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# ON THE SET $a\,x + b\,g^x \pmod{p}$

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**Abstract:** Given nonzero integers $a$, $b$ we prove an asymptotic result for the distribution function of the set $a\,x + b\,g^x \pmod{p}$, as $p$ goes to infinity and $g$ is a primitive root mod $p$.

## 1 – Introduction

Various aspects of the distribution of powers of a primitive root $g$ modulo a large prime number $p$ have been investigated by a number of authors (see for example [2], [3], [4], [6], [7], [8]). In this paper we fix nonzero integers $a$, $b$ and study the distribution function of the set $a\,x + b\,g^x \pmod{p}$, as $p$ goes to infinity and $g$ is a primitive root mod $p$. In particular we are interested in the distance between $x$ and $g^x$ as $x$ runs over the set $\{1, 2, ..., p-1\}$. Throughout this paper $g^x$ means the least positive residue of $g^x$ mod $p$. We also consider a short interval version of the problem, more precisely we fix two intervals $\mathcal{I}$, $\mathcal{J}$ and work only with those integers $x \in \mathcal{I}$ for which $g^x \pmod{p}$ belongs to $\mathcal{J}$. In the following we let $\mathcal{I} = \{0, 1, ..., M-1\}$, $\mathcal{J} = \{0, 1, ..., N-1\}$ with $M$, $N$ positive integers $\le p$ and denote $\mathcal{M} = \{x \in \mathcal{I}\colon\ g^x \in \mathcal{J},\ a\,x + b\,g^x < t\}$. The distribution function is given by $D(t) = D(a,b,p,g,\mathcal{I},\mathcal{J},t) = \#\mathcal{M}$. Replacing if necessary $a$, $b$ and $t$ by $-a, -b$ and $-t$ respectively, we may assume in the following that $b > 0$. We now introduce a function $G(t,a,b,M,N)$ which will appear in the estimation of $D(t)$.

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If $a > 0$ we set

$$G(t,a,b,M,N) \;=\; \begin{cases} 0, & \text{if } t < 0, \\ \dfrac{t^2}{2\,a\,b}, & \text{if } 0 \le t < U, \\ \dfrac{U^2}{2\,a\,b} + \dfrac{U(t-U)}{a\,b}, & \text{if } U \le t < V, \\ MN - \dfrac{(a\,M + b\,N - t)^2}{2\,a\,b}, & \text{if } V \le t < aM + bN, \\ MN, & \text{if } aM + bN \le t\ , \end{cases}$$

where $U = \min\{aM, bN\}$ and $V = \max\{aM, bN\}$. If $a < 0$ then we let

$$G(t,a,b,M,N) = \begin{cases} 0, & \text{if } t < aM, \\ -\dfrac{(t-a\,M)^2}{2\,a\,b}, & \text{if } aM \le t \le W, \\ \left(MN + \dfrac{(W-aM)^2}{a\,b}\right)\dfrac{t-W}{Z-W} - \dfrac{(W-aM)^2}{2\,a\,b}, & \text{if } W < t < Z, \\ MN + \dfrac{(t-b\,N)^2}{2\,a\,b}, & \text{if } Z \le t < bN, \\ MN, & \text{if } bN \le t\ , \end{cases}$$

where $W = \min\{0,\, bN + aM\}$ and $Z = \max\{0,\, bN + aM\}$. We will prove the following

**Theorem 1.** *For any* $a$, $b$, $p$, $g$, $\mathcal{I}$, $\mathcal{J}$, $t$ *as above one has*

$$D(a,b,p,g,\mathcal{I},\mathcal{J},t) \;=\; \frac{G(t,a,b,M,N)}{p} \;+\; O_{a,b}(p^{1/2}\,\log^3 p)\ .$$

It is well established that the discrete exponential map $x \mapsto g^x \bmod p$ is a "random" map, and this is used by random number generators which use the linear congruential method [1]. There are various ways to check this randomness. For instance, if we count those $x \in \{1,2,...,p-1\}$ for which $g^x < x$, respectively those $x$ for which $g^x > x$ there should be no bias towards any one of these inequalities, in other words one would expect that about half of the $x$'s are larger than $g^x$ and half of the $x$'s are smaller than $g^x$. We can actually prove this statement by using Theorem 1.

