 - \frac{1}{c}|\partial_t\psi|^2 + |\partial_x\psi|^2 + b\,|\partial_x w|^4 + c\,d\,|\psi-\partial_x w|^2 \\ &+ \alpha\int_\Omega w\,\partial_t w\;dx + \beta\int_\Omega \psi\,\partial_t\psi\;dx\Big)\;dt\;.\end{aligned}$$

But from the expression of the energy we get

$$\frac{b}{2}|\partial_x w|^4 + c\,d\,|\psi-\partial_x w|^2 + |\partial_x\psi|^2 \;=\; 2\,E(t) - |\partial_t w|^2 - \frac{1}{c}|\partial_t\psi|^2\;.$$

Hence it follows that

$$\begin{aligned}0 = &-\Big[\int_\Omega w\,\partial_t w\;dx\Big]_S^T - \frac{1}{c}\Big[\int_\Omega \psi\,\partial_t\psi\;dx\Big]_S^T \\ &- \int_S^T\Big(-|\partial_t w|^2 - \frac{1}{c}|\partial_t\psi|^2 + \frac{b}{2}|\partial_x w|^4 + 2\,E(t) - |\partial_t w|^2 - \frac{1}{c}|\partial_t\psi|^2 \\ &+ \alpha\int_\Omega w\,\partial_t w\;dx + \beta\int_\Omega \psi\,\partial_t\psi\;dx\Big)\;dt\;.\end{aligned}$$

This gives

$$\begin{aligned}2\int_S^T E(t)\;dt \;=\; &-\Big[\int_\Omega w\,\partial_t w\;dx\Big]_S^T - \frac{1}{c}\Big[\int_\Omega \psi\,\partial_t\psi\;dx\Big]_S^T \\ &+ \int_S^T\Big(-\alpha\int_\Omega w\,\partial_t w\;dx - \beta\int_\Omega \psi\,\partial_t\psi\;dx \\ &- \frac{b}{2}|\partial_x w|^4 + 2\Big(|\partial_t w|^2 + \frac{1}{c}|\partial_t\psi|^2\Big)\Big)\;dt\;.\end{aligned}$$

Note that if we write

$$(\partial_x w)^2 \;=\; (\partial_x w-\psi)\,\partial_x w + \psi\,\partial_x w\;,$$

then

$$\int_\Omega(\partial_x w)^2\,dx \;\le\; \frac{1}{2\varepsilon}\int_\Omega(\partial_x w-\psi)^2\,dx + 2\,\varepsilon\int_\Omega(\partial_x w)^2\,dx + \frac{1}{2\varepsilon}\int_\Omega(\partial_x\psi)^2\,dx\;,\qquad \forall\,\varepsilon,\quad 0<\varepsilon<\frac{1}{2}\;.$$

It follows that

$$\int_\Omega (\partial_x w)^2 \, dx \le K(\varepsilon) \left(\int_\Omega (\partial_x w - \psi)^2 \, dx + \int_\Omega (\partial_x \psi)^2 \, dx \right),$$

where

$$K(\varepsilon) = \frac{1}{2\,\varepsilon\,(1 - 2\,\varepsilon)} .$$

Consequently we obtain

$$2 \int_S^T E(t) \, dt \le c\, E(S) - \alpha \int_S^T \int_\Omega w\, \partial_t w \, dx \, dt - \beta \int_S^T \int_\Omega \psi\, \partial_t \psi \, dx \, dt . \blacksquare$$

Now we will estimate the quantities $\int_S^T \int_\Omega w\, \partial_t w \, dx \, dt$ and $\int_S^T \int_\Omega \psi\, \partial_t \psi \, dx \, dt$. We have

$$\begin{aligned} \alpha \int_\Omega w\, \partial_t w \, dx \, dt &\le \alpha \left(\int_\Omega (\partial_x w)^2 \, dx \right)^{1/2} \left(\int_\Omega (\partial_t w)^2 \, dx \right)^{1/2}, \\ &\le \sqrt{K(\varepsilon) \left(\frac{2}{c\,d} + 2 \right) \alpha} \; E^{1/2} \, |E'|^{1/2} , \end{aligned}$$

and

$$\beta \int_\Omega \psi\, \partial_t \psi \, dx \, dt \le \sqrt{\beta} \sqrt{K(\varepsilon) \left(\frac{2}{c\,d} + 2 \right) + \frac{2}{c\,d}} \; E^{1/2} \, |E'|^{1/2} .$$

This gives

$$\int_S^T \int_\Omega w\, \partial_t w \, dx \, dt + \int_S^T \int_\Omega \psi\, \partial_t \psi \, dx \, dt \le c_1 \int_S^T E^{1/2} \, |E'|^{1/2} \, dt ,$$

where

$$c_1 = \sqrt{K(\varepsilon) \left(\frac{2}{c\,d} + 2 \right) \alpha} + \sqrt{\beta} \sqrt{K(\varepsilon) \left(\frac{2}{c\,d} + 2 \right) + \frac{2}{c\,d}} .$$

From Young's inequality, we have

$$c_1 \, E^{1/2} \, |E'|^{1/2} \le c_2 \, |E'| + \frac{1}{2} E ,$$

where

$$c_2 = \frac{c_1^2}{2} .$$

Finally, we arrive at

$$\begin{aligned}
2\int_S^T E(t)\,dt \;&\leq\; c\,E(S) \,+\, c_2\int_S^T |E'|\,dt \,+\, \frac{1}{2}\int_S^T E\,dt\ ,\\
&\leq\; \frac{4}{3}\,(c_2+c)\,E(S)\ .
\end{aligned}$$

Setting T goes to $+\infty$ we obtain

$$\int_S^{+\infty} E(t)\,dt \;\leq\; \frac{1}{\omega}\,E(S)\ ,$$

where

$$\omega = \frac{3}{2\,(c+c_2)}\ .$$

Using once again Lemma 4.2, we finally obtain

$$E(t) \leq E(0)\,e^{1-\omega\,t}\,, \qquad \forall\,t\geq 0\ . \blacksquare$$

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