PORTUGALIAE MATHEMATICA
Vol. 59 Fasc. 2 – 2002
Nova Série

EXPONENTIAL DECAY AND EXISTENCE OF ALMOST PERIODIC SOLUTIONS FOR SOME LINEAR FORCED DIFFERENTIAL EQUATIONS

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Abstract: We study the existence of almost periodic solutions of some linear evolution equation $u' + A(t)\,u = f$. To obtain these results, we establish an alternative concerning an almost periodic contraction process on $\mathbb{R}^N$. Then we apply these results to a class of second order differential equations.

1 – Introduction

In this paper we study the existence of almost periodic solutions of some linear evolution systems with almost periodic forcing of the form

$$u'(t) + A(t)\,u(t) \;=\; f(t) \;, \tag{1.1}$$

where $u = (u_1, ..., u_N)$, $f\colon \mathbb{R} \to \mathbb{R}^N$ is almost periodic, $A\colon \mathbb{R} \to \mathcal{L}(\mathbb{R}^N)$ is an almost periodic operator-valued function and $A(t) \geq 0$ for all $t \in \mathbb{R}$. We give a necessary and sufficient condition for the equation (1.1) to have an exponentially sable almost periodic solution. In particular if the almost periodic operator-valued function $A(t)$ is symmetric, for all $t \in \mathbb{R}$, this condition is equivalent to positive definiteness of the average $\mathcal{M}\{A(t)\}_t$. We also give an application of this result to the nonlinear differential equation

$$u'(t) + \nabla\Phi\, u(t) \;=\; f(t) \;, \tag{1.2}$$

where $\nabla\Phi$ denotes the gradient of a C^1 convex function $\Phi\colon \mathbb{R}^N \to \mathbb{R}$.

Received: June 26, 2000; *Revised*: April 2, 2001.

To obtain the condition for equation (1.1), we shall establish, for any almost periodic linear contraction process on $\mathbb{R}^N$ in the sense of Dafermos [6] the following alternative: either there is a complete trajectory with constant positive norm, or this process is exponentially damped.

Then we give a necessary and sufficient condition on the almost periodic forcing term for equation (1.1) to generate at least one almost periodic solution (Fredholm alternative-type condition).

At the end we apply these results to the second order differential system

$$u'(t) + L\,u(t) + B(t)\,u'(t) \;=\; f(t) \;, \tag{1.3}$$

where L is a fixed positive definite symmetric operator on $\mathbb{R}^N$ and $B\colon\, \mathbb{R} \to \mathcal{L}(\mathbb{R}^N)$ is an almost periodic operator-valued function with $B(t)$ symmetric and $B(t) \geq 0$.

The existence of almost periodic solutions like (1.1) has been studied extensively in recent years. For example Aulbach and Minh, Minh, Minh and Naiton, Palmer, Seifert, Trachenko ([2], [12], [13], [14], [15], [16]) have given important contributions to the solution of this problem.

For a nonlinear differential equation in a Banach space, Aulbach and Minh in [2] give sufficient conditions for the existence of almost periodic solutions. For that they use the theory of semigroups of linear and nonlinear operators. In the case of linear equations, they establish necessary and sufficient conditions for the homogeneous equation

$$u'(t) + A(t)\,u(t) \;=\; 0 \tag{1.4}$$

to have an exponential dichotomy. In [12], Minh studies the existence of bounded solutions by the same method.

For the equation (1.1) when $A(t)$ is a possibly unbounded linear operator in a Banach space, Seifert [15] gives sufficient conditions for the existence of almost periodic solutions.

In [16], Trachenko studies the existence of almost periodic solutions of (1.4), when $A(t)$ is a skew symmetric matrix.

For almost periodic contraction processes, the question of existence of almost periodic complete trajectories has been studied by Dafermos [6], then Ishii [11]. In the special case of equation (1.1) when $A(t) \geq 0$, the main result of Ishii ([11], Theorem 1) ensures the existence of some almost periodic solution of (1.1) if this equation admits a bounded solution on $\mathbb{R}^+$.

In section 3, we will compare firstly some results of Aulbach and Minh in [2], secondly those of Seifert in [15], with our results.

The paper is organized as follows: in Section 2 we recall some notation and definitions. The results are announced and discussed in Section 3 and compared with those of the above quoted authors when related. In Section 4 we prove Theorem 3.4. In Section 5 we deduce Theorem 3.1 and give the proofs of Proposition 3.3, 3.8, Theorem 3.6 and Corollary 3.2, 3.5, 3.7.

2 – Notation and definitions

The numerical space $\mathbb{R}^N$ is endowed with its standard inner product $(x,y) := \sum_{k=1}^N x_y\, y_k$, and $\|\cdot\|$ denotes the associated euclidian norm. We denote by $\mathcal{L}(\mathbb{R}^N)$ the set of endomorphisms of $\mathbb{R}^N$.

A continuous function $u(\cdot)\colon \mathbb{R} \to \mathbb{R}^N$ is called *almost periodic* if for any sequence $(\sigma_n)_n$ of $\mathbb{R}$, there exists a subsequence $(\sigma'_n)_n$ of $(\sigma_n)_n$ such that the sequence $(u(t+\sigma'_n))_n$ is uniformly convergent in $\mathbb{R}^N$. Every function u almost periodic possesses a *time mean*

$$\mathcal{M}\{u(t)\}_t := \lim_{T\to+\infty} \frac{1}{T}\int_0^T u(t)\,dt$$

and for $\omega \in \mathbb{R}$

$$a(u,\omega) := \mathcal{M}\{u(t)\, e^{-i\omega t}\}_t$$

is the *Fourier–Bohr coefficient* of u associated at ω (cf. [1]). We denote

$$\Lambda(u) := \Big\{\omega \in \mathbb{R};\ a(u,\omega) \neq 0\Big\}$$

the set of exponents of u. The *module* of u denoted by $\mathrm{mod}(u)$, is the additive group generated by $\Lambda(u)$.

