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WEIGHTED NORM INEQUALITY FOR THE POISSON INTEGRAL ON THE SPHERE

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Abstract: We obtain, for each p, $1<p<\infty$, a necessary and sufficient condition for the Poisson integral of functions defined on the sphere S^n, to be bounded from a weighted space $L^p(S^n, W d\sigma)$ into a space $L^p(\mathbb{B}, \nu)$, where σ is the Lebesgue measure on S^n and ν is a positive measure on the unit ball $\mathbb{B}$ of $\mathbb{R}^{n+1}$.

Introduction

In this paper we consider a homogeneous space $X = G/H$ where G is a locally compact Hausdorff topological group and H is a compact subgroup of G which is provided with a quasi-distance d and with a measure μ induced on X by a Haar measure on the topological group G. If $x \in X$ and $r>0$, $B(x,r)$ will denote the ball $\{y \in X : d(x,y) < r\}$ in X. We also write $\widetilde{X} = X \times [0,\infty)$ and if $B = B(x,r)$ we write $\widetilde{B} = B(x,r) \times [0,r]$.

We define the maximal operator $\mathcal{M}$ by

$$\mathcal{M}f(x,r) = \sup_{s\geq r} \frac{1}{\mu(B(x,s))} \int_{B(x,s)} |f(y)|\, d\mu(y)$$

for all real-valued locally integrable function f on X and $(x,r) \in \widetilde{X}$. If $r=0$ the above supremum is taken over all $s>0$ and $\mathcal{M}f(x,0) = f^*(x)$ is the Hardy-Littlewood maximal function.

A weight is a positive locally integrable function $W(x)$ on X and we will write $W(A) = \int_A W\, d\mu$. We say that W is a weight in the class $A_\infty(X)$ if there exist

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positive constants C_W and δ such that

$$\frac{\mu(A)}{\mu(B)} \leq C_W \left(\frac{W(A)}{W(B)}\right)^{\delta},$$

for all ball $B = B(x,r)$, $x \in X$, $r > 0$, and all Borel subsets A of B. We observe that the above inequality is equivalent to a similar one where μ appears instead of W and conversely (see [5, 1]). We write $L^p(W) = L^p(X, W(x)\, d\mu(x))$, $1 \leq p < \infty$.

Let $1 < p < \infty$, p' such that $1/p + 1/p' = 1$, let β be a positive measure on the Borel subsets of $\widetilde{X}$ and W a weight on X. In Section 2 we introduce a maximal operator of dyadic type $\mathcal{M}_d^b$, where b is an integer, using partitions of dyadic type for the homogeneous space X introduced in Section 1.

In Section 3 we prove the following theorem.

Theorem 3.1. *Let G be a compact or an Abelian group, let $1 < p < \infty$ and let W be a weight on X such that $W^{1-p'} \in A_\infty(X)$. Then the following conditions are equivalent:*

(**i**) *There exists a constant $C > 0$, such that, for all $f \in L^p(W)$,*

$$\int_{\widetilde{X}} [\mathcal{M} f(x,r)]^p \, d\beta(x,r) \ \leq \ C \int_X |f(x)|^p \, W(x) \, d\mu(x) \ .$$

(**ii**) *There exists a constant $C > 0$, such that, for all balls $B = B(z,t)$, $0 \leq t < \infty$,*

$$\int_{\widetilde{B}} [\mathcal{M}(W^{1-p'} \chi_B)(x,r)]^p \, d\beta(x,r) \ \leq \ C \int_B W^{1-p'}(x) \, d\mu(x) \ < \ \infty \ .$$

The above result for $X = \mathbb{R}^n$ was proved in Ruiz-Torrea [7]. A similar result for the fractional maximal operator was obtained in Bernardis-Salinas [1]. The condition (ii) of Theorem 3.1 implies the condition

$$\frac{\beta(\widetilde{B})^{1/p}}{\mu(B)} \left(\int_B W^{1-p'}(x) \, d\mu(x)\right)^{1/p'} \leq C \ < \infty$$

for all balls B. It was proved in Ruiz-Torrea [8] that the above condition is a necessary and sufficient condition for $\mathcal{M}$ to be a bounded operator from $L^p(X, W(x)\, d\mu(x))$ into weak $-L^p(\widetilde{X}, \beta)$. In the particular case $W(x) \equiv 1$, the condition (ii) of Theorem 3.1 is equivalent to the Carleson's condition for the homogeneous space X:

$$\beta(\widetilde{B}) \leq C \, \mu(B)$$

for all balls B and for a constant $C > 0$.