**Corollary 1.** *One has*

$$\left|\#\left\{0 \le x \le p-1:\ x > g^x\right\} - \frac{p}{2}\right| \le 7\,p^{1/2}(1+\log p)^3 \ .$$

As another application of Theorem 1 we have the following asymptotic result for all even moments of the distance between $x$ and $g^x$.

**Corollary 2.** *Let $k$ be a positive integer. Then we have*

$$M(p,g,2k) := \sum_{x=0}^{p-1}(g^x-x)^{2k} = \frac{p^{2k+1}}{(k+1)\,(2k+1)} + O_k(p^{2k+1/2}\ \log^3 p) \ .$$

In particular, for $k=1$ one has

$$M(p,g,2) = \frac{p^3}{6} + O(p^{5/2}\ \log^3 p) \ .$$

This says that in quadratic average $|g^x-x|$ is $\sim \frac{p}{\sqrt{6}}$.

## 2 – Setting the problem

We will need a bound for the exponential sum

$$S(m,n,g,p) = \sum_{z=0}^{p-1} e_p(m\,z+n\,g^z) \ ,$$

where $m,n$ are integers and $e_p(t)=e^{\frac{2\pi i t}{p}}$. This problem was handled by Mordell [5].

**Lemma 1** (Mordell). *Let $p$ be a prime, $g$ a primitive root mod $p$ and $m,n$ integers, not both multiples of $p$. Then*

$$|S(m,n,g,p)| < 2\,p^{1/2}(1+\log p) \ . \ \blacksquare$$

The next lemma allows us to compute quite general sums involving $x$ and $g^x$.

**Lemma 2.** *Let $\mathcal{U}$, $\mathcal{V}$ be subsets of $\{0,1,...,p-1\}$, let $f$ be a complex valued function defined on $\mathcal{U}\times\mathcal{V}$ and consider the transform*

$$\check{f}(m,n) = \sum_{(x,y)\in\mathcal{U}\times\mathcal{V}} f(x,y)\, e_p(m\,x+n\,y) \ .$$

*Then*

$$\sum_{\substack{(x,y)\in\mathcal{U}\times\mathcal{V}\\ y\equiv g^x \pmod p}} f(x,y) = \frac{1}{p^2}\sum_{m=0}^{p-1}\sum_{n=0}^{p-1} \check{f}(m,n)\ S(-m,-n,g,p) \ .$$

**Proof:** Using the definition, the right hand side can be written as

$$\begin{aligned}
\frac{1}{p^2}\sum_{m=0}^{p-1}\sum_{n=0}^{p-1} \check{f}(m,n)\, S(-m,-n,g,p) &= \\
&= \frac{1}{p^2}\sum_{m=0}^{p-1}\sum_{n=0}^{p-1}\sum_{(x,y)\in\mathcal{U}\times\mathcal{V}} f(x,y)\, e_p(mx+ny)\sum_{z=0}^{p-1} e_p(-m\,z-n\,g^z)\\
&= \frac{1}{p^2}\sum_{(x,y)\in\mathcal{U}\times\mathcal{V}} f(x,y)\sum_{z=0}^{p-1}\sum_{m=0}^{p-1} e_p(m(x-z))\sum_{n=0}^{p-1} e_p(n(y-g^z)) \ .
\end{aligned}$$

Here the sum over $n$ is zero unless $y\equiv g^z \pmod p$ when it equals $p$. Similarly, since $0<x,z\le p-1$ the sum over $m$ is zero unless $x=z$ when it equals $p$. Thus the sum over $z$ is zero if $y\not\equiv g^x \pmod p$ and it equals $p^2$ if $y\equiv g^x \pmod p$, which proves the statement of the lemma. ∎

We will apply Lemma 2 with $\mathcal{U}=\mathcal{I}$, $\mathcal{V}=\mathcal{J}$ and

$$f(x,y) = f(t,x,y,a,b) = \begin{cases} 1, & \text{if } a\,x+b\,y<t,\\ 0, & \text{if } a\,x+b\,y\ge t \ . \end{cases} \tag{1}$$

Then the distribution function is given by

$$D(t) = \sum_{\substack{(x,y)\in\mathcal{I}\times\mathcal{J}\\ y\equiv g^x \pmod p}} f(x,y) \tag{2}$$

and this is a sum as in Lemma 2. The coefficients $\check{f}(m,n)$ can be estimated accurately, as we will see in the next section.