Recall that the linear system (1.4) has an exponential dichotomy if there exist $C>0$, $\alpha>0$ and P a projection in $\mathbb{R}^N$ such that

$$\|X(t)\,P\,X^{-1}(s)\| \le C\exp\Big(-\alpha\,(t-s)\Big) \quad \text{for } t \ge s$$

and

$$\|X(t)\,(I-P)\,X^{-1}(s)\| \le C\exp\Big(-\alpha\,(s-t)\Big) \quad \text{for } s \ge t\ ,$$

where $X(t)$ denotes the fundamental matrix for (1.4) such that $X(0) = I$. In the case where $A(t) \ge 0$ the linear system (1.4) has an exponential dichotomy if and only if $X(t)$ is exponentially damped, which means: there exist $C>0$, $\alpha>0$ such that

$$\|X(t)\,X^{-1}(s)\| \le C\exp\Big(-\alpha\,(t-s)\Big) \quad \text{for } t \ge s\ .$$

Following [6] we introduce some classes of process. A *process* on $\mathbb{R}^N$ is a two parameter family of maps $U(t,\tau)\colon \mathbb{R}^N \to \mathbb{R}^N$ defined for $(t,\tau) \in \mathbb{R}\times\mathbb{R}^+$ satisfying

(i) $\forall\, t \in \mathbb{R}$, $\forall\, x \in \mathbb{R}^N$, $U(t,0)\,x = x$;

(ii) $\forall\,(t,\sigma,\tau) \in \mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+$, $\forall\, x \in \mathbb{R}^N$, $U(t,\sigma+\tau)\,x = U(t+\tau,\sigma)\,U(t,\tau)\,x$;

(iii) $\forall\,\tau \in \mathbb{R}^+$, the one parameter family of maps $U(t,\tau)\colon \mathbb{R}^N \to \mathbb{R}^N$ with parameter $t \in \mathbb{R}$ is equicontinuous.

A process U on $\mathbb{R}^N$ is said to be *contractive* if

$$\forall\,(t,\tau) \in \mathbb{R}\times\mathbb{R}^+, \quad \forall\,(x,y) \in \mathbb{R}^N\times\mathbb{R}^N, \qquad \|U(t,\tau)\,x - U(t,\tau)\,y\| \le \|x-y\| \;.$$

We define the σ-translate U_σ by $U_\sigma(t,\tau) = U(t+\sigma,\tau)$. A process U on $\mathbb{R}^N$ is called *almost periodic* if for any sequence $(\sigma_n)_n$ of $\mathbb{R}$, there exists a subsequence $(\sigma'_n)_n$ of $(\sigma_n)_n$ such that the sequence $(U_{\sigma'_n}(t,\tau)\,x)_n$ converges to some $V(t,\tau)\,x$ in $\mathbb{R}^N$ uniformly in $t \in \mathbb{R}$ and pointwise in $(\tau,x) \in \mathbb{R}^+\times\mathbb{R}^N$.

We denote by $H(U)$ the *hull* of U the set of all processes V on $\mathbb{R}^N$ for which there exists a sequence $(\sigma_n)_n$ of $\mathbb{R}$ such that $U_{\sigma_n}(t,\tau)\,x \to V(t,\tau)\,x$ uniformly in $t \in \mathbb{R}$ and pointwise in $(\tau,x) \in \mathbb{R}^+\times\mathbb{R}^N$.

Let U a process on $\mathbb{R}^N$.

The *positive trajectory* trough $(t,x) \in \mathbb{R}\times\mathbb{R}^N$ is the map $U(t,\cdot)\,x\colon \mathbb{R}^+ \to \mathbb{R}^N$.

A *complete trajectory* through $(t,x) \in \mathbb{R}\times\mathbb{R}^N$ is a map $u(\cdot)\colon \mathbb{R} \to \mathbb{R}^N$ such that $u(t) = x$ and

$$\forall\,(s,\tau) \in \mathbb{R}\times\mathbb{R}^+\,, \qquad u(s+\tau) = U(s,\tau)\,u(s) \;.$$

3 – Statement of the results

In the case of the simple first order system (1.1) with $A(t)$ symmetric, $A(t) \ge 0$ and $A(t)$ periodic in t, the first author has announced, without proof, in [4], and has established in ([5], Theorem 4, p. 48) the existence of a unique exponentially stable almost periodic solution for any almost periodic forcing term f it the average $\mathcal{M}\{A(t)\}_t$ is positive definite. However if $N=1$, a trivial calculation indicates that this result remains valid in the general almost periodic case. In this paper we prove the following generalization for any dimension $N \ge 1$.

Theorem 3.1. *Assume that $A\colon \mathbb{R} \to \mathcal{L}(\mathbb{R}^N)$ is an almost periodic operator-valued function, such that for all $t \in \mathbb{R}$, $A(t)$ is symmetric and $A(t) \ge 0$. If the average $\mathcal{M}\{A(t)\}_t$ is positive definite (i.e.* $\operatorname{Ker}\mathcal{M}\{A(t)\}_t = \{0\}$*), then for each* *almost periodic forcing term* $f : \mathbb{R} \to \mathbb{R}^N$ *there exists a unique, exponentially stable almost periodic solution* u *of (1.1). In addition we have*

$$\operatorname{mod}(u) \subset \operatorname{mod}(A, f) \subset \operatorname{mod}(A) + \operatorname{mod}(f) .$$

Remark. $\operatorname{Ker} \mathcal{M}\{A(t)\}_t = \bigcap_{t\in\mathbb{R}} \operatorname{Ker} A(t)$ ([5], Theorem 3, p. 30). ◻

Theorem 3.1 will be a consequence of the general result: Theorem 3.4.

If the kernel of $\mathcal{M}\{A(t)\}_t$ is not trivial, we shall study the conditions on the almost periodic term f, to obtain the existence of almost periodic solutions of (1.1). If u is an almost periodic solution of (1.1), then for all $c \in \operatorname{Ker} \mathcal{M}\{A(t)\}_t$, we have $(f(t), c) = \frac{d}{dt}(u(t), c)$, therefore $[t \to \int_0^t (f(s), c)\, ds]$ is almost periodic. We shall prove that this condition is sufficient. Recall that the set of almost periodic solutions of homogeneous equation (1.4) is the set of constant functions $u(\cdot) \equiv c$ with $c \in \operatorname{Ker} \mathcal{M}\{A(t)\}_t$ ([4], Corollary 2 or [5], Theorem 1, p. 35). The existence of almost periodic solutions of (1.1) is established by Proposition 3.3.