Now, if $x \in \mathbb{R}^{n+1}$, we write $|x| = (x \cdot x)^{1/2}$ and $d(x,y) = |x-y|$, where $x \cdot y$ is the usual scalar product of x and y in $\mathbb{R}^{n+1}$. Here S^n will denote the unit n-sphere $\{y \in \mathbb{R}^{n+1} : |y| = 1\}$ in $\mathbb{R}^{n+1}$, σ the normalized Lebesgue measure on S^n and $h : [1-\sqrt{2}, 1] \to [0,2]$ will be the function defined by $h(r) = \sqrt{2}(1-r)$.

The Poisson kernel for the sphere S^n is given by

$$P_{ry}(x) = \frac{1}{\omega_n} \frac{1-r^2}{|ry-x|^{n+1}}$$

for $x, y \in S^n$ and $0 \leq r < 1$, where ω_n is the area of the sphere S^n. For a real-valued integrable function f we denote by $u_f(ry)$ the Poisson integral

$$u_f(ry) = \int_{S^n} P_{ry}(x) f(x) \, d\sigma(x) \ .$$

and we define the maximal function u_f^* by

$$u_f^*(ry) = \sup_{0 \leq s \leq r} |u_f(sy)| \, , \qquad 0 \leq r < 1, \ \ y \in S^n \ .$$

If B is the open ball $B(z,t) = \{x \in S^n : \ |x-z| < t\}, \ 0 < t \leq 2$, we define

$$\bar{B} = \{sx \ : \ h^{-1}(t) \leq s \leq 1, \ x \in B\} \quad \text{if } \ 0 < t \leq \sqrt{2} \ ;$$

$$\bar{B} = \{sx \ : \ 0 \leq s \leq 1, \ x \in B\} \quad \text{if } \ \sqrt{2} \leq t \leq 2 \ .$$

We observe that $\bar{B}$ is a truncated cone in the ball $\mathbb{B} = \{y \in \mathbb{R}^{n+1} : \ |y| \leq 1\}$ in $\mathbb{R}^{n+1}$ if $0 < t \leq \sqrt{2}$ and a cone if $\sqrt{2} \leq t \leq 2$.

In Section 4 we prove the following result.

Theorem 4.1. *Let $1 < p < \infty$, let W be a weight on S^n such that $W^{1-p'} \in A_\infty(S^n)$ and let ν be a Borel positive measure on S^n. Then the following conditions are equivalent:*

(i) *There exists a constant $C > 0$, such that, for all $f \in L^p(W)$,*

$$\int_{\mathbb{B}} [u_f^*(y)]^p \, d\nu(y) \ \leq \ C \int_{S^n} |f(x)|^p \, W(x) \ d\sigma(x) \ .$$

(ii) *There exists a constant $C > 0$, such that, for all balls $B = B(z,t)$, $0 < t \leq 2$,*

$$\int_{\bar{B}} [u^*_{W^{1-p'}\chi_B}(y)]^p \, d\nu(y) \ \leq \ C \int_B W^{1-p'}(x) \, d\sigma(x) \ < \ \infty \ .$$

We point out that the Theorem 4.1 for $W \equiv 1$ and $n = 1$ was proved in Carleson [2].

1 – Preliminaries

In this section we introduce some notations, definitions and basic facts.

Let G be a locally compact Hausdorff topological group with unit element e, H be a compact subgroup and $\pi : G \mapsto G/H$ the canonical map. Let dg denote a left Haar measure on G, which we assume to be normalized in the case of G to be compact. If A is a Borel subset of G we will denote by $|A|$ the Haar measure of A. The homogeneous space $X = G/H$ is the set of all left cosets $\pi(g) = gH, g \in G$, provided with the quotient topology. The Haar measure dg induces a measure μ on the Borel σ-field on X. For $f \in L^1(X)$,

$$\int_X f(x)\,d\mu(x) \;=\; \int_G f \circ \pi(g)\,dg\ .$$

We observe that the group G acts transitively on X by the map $(g, \pi(h)) \mapsto g\pi(h) = \pi(gh)$, that is, for all $x, y \in X$, there exists $g \in G$ such that $gx = y$. We also observe that the measure μ on X is invariable on the action of G, that is, if $f \in L^1(X)$, $g \in G$ and $R_g f(x) = f(g^{-1}x)$, then

$$\int_X f(x)\,d\mu(x) \;=\; \int_X R_g f(x)\,d\mu(x)\ .$$

Definition 1.1. A quasi-distance on X is a map $d : X \times X \mapsto [0, \infty)$ satisfying:

(i) $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(iii) $d(gx, gy) = d(x, y)$ for all $g \in G,\ x, y \in X$;

(iv) there exists a constant $K \geq 1$ such that, for all $x, y, z \in X$,

$$d(x, y) \leq K[d(x, z) + d(z, y)]\ ;$$

(v) the balls $B(x, r) = \{y \in X : d(x, y) < r\},\ x \in X,\ r > 0$, are relatively compact and measurable, and the balls $B(\mathbb{1}, r),\ r > 0$, form a basis of neighborhoods of $\mathbb{1} = \pi(e)$;

(vi) there exists a constant $A \geq 1$ such that, for all $r > 0$ and $x \in X$,

$$\mu(B(x, 2r)) \leq A\mu(B(x, r))\ . \ \square$$

In this paper X will denote a homogeneous space provided with a quasi-distance d.

Given a quasi-distance d on X, there exists a distance ρ on X and a positive real number γ such that d is equivalent to ρ^γ (see [5]). Therefore the family of d-balls is equivalent to the family of ρ^γ-balls and ρ^γ-balls are open sets.

It follows by Definition 1.1(iii) that $B(gx,r) = gB(x,r)$ for all $g \in G$, $x \in X$ and $r > 0$, and hence $\mu(B(gx,r)) = \mu(B(x,r))$. Thus we can write $X = \bigcup_{j\geq 1} g_j B(x,r)$ where (g_j) is a sequence of elements of G and consequently $\mu(B(x,r)) > 0$. In particular, X is separable.

Lemma 1.1. *Let b be a positive integer and let $\lambda = 8K^5$. Then for each integer k, $-b \leq k \leq b$, there exist an enumerable Borel partition $\mathcal{A}_k^b$ of X and a positive constant C depending only on X, such that:*

(i) *for all $Q \in \mathcal{A}_k^b$, $-b \leq k \leq b$, there exists $x_Q \in Q$ such that*

$$B(x_Q, \lambda^k) \subset Q \subset B(x_Q, \lambda^{k+1})$$

and

$$\mu(B(x_Q, \lambda^{k+1})) \leq C\mu(Q) \ ;$$

(ii) *if $-b \leq k < b$, $Q_1 \in \mathcal{A}_{k+1}^b$, $Q_2 \in \mathcal{A}_k^b$ and $Q_1 \cap Q_2 \neq \emptyset$, then $Q_2 \subset Q_1$, and*

$$0 < \mu(Q_1) \leq C\mu(Q_2) \ ;$$

(iii) *for all $x \in X$ and r, $\lambda^{-b-1} \leq r \leq \lambda^b$, there exist $Q \in \mathcal{A}_k^b$ for some $-b \leq k \leq b$ and $g \in G$ such that $d(gx,x) \leq \lambda^{k+1}$, $B(x,r) \subset gQ$ and*

$$\mu(Q) \leq C\mu(B(x,r)) \ .$$

Proof: The properties (i) and (ii) follow by Lemma 3.21, p. 852 of [9] and by Definition 1.1.

Let us prove (iii). Given $x \in X$ and $\lambda^{-b-1} \leq r \leq \lambda^b$, let $-b \leq k \leq b$ such that $\lambda^{k-1} \leq r \leq \lambda^k$. There exists an unique $Q \in \mathcal{A}_k^b$ such that $x \in Q$. Consider x_Q as in (i) and $g \in G$ such that $x = g\,x_Q$. If a is an integer such that $2^{a-1} < \lambda \leq 2^a$, then by (i) we have

$$B(x,r) \subset B(gx_Q, \lambda^k) \subset gQ \subset B(x, \lambda^{k+1})$$

and hence by Definition 1.1(vi) we have

$$\begin{aligned} \mu(Q) &\leq \mu(B(x_Q, 2^a\lambda^k)) \\ &\leq A^a\mu(B(x_Q, \lambda^k)) \\ &\leq A^{2a}\mu(B(x, r)) \ . \end{aligned}$$