## 3 – Proof of Theorem 1

In what follows we assume that $0 \le m, n \le p-1$. We find an upper bound for $\check{f}(m,n) = \check{f}(t,m,n,a,b)$ which is independent of $t$ and then calculate explicitly $\check{f}(0,0)$, which gives the main term of $D(t)$. There are four cases.

**I.** $m = 0$, $n \ne 0$. We have

$$\check{f}(t,0,n,a,b) = \sum_{(x,y)\in \mathcal{I}\times\mathcal{J}} f(x,y)\; e_p(n\,y)\;.$$

By the definition of $f(x,y)$ it follows that for each $x \in \mathcal{I}$ we have a sum of $e_p(ny)$ with $y$ running in a subinterval of $\mathcal{J}$, that is a sum of a geometric progression with ratio $e_p(n)$. The absolute value of such a sum is $\le \frac{2}{|e_p(n)-1|}$ and consequently

$$|\check{f}(t,0,n,a,b)| \le |\mathcal{I}|\,\frac{2}{|e_p(n)-1|} = \frac{M}{\sin\frac{n\,\pi}{p}} \le \frac{M}{2\left\|\frac{n}{p}\right\|}\,, \tag{3}$$

where $\|\cdot\|$ denotes the distance to the nearest integer.

**II.** $m \ne 0$, $n = 0$. Similarly, as in case **I**, we have

$$|\check{f}(t,m,0,a,b)| \le \frac{N}{2\left\|\frac{m}{p}\right\|}\;. \tag{4}$$

**III.** $m \ne 0$, $n \ne 0$. We need the following lemma.

**Lemma 3.** *Let $h, k \not\equiv 0 \pmod p$, $L$, $T$ and $u \ge 0$ be integers. Let $S = \sum_{x=0}^{L}\sum_{y=0}^{ux+T} e_p(hx)\,e_p(ky)$. Then one has*

$$|S| \le \frac{1}{4\left\|\frac{k}{p}\right\|}\,\min\left\{L,\,\frac{1}{2\left\|\frac{h+u\,k}{p}\right\|}\right\} + \frac{1}{4\left\|\frac{k}{p}\right\|}\cdot\frac{1}{2\left\|\frac{h}{p}\right\|}\;. \blacksquare$$

The proof is left to the reader. We now return to the estimation of $\check{f}(m,n)$. Writing

$$\check{f}(m,n) = \sum_{\substack{(x,y)\in\mathcal{I}\times\mathcal{J}\\ ax+by<t}} e_p(m\,x + n\,y)$$

as a sum of $b$ sums according to the residue of $x$ modulo $b$, one arrives at sums as in Lemma 3, with $h = m\,b$, $k = n$, $u = -a$. It follows that

$$(5) \qquad |\check{f}(t,m,n,a,b)| \ll_{a,b} \frac{1}{2\left\|\frac{n}{p}\right\|} \min\left\{M, \frac{1}{2\left\|\frac{m\,b-a\,n}{p}\right\|}\right\} + \frac{1}{2\left\|\frac{n}{p}\right\|} \cdot \frac{1}{2\left\|\frac{mb}{p}\right\|} .$$

**IV.** $m, n = 0$. By definition, we have

$$\check{f}(t,0,0,a,b) = \sum_{(x,y)\in\mathcal{I}\times\mathcal{J}} f(t,x,y,a,b) .$$

Let $\mathcal{D}$ be the set of real points from the rectangle $[0,M)\times[0,N)$ which lie below the line $a\,x + b\,y = t$. Then $\check{f}(t,0,0,a,b)$ equals the number of integer points from $\mathcal{D}$. Therefore

$$\check{f}(t,0,0,a,b) = \mathrm{Area}(\mathcal{D}) + O(\mathrm{length}(\partial\mathcal{D})) .$$