Before we give an application of Theorem 3.1 to the nonlinear differential equation (1.2) by the inverse mapping theorem. This result can be viewed as a property of structural stability of almost periodic solutions.

Corollary 3.2. *Let* $\Phi \in C^2(\mathbb{R}^N, \mathbb{R})$ *a convex mapping. Let* u_0 *an almost periodic solution of (1.2) with some almost periodic forcing term* $f = f_0$. *If the average* $\mathcal{M}\{\nabla^2\Phi(u_0(t))\}_t$ *is positive definite (*$\nabla^2\Phi$ *denotes the Hessian operator of* Φ*), then there exists* $\epsilon > 0$ *such that for each almost periodic forcing term* $f : \mathbb{R} \to \mathbb{R}^N$ *satisfying* $\sup_{t\in\mathbb{R}} \|f(t) - f_0(t)\| < \epsilon$, *the equation (1.2) has one and only one almost periodic solution.*

Proposition 3.3. *Let* $f : \mathbb{R} \to \mathbb{R}^N$ *an almost periodic function. Assume that* $A : \mathbb{R} \to \mathcal{L}(\mathbb{R}^N)$ *is an almost periodic operator-valued function, such that for all* $t \in \mathbb{R}$, $A(t)$ *is symmetric and* $A(t) \geq 0$.

The equation (1.1) has at least one almost periodic solution, if and only if for all $c \in \operatorname{Ker} \mathcal{M}\{A(t)\}_t$ *the function* $[t \to \int_0^t (f(s), c)\, ds]$ *is almost periodic.*

The following result extends to a nonautonomous almost periodic framework a well-known alternative concerning contraction semigroups, cf. e.g. [9], [10].

Theorem 3.4. *Let* $U = U(t, \tau)$ *be an almost periodic linear contraction process on* $\mathbb{R}^N$. *Then one of the following alternative is fullfilled:*

(i) *There is a complete trajectory* $z = z(s)$ *of* U *with constant positive norm.*

(ii) *There are two constants $C \geq 0$, $\delta > 0$ such that $\|U(t,\tau)\|_{\mathcal{L}(\mathbb{R}^N)} \leq C\, e^{-\delta\tau}$, for all $t \in \mathbb{R}$ and $\tau > 0$.*

Theorem 3.4 is applicable to any equation of the form (1.1) with $A(t) \geq 0$ (we don't assume that $A(t)$ is symmetric). We denote by

$$S(t) := \frac{1}{2}\Big(A(t) + A^*(t)\Big) \quad \text{and} \quad K(t) := \frac{1}{2}\Big(A(t) - A^*(t)\Big) .$$

The following result is a direct consequence of Theorem 3.4.

Corollary 3.5. *Assume that $A\colon \mathbb{R} \to \mathcal{L}(\mathbb{R}^N)$ is an almost periodic operator-valued function, such that for all $t \in \mathbb{R}$, $A(t) \geq 0$. Then one of the following alternative is fullfilled:*

(i) *There is a solution $v = v(t) \not\equiv 0$ of*

$$v'(t) + K(t)\, v(t) \;=\; 0 \tag{3.1}$$

on $\mathbb{R}$ with $S(t)\, v(t) \equiv 0$. In this case $u(t) = t\, v(t)$ is an unbounded solution of (1.1) with $f = v$. Therefore (1.1) has no bounded solution for $f = v$.

(ii) *There is* **no** *solution $v = v(t) \not\equiv 0$ of (3.1) with $S(t)\, v(t) \equiv 0$. In this case (1.1) has a unique exponentially stable almost periodic solution for any almost periodic f.*

When $A(t) \geq 0$, our aim is to study necessary and sufficient conditions on the almost periodic forcing term f, for the existence of almost periodic solutions of (1.1). If u is an almost periodic solution of (1.1), then for any almost periodic solution v of (3.1) with $S(t)\, v(t) \equiv 0$ on $\mathbb{R}$, we have

$$\begin{aligned} \Big(f(t), v(t)\Big) &= \Big(u'(t) + A(t)\, u(t), v(t)\Big) \\ &= \Big(u'(t), v(t)\Big) + \Big(u(t), v'(t)\Big) \;=\; \frac{d}{dt}\Big(u(t), v(t)\Big) \end{aligned}$$

therefore $[t \to \int_0^t (f(s), v(s))\, ds]$ is almost periodic. In the case where $K(t)$ is periodic, we shall prove that this condition is sufficient; for that, we use Proposition 3.3.

Theorem 3.6. *Let $f : \mathbb{R} \to \mathbb{R}^N$ an almost periodic function. Assume that $A\colon \mathbb{R} \to \mathcal{L}(\mathbb{R}^N)$ is an almost periodic operator-valued function, such that for all $t \in \mathbb{R}$, $A(t) \geq 0$. We also assume that $K\colon \mathbb{R} \to \mathcal{L}(\mathbb{R}^N)$ is periodic.*

The equation (1.1) has at least one almost periodic solution, if and only if for all solution v of (3.1) with $S(t)\,v(t) \equiv 0$ on $\mathbb{R}$, the function

$$\left[t \to \int_0^t \Big(f(s), v(s)\Big)\, ds\right]$$

is almost periodic.

In particular, the last result is applicable to the following second order differential system (1.3) where L is a fixed positive definite symmetric operator, $B\colon \mathbb{R} \to \mathcal{L}(\mathbb{R}^N)$ is an almost periodic operator-valued function with $B(t)$ symmetric and $B(t) \geq 0$, we obtain the following result:

Corollary 3.7. *Let $f\colon \mathbb{R} \to \mathbb{R}^N$ an almost periodic function. The equation (1.3) has at least one almost periodic solution, if and only if for all solution z of $z''(t) + Lz(t) = 0$ with $B(t)\,z'(t) \equiv 0$ on $\mathbb{R}$, the function*

$$\left[t \to \int_0^t \Big(f(s), z'(s)\Big)\, ds\right]$$

is almost periodic.