We also have that $d(gx, x) = d(x, x_Q) \leq \lambda^{k+1}$. ∎

Let $(\Omega, \mathcal{F}, \nu)$ be a σ-finite measure space, let $(\mathcal{F}_k)_{k\in\mathbb{Z}}$ be an increasing sequence of sub-σ-fields of $\mathcal{F}$, and for each $k \in \mathbb{Z}$, consider a real-valued $\mathcal{F}_k$-measurable function f_k. We say that the sequence $(f_k)_{k\in\mathbb{Z}}$ is a martingale with respect to the sequence $(\mathcal{F}_k)_{k\in\mathbb{Z}}$ if, for all $k \in \mathbb{Z}$ and all $A \in \mathcal{F}_k$ such that $\nu(A) < \infty$, we have that

$$\int_A |f_k|\, d\nu < \infty\,, \qquad \int_A f_k\, d\nu = \int_A f_{k+1}\, d\nu \ .$$

Now, consider a σ-finite measure ν on the Borel σ-field of X and let $\mathcal{F}_k$ be the σ-field generated by the partition $\mathcal{A}^b_{-k}$ for $-b \leq k \leq b$, by $\mathcal{A}^b_{-b}$ for $k \geq b$ and by $\mathcal{A}^b_b$ for $k \leq -b$. If $f \in L^1(X, \nu)$,

$$f_k(x) \ = \ E[f|\mathcal{F}_k](x) \ = \sum_{Q\in\mathcal{A}^b_k} \left(\frac{1}{\nu(Q)}\int_Q f(y)\, d\nu(y)\right) \chi_Q(x), \quad -b \leq k \leq b \ ;$$

and $f_k = f_b$ for $k \geq b$, $f_k = f_{-b}$ for $k \leq -b$, then $(f_k)_{k\in\mathbb{Z}}$ is a martingale with respect to the sequence $(\mathcal{F}_k)_{k\in\mathbb{Z}}$. We define the maximal operator M^b_ν, for all $f \in L^1(X, \nu)$ by

$$M^b_\nu f(x) = \sup_{k\in\mathbb{Z}} E[|f| \mid \mathcal{F}_k](x) \ = \ \sup_{\substack{x\in Q \\ Q\in\mathcal{A}^b}} \frac{1}{\nu(Q)}\int_Q |f(y)|\, d\nu(y) \ .$$

where $\mathcal{A}^b = \bigcup_{-b\leq k\leq b} \mathcal{A}^b_k$.

The next result can be found in Dellacherie-Meyer [3], number 40, p. 37.

Theorem 1.1. *If* $1 < p < \infty$ *and* $f \in L^p(X, \nu)$, *then*

$$\|M^b_\nu f\|_{L^p(X,\nu)} \leq p'\|f\|_{L^p(X,\nu)} \ .$$

2 – A maximal operator of dyadic type

Let b be a fixed positive integer. Given $Q \in \mathcal{A}^b = \bigcup_{-b \le k \le b} \mathcal{A}_k^b$, where $\mathcal{A}_k^b$ are the partitions of X in Lemma 1.1, $\widetilde{Q}$ will denote the subset $Q \times [0, \alpha^{-1}(\mu(Q))]$ of $\widetilde{X} = X \times [0,\infty)$, where $\alpha : [0,\infty) \to [0,\infty)$ is the function defined by $\alpha(r) = \mu(B(\mathbb{1}, r))$, $\mathbb{1} = \pi(e)$.

If f is a real-valued locally integrable function on X, we define, for each $(x,r) \in \widetilde{X}$,

$$\mathcal{M}_d^b f(x,r) = \sup_{\substack{x \in Q \in \mathcal{A}^b \\ \mu(Q) \ge \alpha(r)}} \frac{1}{\mu(Q)} \int_Q |f(y)| \, d\mu(y) \ .$$

If $\mu(Q) < \alpha(r)$ for all $Q \in \mathcal{A}^b$ such that $x \in Q$, we define $\mathcal{M}_d^b f(x,r) = 0$.

Lemma 2.1. *Let W be a weight and let A be a measurable subset of X. If $1 < p < \infty$ and $W^{-1}\chi_A \notin L^{p'}(W)$, then there exists a positive function $f \in L^p(W)$ such that*

$$\int_A f(x) \, d\mu(x) = \infty \ .$$

Proof: Let ψ be the linear functional on $L^p(W)$ given by $\psi(g) = \int_A g \, d\mu$. Since $W^{-1}\chi_A \notin L^{p'}(W)$, it follows by the Riesz representation theorem that ψ is not continuous. Therefore, there exists $\varepsilon > 0$, such that, for each positive integer m, there exists $g_m \in L^p(W)$ such that $\|g_m\|_{L^p(W)} \le 2^{-m}$ and $|\psi(g_m)| \ge \varepsilon$. We set $f_m(x) = |g_1(x)| + \cdots + |g_m(x)|$ and then, for all $m, k \ge 1$,

$$\|f_{m+k} - f_m\|_{L^p(W)} \le \|g_{m+1}\|_{L^p(W)} + \cdots + \|g_{m+k}\|_{L^p(W)} < 2^{-m} \ .$$