An easy computation shows that $\mathrm{Area}(\mathcal{D})$ equals the expression $G(t,a,b,M,N)$ defined in the Introduction, while the length of the boundary $\partial\mathcal{D}$ is $\le 2\,M+2\,N \le 4\,p$. Hence

$$\check{f}(t,0,0,a,b) = G(t,a,b,M,N) + O(p) .$$

By (2) and Lemma 2 we know that

$$\left|D(t) - \frac{1}{p^2}\,\check{f}(0,0)\,S(0,0,g,p)\right| \le D_1 + D_2 + D_3 ,$$

where

$$D_1 = \frac{1}{p^2}\sum_{m=1}^{p-1} |\check{f}(m,0)|\;|S(m,0,g,p)| , \qquad D_2 = \frac{1}{p^2}\sum_{n=1}^{p-1} |\check{f}(0,n)|\;|S(0,n,g,p)|$$

and

$$D_3 = \frac{1}{p^2}\sum_{m=1}^{p-1}\sum_{n=1}^{p-1} |\check{f}(m,n)|\;|S(m,n,g,p)| .$$

One has

$$\frac{1}{p^2}\,\check{f}(0,0)\,S(0,0,g,p) = \frac{\check{f}(0,0)}{p} = \frac{G(t,a,b,M,N)}{p} + O(1) .$$

Next, since $S(m,0,g,p) = \sum_{x=0}^{p-1} e_p(mx) = 0$ for $1 \le m \le p-1$, it follows that $D_1 = 0$. By (3) and Lemma 1 we have

$$D_2 \le \frac{1}{p^2}\sum_{n=1}^{p-1} \frac{M}{\left\|\frac{n}{p}\right\|}\, p^{1/2}(1+\log p) = 2\,M\,p^{-3/2}(1+\log p)\sum_{n=1}^{\frac{p-1}{2}} \frac{p}{n}$$
$$\le 2\,p^{1/2}(1+\log p)^2 .$$

In order to estimate $D_3$ we first use Lemma 1 and (5) to obtain

$$(6)\quad D_3 \ll_{a,b} \frac{\log p}{p^{3/2}} \sum_{m=1}^{p-1}\sum_{n=1}^{p-1} \frac{1}{\left\|\frac{n}{p}\right\|} \min\left\{M, \frac{1}{\left\|\frac{mb-an}{p}\right\|}\right\} + \frac{\log p}{p^{3/2}} \sum_{m=1}^{p-1}\sum_{n=1}^{p-1} \frac{1}{\left\|\frac{n}{p}\right\|}\cdot\frac{1}{\left\|\frac{mb}{p}\right\|} .$$

The first double sum in (6) is

$$\sum_{m=1}^{p-1}\sum_{n=1}^{p-1} \frac{1}{\left\|\frac{n}{p}\right\|} \min\left\{M, \frac{1}{\left\|\frac{mb-an}{p}\right\|}\right\} \le$$

$$\le \sum_{n=1}^{p-1} \frac{1}{\left\|\frac{n}{p}\right\|} \sum_{\substack{m=1\\ mb-an\equiv 0 \,(\mathrm{mod}\, p)}}^{p-1} p \;+\; \sum_{n=1}^{p-1} \frac{1}{\left\|\frac{n}{p}\right\|} \sum_{\substack{m=1\\ mb-an\not\equiv 0 \,(\mathrm{mod}\, p)}}^{p-1} \frac{1}{\left\|\frac{mb-an}{p}\right\|}$$

$$\le p \sum_{n=1}^{\frac{p-1}{2}} \frac{p}{n} + \sum_{n=1}^{p-1} \frac{1}{\left\|\frac{n}{p}\right\|} \sum_{m'=1}^{p-1} \frac{1}{\left\|\frac{m'}{p}\right\|} \le p^2(1+\log p) + 4p^2(1+\log p)^2 ,$$

while the second double sum is

$$\sum_{m=1}^{p-1}\sum_{n=1}^{p-1} \frac{1}{\left\|\frac{n}{p}\right\|}\cdot\frac{1}{\left\|\frac{mb}{p}\right\|} = 4\sum_{m=1}^{\frac{p-1}{2}} \frac{p}{m} \sum_{n=1}^{\frac{p-1}{2}} \frac{p}{n} \le 4p^2(1+\log p)^2 .$$