In addition we can distinguish the following two cases:

- **(i)** *There is a solution $z = z(t) \not\equiv 0$ of $z'' + Lz(t) = 0$ on $\mathbb{R}$ with $B(t)\,z'(t) \equiv 0$ on $\mathbb{R}$. In this case $v(t) = t\,z(t)$ is an unbounded solution of (1.3) with $f(t) = 2\,z'(t) + B(t)\,z(t)$. Therefore (1.3) has no almost periodic solution for $f = 2\,z' + B(t)\,z$.*
- **(ii)** *There is* **no** *solution $z = z(t) \not\equiv 0$ of $z'' + Lz(t) = 0$ on $\mathbb{R}$ with $B(t)\,z'(t) \equiv 0$. In this case (1.3) has a unique exponentially stable almost periodic solution for any almost periodic f.*

It is natural comparing with problem (1.1), to wander whether it is sufficient, in order for (ii) to happen, to assume that the average $\mathcal{M}\{B(t)\}_t$ of the almost periodic operator-valued function $B(t)$ is positive definite. The situation is in fact more complicated. We shall prove the following.

Proposition 3.8.

- **(i)** *If $N = 1$ and the average $\mathcal{M}\{B(t)\}_t$ of $B(t)$ is positive, (1.3) has a unique exponentially stable almost periodic solution for any almost periodic f.*
- **(ii)** *If $N \geq 2$, in order for (1.3) to have an almost periodic solution for any almost periodic f, it is not sufficient that the average $\mathcal{M}\{B(t)\}_t$ of the almost periodic operator-valued function $B(t)$ be positive definite.*

To close this section we compare our results (Theorem 3.1 and Corollary 3.5) first with some results of Aulbach and Minh in [2], then with Seifert [15].

Remark 3.9. Aulbach and Minh in [2] give necessary and sufficient conditions for the equation (1.4) to have an exponential dichotomy ([2], Proposition 4 and Corollary 1). This result is as follows: let $A \in AP^0(\mathcal{L}(\mathbb{R}^N))$ and let T^h, $h > 0$ be the evolution operators associated with equation (1.4) acting on $AP^0(\mathbb{R}^N)$, i.e. $T^h u(t) = X(t)\,X^{-1}(t-h)\,u(t-h)$ for all $t \in \mathbb{R}$ and $u \in AP^0(\mathbb{R}^N)$ where $X(t)$ denotes the fundamental matrix for (1.4) such that $X(0) = I$. Then the equation (1.4) has an exponential dichotomy if and only if for some $h > 0$, the evolution operator T^h is hyperbolic, i.e. if $\lambda \in \mathrm{sp}(T^h)$ (spectrum of T^h), then $|\lambda| \neq 1$, and if and only if the difference equation $u(t) = X(t)\,X^{-1}(t-h)\,u(t-h) + f(t-h)$ have a unique solution in $AP^0(\mathbb{R}^N)$ for every $f \in AP^0(\mathbb{R}^N)$. To compare this result with ours, we suppose that $A(t) \geq 0$ for all $t \in \mathbb{R}$. With this assumption the equation (1.4) have an exponential dichotomy if and only if $X(t)$ is exponentially damped. By definition of T^h, we have

$$\|T^h\| \;=\; \sup\left\{\sup_{t\in\mathbb{R}} \|X(t)\,X^{-1}(t-h)\,v(t-h)\|;\; v \in AP^0(\mathbb{R}^N) \;\text{ et }\; \|v\|_\infty \leq 1\right\},$$

then $\|T^h\| = \sup_{t\in\mathbb{R}} \|X(t)\,X^{-1}(t-h)\|$. With the assumption $A(t) \geq 0$, we have $\|T^h\| \leq 1$, then the equation (1.4) have an exponential dichotomy if and only if there exists $h > 0$ such that

$$\sup_{t\in\mathbb{R}} \|X(t)\,X^{-1}(t-h)\| < 1\ .$$

If we use this latter result to establish Theorem 3.1 and Corollary 3.5, we must prove that the condition: for all $h > 0$

$$\sup_{t\in\mathbb{R}} \|X(t)\,X^{-1}(t-h)\| = 1$$

implies the existence of a solution u of (1.4) such that $\|u(t)\|$ is constant. This is in fact exactly the object of our Lemma 4.1, therefore the result of [2] is of no help for us. Moreover, our results are a consequence of the more general Theorem 3.3 valid for all almost periodic linear contraction processes on $\mathbb{R}^N$. □

Remark 3.10. For the equation (1.1) when $A(t)$ is a possibly unbounded linear operator in a Banach space, Seifert gives sufficient conditions for the existence of almost periodic solutions. When $A\colon \mathbb{R} \to \mathcal{L}(\mathbb{R}^N)$ is an almost periodic operator such that $A(t) \geq 0$ for all $t \in \mathbb{R}$, the result of Seifert ([15], Theorem 4) becomes "if A satisfies the following condition:

(**C**) *there exists* $\omega\colon \mathbb{R} \to \mathbb{R}$ *almost periodic such that* $\mathcal{M}\{\omega(t)\}_t < 0$ *and for all* $t \in \mathbb{R}$*, and for all* $x \in \mathbb{R}^N$ *such that* $x \neq 0$*, one has*

$$\lim_{\theta \to 0^+} \frac{\|x\| - \|x + \theta\, A(t)\, x\|}{\theta\, \|x\|} \leq \omega(t) \ ,$$

then (1.1) has a unique almost periodic solution for any almost periodic forcing term f".

To compare this resut with ours, we must study the condition (C). It is easily checked that

$$\lim_{\theta \to 0^+} \frac{\|x\| - \|x + \theta\, A(t)\, x\|}{\theta\, \|x\|} = -\frac{(A(t)\, x,\, x)}{\|x\|^2} \ .$$

Recall that

$$\inf_{x \neq 0} \frac{\langle A(t)\, x \mid x\rangle}{|x|^2} = \lambda(t) \ ,$$

where $\lambda(t)$ denotes the smallest eigenvalue of $S(t) := \frac{1}{2}(A(t) + A^*(t))$. Moreover λ is almost periodic, then the condition (C) becomes

$$\mathcal{M}\{\lambda(t)\}_t > 0 \ .$$

Using the fact that a solution v of (3.1) satisfies $\|v(t)\| \equiv \text{constant}$, and $\langle S(t)\, v(t) \,|\, v(t)\rangle \geq \lambda(t)\, \|v(t)\|^2$, we deduce that the condition (C) implies there is no solution $v = v(t) \not\equiv 0$ of (3.1) with $S(t)\, v(t) \equiv 0$. Now, if

$$S(t) := \begin{pmatrix} \cos^2(t) & \sin(t)\cos(t) \\ \sin(t)\cos(t) & \sin^2(t) \end{pmatrix}$$

and

$$K(t) := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

our Corollary 3.5 gives the existence of almost periodic solutions, because there is no solution $v = v(t) \not\equiv 0$ of (3.1) with $S(t)\, v(t) \equiv 0$, but since $\lambda(t) \equiv 0$ we cannot conclude with the result of Seifert.

Even in the case where $A(t)$ is symmetric, the condition (C) becomes

$$\bigcap_{t \in \mathbb{R}} \operatorname{Ker} A(t) = \{0\} \ ,$$

therefore $\operatorname{Ker} \mathcal{M}\{A(t)\}_t = \{0\}$ (cf. the remark of Theorem 3.1). If

$$A(t) = S(t) := \begin{pmatrix} \cos^2(t) & \sin(t)\cos(t) \\ \sin(t)\cos(t) & \sin^2(t) \end{pmatrix}$$

our Theorem 3.1 gives the existence of almost periodic solutions since $\mathcal{M}\{A(t)\}_t = \frac{1}{2} I_2$, but we cannot conclude with the result of Seifert because $\lambda(t) \equiv 0$.

As a conclusion, when $A\colon \mathbb{R} \to \mathcal{L}(\mathbb{R}^N)$ is an almost periodic operator such that $A(t) \geq 0$ for all $t \in \mathbb{R}$, our results are systematically better. □

4 – Proof of Theorem 3.4

The object of this section is to prove Theorem 3.4. We start by the following lemma.

Lemma 4.1. *Let $U = U(t,\tau)$ be an almost periodic linear contraction process on $\mathbb{R}^N$. Then one of the following alternatives is fullfilled: either there is some $\tau_0 > 0$ for which*

$$\sup_{t\in\mathbb{R}} \|U(t,\tau_0)\|_{\mathcal{L}(\mathbb{R}^N)} < 1 \tag{4.1}$$

or there is a complete trajectory $z = z(s)$ of U with constant positive norm.

Proof of Lemma 4.1: Let us recall that, if U is an almost periodic linear contraction process on $\mathbb{R}^N$, then

$$\forall\,\tau \geq 0\,, \quad \sup_{t\in\mathbb{R}} \|U(t,\tau)\|_{\mathcal{L}(\mathbb{R}^N)} \leq 1\ .$$

Assuming that (4.1) is not satisfied for any $\tau > 0$, there exists a sequence $(t_n)_n$ of real numbers and a sequence of vectors $(x_n)_n$ in $\mathbb{R}^N$ such that

$$\forall\, n \in \mathbb{N}^*\,, \quad \|x_n\| = 1 \ \text{ and } \lim_{n\to+\infty} \|U(t_n,n)\,x_n\| = 1\ .$$

Assuming in addition (up to a subsequence) that $(x_n)_n$ converges to a limit $x \in \mathbb{R}^N$. It follows from

$$\forall\,\tau \in \mathbb{R}^+\,, \quad U(t_n,n)\,x_n = U(t_n+\tau, n-\tau)\,U(t_n,\tau)\,x_n$$

the inequalities

$$\|U(t_n,n)\,x_n\| \;\leq\; \|U(t_n,\tau)\,x_n\| \;\leq\; \|x_n\| \;=\; 1\ ,$$

therefore

$$\|x_n\| = 1 \quad \text{and} \quad \forall\,\tau \geq 0 \ \lim_{n\to+\infty} \|U(t_n,\tau)\,x_n\| = 1\ .$$

Using inequalities

$$\|U(t_n,\tau)\,x_n\| \;\leq\; \|x_n - x\| + \|U(t_n,\tau)\,x\| \;\leq\; \|x_n - x\| + 1\ ,$$

we obtain

$$\|x\| = 1 \quad \text{and} \quad \forall\,\tau \geq 0 \ \ \lim_{n\to+\infty} \|U(t_n,\tau)\,x\| = 1\ .$$

Passing again to a subsequence, we may assume that $\forall\,\tau \in \mathbb{R}$, $U(t_n+s,\tau)\,x$ converges to $V(s,\tau)\,x$ uniformly in $s \in \mathbb{R}$ for some $V \in H(U)$. Then we have

$$\|x\| = 1 \quad \text{and} \quad \forall\,\tau \geq 0 \ \ \|V(0,\tau)\,x\| = 1\ . \tag{4.2}$$

By using Corollary 3.5 and Lemma 3.6 of ([6], p. 49) we deduce the existence of a sequence $(a_n)_n$ of positive real numbers tending to infinity such that $V(a_n+t,\tau)\,y$ converges to $U(t,\tau)\,y$ uniformly in $t \in \mathbb{R}$ and pointwise $(\tau,y) \in \mathbb{R}^+ \times \mathbb{R}^N$. By Lemma 3.7 of ([6], p. 50), there is a subsequence of $(a_n)_n$, denoted again by $(a_n)_n$ such that $V(0,a_n+s)\,x$ converges for all $s \in \mathbb{R}$ to a complete trajectory $z(s)$ of U. With the relation (4.2), we deduce

$$\forall\,s \in \mathbb{R} \quad \|z(s)\| = 1\ . \ \blacksquare$$

Proof of Theorem 3.4: We introduce

$$\rho := \sup_{t\in\mathbb{R}} \|U(t,\tau_0)\|_{\mathcal{L}(\mathbb{R}^N)}\ .$$

According to the result of Lemma 4.1, if (i) is not fullfilled we have $0 \leq \rho < 1$. Let now $t \in \mathbb{R}$ be arbitrary and $\tau > 0$. We set $\tau = n\,\tau_0 + \sigma$ with $n \in \mathbb{N}$ and $0 \leq \sigma < \tau_0$. We obtain

$$U(t,\tau) \;=\; U(t+n\,\tau_0,\sigma)\,\prod_{j=0}^{n-1} U(t+j\,\tau_0,\tau_0)\ ,$$

therefore

$$\|U(t,\tau)\|_{\mathcal{L}(\mathbb{R}^N)} \;\leq\; \left\|\prod_{j=0}^{n-1} U(t+j\,\tau_0,\tau_0)\right\|_{\mathcal{L}(\mathbb{R}^N)} \leq\; \rho^n\ .$$