Hence $D_3 \ll_{a,b} p^{1/2} \log^3 p$. Putting all these together, Theorem 1 follows. ∎

## 4 – Proof of the Corollaries

For the proof of the first Corollary, let us notice that

$$\#\left\{0 \le x \le p-1\colon\; x > g^x\right\} = D(a=-1,\, b=1,\, p, g, \mathcal{I}, \mathcal{J}, t=0)$$

with $\mathcal{I} = \mathcal{J} = \{0,1,...,p-1\}$. Here $M = N = p$, $W = Z = 0$ and so

$$G(t=0,\, a=-1,\, b=1,\, M=p,\, N=p) = -\frac{(a\,M-t)^2}{2\,a\,b} = \frac{p^2}{2} .$$

Thus

$$\#\left\{0 \le x \le p-1\colon\; x > g^x\right\} = \frac{p}{2} + O(p^{\frac12}\log^3 p) .$$

One obtains the more precise upper bound $7p^{\frac12}\log^3 p$ for the error term by following the proof of Theorem 1 in this particular case.

To prove Corollary 2 note that

$$\begin{aligned} M(p,g,2k) &= \sum_{x=0}^{p-1}(g^x - x)^{2k} \\ &= \sum_{-p<t<p} t^{2k}\ \#\Big\{0 \le x,\ y \le p-1\colon\ y \equiv g^x \ (\operatorname{mod} p),\ \ y - x = t\Big\}\ . \end{aligned}$$

This equals

$$\sum_{-p<t<p} t^{2k}\Big(D(t+1) - D(t)\Big) \;=\; D(p)\,(p-1)^{2k} \;+ \sum_{-p<t<p} D(t)\,\Big((t-1)^{2k} - t^{2k}\Big)$$

where $D(t) = D(a{=}-1,\, b{=}1,\, p, g, \mathcal{I}, \mathcal{J}, t)$ with $\mathcal{I} = \mathcal{J} = \{0,1,...,p{-}1\}$. From Theorem 1 it follows that

$$\begin{aligned} M(p,g,2k) \;=\; & p^{2k-1}\, G(p,-1,1,p,p) \;+\; \frac{1}{p}\sum_{-p<t<p} G(t,-1,1,p,p)\,\Big((t-1)^{2k} - t^{2k}\Big) \\ & +\; O_k\Big(p^{2k+\frac12}\,\log^3 p\Big) \;+\; O\Big(p^{1/2}\,\log^3 p \sum_{-p<t<p}\Big|(t-1)^{2k} - t^{2k}\Big|\Big)\ . \end{aligned}$$

Since $(t-1)^{2k} - t^{2k} = -2\,k\,t^{2k-1} + O_k(p^{2k-2})$ and $0 \le G(t,-1,1,p,p) \le p^2$ we derive

$$\begin{aligned} M(p,g,2k) \;=\; & p^{2k-1}\, G(p,-1,1,p,p) \\ & -\; \frac{2\,k}{p}\sum_{-p<t<p} t^{2k-1}\, G(t,-1,1,p,p) \;+\; O_k\Big(p^{2k+\frac12}\,\log^3 p\Big)\ . \end{aligned}$$

From the definition of $G$ we see that

$$G(t,-1,1,p,p) \;=\; \begin{cases} 0, & \text{if } t < -p, \\ \dfrac{(p+t)^2}{2}, & \text{if } -p \le t \le 0, \\ p^2 - \dfrac{(p-t)^2}{2}, & \text{if } 0 < t < p, \\ p^2, & \text{if } p \le t\ . \end{cases}$$

Using the fact that for any positive integer $r$ one has $\sum_{-p<t<p} t^r = \frac{2\,p^{r+1}}{r+1} + O_r(p^r)$ if $r$ is even and $\sum_{-p<t<p} t^r = 0$ if $r$ is odd, the statement of Corollary 2 follows after a straightforward computation. ∎

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