The case $\rho = 0$ is trivial. If ρ is positive, we have

$$\|U(t,\tau)\|_{\mathcal{L}(\mathbb{R}^N)} \leq C\,e^{-\delta\tau}, \quad \text{with}\ \ C := \frac{1}{\rho}\ \ \text{and}\ \ \delta := \frac{1}{\tau_0}\ln\!\left(\frac{1}{\rho}\right)\ ,$$

hence (ii) is fullfilled. Moreover if z is a complete trajectory, then

$$\forall \tau > 0 \quad z(\tau) = U(0,\tau)\, z(0) \ ,$$

and if (ii) is fullfilled, then

$$\forall \tau > 0 \quad \|z(\tau)\| \leq C\, \|z(0)\|\, e^{-\delta\tau} \ ,$$

therefore (i) and (ii) are compatible, this concludes the proof of Theorem 3.4. ∎

5 – Consequences of Theorem 3.4

The object of this section is to prove Theorems 3.1, 3.6, propositions 3.3, 3.8 and Corollary 3.2, 3.5, 3.7. In the proof of Theorem 3.1 a crucial role will be played by the following lemma.

Lemma 5.1. *Let $A \in C([0,T]; \mathcal{L}(\mathbb{R}^N))$ be such that $A(t)$ is symmetric and $A(t) \geq 0$ for all $t \in [0,T]$. We assume that*

$$\mathrm{Ker}\left\{\int_0^T A(t)\, dt\right\} = \{0\} \ . \tag{5.1}$$

Then any solution y of homogeneous equation (1.4) on $[0,T]$ with $y(0) \neq 0$ is such that: $\|y(T)\| < \|y(0)\|$.

Proof of Lemma 5.1: Assuming $\|y(T)\| = \|y(0)\|$, since

$$\frac{d}{dt}\left(\frac{1}{2}\,\|y(t)\|^2\right) \;=\; \Big(y'(t), y(t)\Big) \;=\; -\Big(A(t)\, y(t), y(t)\Big) \;\leq\; 0 \ ,$$

we have $\|y(t)\| = \|y(0)\|$ for all $t \in [0,T]$ and therefore $(A(t)\, y(t), y(t)) \equiv 0$ on $[0,T]$. Since $(A(t))^* = A(t) \geq 0$, we have $A(t)\, y(t) \equiv 0$, which implies in fact $y'(t) \equiv 0$. Hence $y(t) \equiv y(0)$ on $[0,T]$ and in particular

$$\left\{\int_0^T A(t)\, dt\right\} y(0) \;=\; 0 \ ,$$

which by hypothesis (5.1) implies $y(0) = 0$. ∎

Proof of Theorem 3.1: Let $f\colon \mathbb{R} \to \mathbb{R}^N$ an almost periodic function. For any $(t,\tau) \in \mathbb{R}{\times}\mathbb{R}^+$, let $V(t,\tau)$ denote the solution operator which assigns to each

$x \in \mathbb{R}^N$ the value $v(t+\tau)$ at time $t+\tau$ of the unique solution v of (1,1) such that $v(t) = x$. By using the inequality for $s_1 \leq s_2$

$$\|v(s_2)\| \leq \|v(s_1)\| + \int_{s_1}^{s_2} \|f(\sigma)\| \, d\sigma$$

and linear dependence with respect to (x, f) it is easily checked that V is an almost periodic process on $\mathbb{R}^N$. As a consequence of Corollary 2.9 and Theorem 2.5 of ([6], pp. 46–47) our results will be established if we prove that V has precisely one complete trajectory with relatively compact range in $\mathbb{R}^N$.

For any $(s,t) \in \mathbb{R} \times \mathbb{R}^+$, we define the map $U(t,\tau)\colon \mathbb{R}^N \to \mathbb{R}^N$ by $U = V$ with $f = 0$, in other terms we have $U(t,\tau) = \Phi(t+\tau, t)$ for all $(t,\tau) \in \mathbb{R} \times \mathbb{R}^+$ where Φ is the fundamental matrix for (1.4). It is clear that U is an almost periodic linear contraction process on $\mathbb{R}^N$. We shall establish that U is an exponential damped process, which means:

$$\exists\, C \geq 0, \quad \exists\, \delta > 0, \quad \forall\, t \in \mathbb{R}, \quad \forall\, \tau \in \mathbb{R}^+, \qquad \|U(t,\tau)\|_{\mathcal{L}(\mathbb{R}^N)} \leq C\, e^{-\delta \tau} .$$

As a consequence of Theorem 3.4, t is sufficient to show that (i) is impossible. In order to do that, first we recall that

$$\lim_{T \to +\infty} \frac{1}{T} \int_0^T A(t)\, dt = \mathcal{M}\{A(t)\}_t .$$

In particular there is $T > 0$ such that

$$\mathrm{Ker}\left\{ \int_0^T A(t)\, dt \right\} = \{0\} .$$

Then by Lemma 5.1, (i) of Theorem 3.4 is impossible It is now clear that $U(t,\tau)$ is an exponentially damped process.

First U has precisely one complete trajectory u with precompact range: namely $u(s) \equiv 0$; this implies the uniqueness of complete trajectory of V with precompact range.

By the relation

$$V(0,\tau)\, 0 = \int_0^\tau U(\sigma, \tau - \sigma)\, f(\sigma)\, d\sigma$$

we obtain for all $\tau \in \mathbb{R}^+$

$$\|V(0,\tau)\, 0\| \leq \frac{C}{\delta} \sup_{s \in \mathbb{R}} \|f(s)\| .$$

Hence V has some positive trajectory with precompact range, by Theorem 2.7 of ([6], p. 47), this implies the existence of complete trajectories of V with precompact range.

Therefore for each f bounded on $\mathbb{R}$ with values in $\mathbb{R}^N$, (1.1) has a unique bounded complete trajectory $u = u(f)$. When f is almost periodic, so is $u(f)$ and moreover we have $\mathrm{mod}(u) \subset \mathrm{mod}(A, f) \subset \mathrm{mod}(A) + \mathrm{mod}(f)$. This concludes the proof of Theorem 3.1. ∎

Proof of Corollary 3.2: We denote by $AP^0(\mathbb{R}^N)$ the space of Bohr-almost periodic functions from $\mathbb{R}$ to $\mathbb{R}^N$ and $AP^1(\mathbb{R}^N)$ the space of functions in $AP^0(\mathbb{R}^N) \cap C^1(\mathbb{R}, \mathbb{R}^N)$ such that their derivatives are in $AP^0(\mathbb{R}^N)$. Recall that $AP^0(\mathbb{R}^N)$ (resp. $AP^1(\mathbb{R}^N)$) endowed with the norm

$$\|u\|_\infty := \sup_{t\in\mathbb{R}} \|u(t)\| \qquad \Big(\text{resp. } \|u\|_{C^1} := \|u\|_\infty + \|u'\|_\infty\Big)$$

is a Banach space.

Recall also that the Nemitski operator built on $\nabla\Phi$ is the mapping $\mathcal{N}_{\nabla\Phi}$ from $AP^0(\mathbb{R}^N)$ in $AP^0(\mathbb{R}^N)$ defined by $\mathcal{N}_{\nabla\Phi}(u) := \nabla\Phi \circ u$. Since $\nabla\Phi \in C^1(\mathbb{R}^N, \mathbb{R}^N)$, then $\mathcal{N}_{\nabla\Phi}$ is of class C^1 on $AP^0(\mathbb{R}^N)$ and $(D\mathcal{N}_{\nabla\Phi}(u)\, h)(t) = \nabla^2\Phi(u(t))\, h(t)$ for all $t \in \mathbb{R}$, u and $h \in AP^0(\mathbb{R}^N)$ ([3], Lemma 7).

Now we consider the nonlinear operator F from $AP^1(\mathbb{R}^N)$ in $AP^0(\mathbb{R}^N)$ defined by $F(u) := u' + \nabla\Phi \circ u$. We note that $F = D + \mathcal{N}_{\nabla\Phi} \cap I$, where D is the derivative operator from $AP^1(\mathbb{R}^N)$ in $AP^0(\mathbb{R}^N)$ and I is the canonical injection from $AP^1(\mathbb{R}^N)$ in $AP^0(\mathbb{R}^N)$. Since the linear maps D and I are bounded, and since $\mathcal{N}_{\nabla\Phi}$ is of class C^1, we see that F is of class C^1, and $(DF(u)\, h)(t) = h'(t) + \nabla^2\Phi(u(t))\, h(t)$ for all $t \in \mathbb{R}$, u and $h \in AP^1(\mathbb{R}^N)$. With the assumption $\mathcal{M}\{\nabla^2\Phi(u_0(t))\}_t = \{0\}$ and using Theorem 3.1 with $A(t) := \nabla^2\Phi(u_0(t))$, we see that for each almost periodic forcing term f, there exists a unique almost periodic solution h of (1.1), i.e. $DF(u_0)h = f$; and so $DF(u_0) \in \mathrm{Isom}(AP^1(\mathbb{R}^N), AP^0(\mathbb{R}^N))$.

Using the inverse mapping theorem, we see that there exists U an open neighbourhood of u_0 in $AP^1(\mathbb{R}^N)$ such that $F\colon U \to F(U)$ is a C^1-diffeomorphism and $F(U)$ an open neighbourhood of $f_0 = F(u_0)$ in $AP^0(\mathbb{R}^N)$; and so for all f in the neighbourhood $F(U)$ of f_0, the equation (1.2) has at least one almost periodic solution. Since the set $\mathcal{S}$ of almost periodic solutions of (1.2) is convex ([8], Theorem 37, p. 72) and $\mathcal{S} \cap U$ contains only one element (local uniqueness of almost periodic solution), we obtain global uniqueness of almost periodic solution of (1.2). ∎

Proof of Proposition 3.3: The sufficiency of the condition remains to be proved. We denote $H^1 := \operatorname{Ker} \mathcal{M}\{A(t)\}_t$ and H_2 the orthogonal of H_1 in $\mathbb{R}^N$. Equation (1.1) can be put in the form

$$\begin{pmatrix} u_1'(t) \\ u_2'(t) \end{pmatrix} + \begin{pmatrix} A_{1,1}(t) & A_{1,2}(t) \\ A_{2,1}(t) & A_{2,2}(t) \end{pmatrix} \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}$$

by introducing the direct sum $\mathbb{R}^N := H_1 \oplus H_2$ and letting $u(t) = (u_1(t), u_2(t))$ and $f(t) = (f_1(t), f_2(t))$. Since $\operatorname{Ker} \mathcal{M}\{A(t)\}_t = \bigcap_{t\in\mathbb{R}} \operatorname{Ker} A(t)$, one has

$$A(t) = \begin{pmatrix} 0 & 0 \\ 0 & A_{2,2}(t) \end{pmatrix}$$

and equation (1.1) reduces to

$$\begin{cases} u_1'(t) & = f_1(t) \\ u_2'(t) + A_{2,2}(t)\, u_2(t) & = f_2(t)\ . \end{cases}$$

To conclude the proof of Proposition 3.3, we just remark that

$$\forall\, c \in \operatorname{Ker} \mathcal{M}\{A(t)\}_t\,, \qquad \int_0^t \Big(f(s), c\Big)\, ds = \Big(\int_0^t f_1(s)\, ds,\ c\Big)$$

then the function $[t \to \int_0^t f_1(s)\, ds]$ is almost periodic. Moreover $A_{2,2}(t)$ is symmetric, $A_{2,2}(t) \geq 0$ and $\operatorname{Ker} \mathcal{M}\{A_{2,2}(t)\}_t = \{0\}$, by Theorem 3.1, we obtain the conclusion. ∎

Proof of Corollary 3.5: For any $(t,\tau) \in \mathbb{R}\times\mathbb{R}^+$, let $U(t,\tau)$ denote the solution operator which assigns to each $x \in \mathbb{R}^N$ the value $u(t+\tau)$ at time $t+\tau$ of the unique solution u of (1.4) such that $u(t) = x$; therefore U is an almost periodic process on $\mathbb{R}^N$ (see proof of Theorem 3.1).

A solution v of $v'(t) + A(t)\, v(t) = 0$ with constant norm satisfies both conditions:

$$v'(t) + K(t)\, v(t) + S(t)\, v(t) = 0 \quad \text{and} \quad \|v(t)\| = \text{constant}\ .$$

By differentiating

$$0 = \frac{d}{dt}\Big(\|v(t)\|^2\Big) = 2\,\Big(v(t), v'(t)\Big)\ ,$$

the equation gives at once $(K(t)\, v(t) + S(t)\, v(t), v(t)) = 0$ identically. Because $K(t)$ is skew symmetric, it follows that $(S(t)\, v(t), v(t)) = 0$ identically, moreover $S(t)$ is symmetric and $S(t) \geq 0$, then $S(t)\, v(t) = 0$, and finally also $v'(t) + K(t)\, v(t) \equiv 0$.

The alternative (i) or (ii) is now an immediate consequence of Theorem 3.4. To conclude the proof of Corollary 3.5, we just remark that with z as above:

$$(t\,v(t))' + A(t)\,t\,v(t) \;=\; v(t)\ .$$

And clearly, by monotonicity, in such a case (1.1) has no bounded solution for $t \geq 0$. ∎

Proof of Theorem 3.6: The sufficiency of the condition remains to be proved. Denote by V the fundamental matrix of (3.1) such that $V(0) = I$. Because K is periodic and skew symmetric, it follows that V is almost periodic ([7], Theorem 6.13, p. 112). Recall that $V^*(t)\,V(t) = I$, for all $t \in \mathbb{R}$. If we set $u = V(t)\,w$, the equation (1.1) is equivalent to the equation

$$w'(t) + V^*(t)\,S(t)\,V(t)\,w(t) \;=\; V^*(t)\,f(t)\ . \tag{5.2}$$

V^* is also almost periodic, so for f almost periodic, the function g defined by $g(t) = V^*(t)\,f(t)$ is also almost periodic. Moreover

$$V^*(\cdot)\,S(\cdot)\,V(\cdot)\colon\ \mathbb{R} \to \mathcal{L}(\mathbb{R}^N)$$

is an almost periodic operator-valued function such that for all $t \in \mathbb{R}$, $V^*(t)\,S(t)\,V(t)$ is symmetric and $V^*(t)\,S(t)\,V(t) \geq 0$, it follows

$$\operatorname{Ker} \mathcal{M}\{V^*(t)\,S(t)\,V(t)\}_t = \bigcap_{t\in\mathbb{R}} \operatorname{Ker} V^*(t)\,S(t)\,V(t) \tag{5.3}$$

([5], Theorem 3, p. 30).

The assumption of Theorem 3.6 is equivalent to: for all $c \in \bigcap_{t\in\mathbb{R}} \operatorname{Ker} S(t)\,V(t)$ the function $[t \to \int_0^t (f(s), V(s)\,c)\,ds]$ is almost periodic. With

$$\int_0^t \Big(f(s), V(s)\,c\Big)\,ds \;=\; \int_0^t \Big(V^*(s), f(s)\,c\Big)\,ds\ ,$$

and

$$\bigcap_{t\in\mathbb{R}} \operatorname{Ker} S(t)\,V(t) = \bigcap_{t\in\mathbb{R}} \operatorname{Ker} V^*(t)\,S(t)\,V(t)\ ,$$

and (5.3), the assumption of Theorem 3.6 is also equivalent to

$$\forall\, c \,\in\, \operatorname{Ker} \mathcal{M}\{V^*(t)\,S(t)\,V(t)\}_t$$

the function $[t \to \int_0^t (V(s)^*\,f(s),\,c)\,ds]$ is almost periodic. The existence of an almost periodic solution (5.2) is now an immediate consequence of Proposition 3.3; therefore there exists an almost periodic solution of (1.1). ∎

Proof of Corollary 3.7: Equation (1.3) can be put in the form (1.1) by introducing the product space $\mathbb{R}^N \times \mathbb{R}^N \simeq \mathbb{R}^{2N}$ endowed with the inner product associated to the quadratic form Φ given by

$$\Phi(u,v) := \|L^{\frac{1}{2}} u\|^2 + \|v\|^2 ,$$

and letting $v(t) = u'(t)$, $U(t) = (u(t), v(t)) = (u(t), u'(t))$. In this framework (1.3) reduces to

$$U' + A(t)\, U(t) = F(t)$$

with

$$F(t) = \Big(0, f(t)\Big) \quad \text{and} \quad A(t) = \begin{pmatrix} 0 & -I \\ L & B(t) \end{pmatrix} .$$

By computing, we obtain, with the respect of the inner product associated to Φ:

$$K(t) = \begin{pmatrix} 0 & -I \\ L & 0 \end{pmatrix} \quad \text{and} \quad S(t) = \begin{pmatrix} 0 & 0 \\ 0 & B(t) \end{pmatrix} .$$

Moreover S is an almost periodic operator-valued function, such that for all $t \in \mathbb{R}$, $S(t) \geq 0$ with respect to the inner product. The conclusion is now an immediate consequence of Theorem 3.6 and Corollary 3.5. ∎

Proof of Proposition 3.8:

(**i**) If $N = 1$, the solutions of $z'' + Lz = 0$ can be written as $z(t) = \rho \cos(\omega t + \varphi)$ with $L = \omega^2 > 0$, and we derive

$$z'(t) = \rho\, \omega \cos(\omega t + \psi)$$

with $\psi := \varphi + \frac{\pi}{2}$. In addition here $B(t)\, x = b(t)\, x$ for some real-valued function $b(t) \geq 0$. The condition $b(t)\, z'(t) \equiv 0$ is here equivalent to $\rho\, \omega\, b(t) \equiv 0$. If $\mathcal{M}\{b(t)\}_t > 0$, this implies $\rho = 0$.

(**ii**) Already for $N = 2$ there may exist non trivial solutions of $z'' + Lz = 0$, even when $\mathcal{M}\{B(t)\} > 0$. As a simple example we can choose

$$L = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B(t) = \begin{pmatrix} \cos^2 t & \sin t\, \cos t \\ \sin t\, \cos t & \sin^2 t \end{pmatrix}$$

hence

$$\mathcal{M}\{B(t)\} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} > 0 .$$

The solution

$$Z(t) = \Big(z(t), z'(t)\Big) = \Big((\cos t, \sin t); (-\sin t, \cos t)\Big)$$

satisfies all the conditions. The proof of (i) is now complete. ∎

ACKNOWLEDGEMENT – The authors are indebted to the referee for pointing out the references [2] and [12]–[16] to their attention.

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