Hence (f_m) is a Cauchy sequence in $L^p(W)$ and therefore there exists $f \in L^p(W)$ such that $f_m \to f$ in $L^p(W)$. On the other hand

$$\psi(f_m) \ge |\psi(g_1)| + \cdots + |\psi(g_m)| \ge m\,\varepsilon \ .$$

But $f_m \uparrow f$ a.e. and thus by the monotone convergence theorem we obtain

$$\int_A f \, d\mu = \lim_{m\to\infty} \psi(f_m) = \infty \ . \ \blacksquare$$

Theorem 2.1. *Given a weight W on X, a positive measure β on $\widetilde{X}$, and $1 < p < \infty$, the following conditions are equivalent:*

(**i**) *There exists a constant $C > 0$, such that, for all $f \in L^p(W)$ and all positive integer b,*

$$\int_{\widetilde{X}} [\mathcal{M}_d^b f(x,r)]^p \, d\beta(x,r) \;\leq\; C \int_X |f(x)|^p \, W(x) \, d\mu(x) \;.$$

(**ii**) *There exists a constant $C > 0$, such that, for all $Q \in \mathcal{A}^b$ and all positive integer b,*

$$\int_{\widetilde{Q}} \Big[\mathcal{M}_d^b (W^{1-p'} \chi_Q)(x,r)\Big]^p \, d\beta(x,r) \;\leq\; C \int_Q W^{1-p'}(x) \, d\mu(x) \;<\; \infty \;.$$

Proof: The proof of (i)⇒(ii) is exactly as the proof of (i)⇒(ii) in Theorem 3.1.

Proof of (ii)⇒(i): Let us fix $f \in L^p(W)$ and for each $k \in \mathbb{Z}$, let Ω_k be the set

$$\Omega_k = \Big\{ (x,r) \in \widetilde{X} \colon \; \mathcal{M}_d^b f(x,r) > 2^k \Big\} \;.$$

For each $k \in \mathbb{Z}$, we denote by C_k^0 the family formed by all $Q \in \mathcal{A}^b$ such that

$$|f|_Q \;=\; \frac{1}{\mu(Q)} \int_Q |f(y)| \, d\mu(y) \;>\; 2^k \;.$$

Since for every $Q \in \mathcal{A}_k^b$, $-b \leq k < b$, there exists $Q' \in \mathcal{A}_{k+1}^b$ such that $Q \subset Q'$, then every element $Q \in C_k^0$ is contained in a maximal element $Q' \in C_k^0$. We denote by C_k the family $\{Q_j^k : j \in J_k\}$ formed by all maximal elements $Q \in C_k^0$. Since $\mathcal{A}_k^b$ is a partition of X and all elements of C_k are maximal, we can conclude that the sets $Q_j^k, j \in J_k$, are pairwise disjoint. Therefore the sets $\widetilde{Q}_j^k$, $j \in J_k$, are also pairwise disjoint and,

$$\Omega_k \;=\; \bigcup_{j \in J_k} \widetilde{Q}_j^k \;.$$

Now, for each $k \in \mathbb{Z}$ and each $j \in J_k$, let

$$E_j^k \;=\; \widetilde{Q}_j^k \setminus \Omega_{k+1} \;.$$

Then the sets E_j^k and E_i^h are disjoint for $(k,j) \neq (h,i)$ and

$$\Big\{ (x,r) \colon \; \mathcal{M}_d^b f(x,r) > 0 \Big\} \;=\; \bigcup_{k \in \mathbb{Z}} (\Omega_k \setminus \Omega_{k+1}) \;=\; \bigcup_{k \in \mathbb{Z}} \bigcup_{j \in J_k} E_j^k \;.$$

Therefore

$$\begin{aligned}\int_{\widetilde{X}}[\mathcal{M}_d^b f(x,r)]^p\,d\beta(x,r) &= \sum_{k,j}\int_{E_j^k}[\mathcal{M}_d^b f(x,r)]^p\,d\beta(x,r)\\ &\le 2^p\sum_{k,j}\beta(E_j^k)\,(2^k)^p\\ &\le 2^p\sum_{k,j}\beta(E_j^k)\left(\frac{1}{\mu(Q_j^k)}\int_{Q_j^k}|f(x)|\,d\mu(x)\right)^p . \qquad (2.1)\end{aligned}$$

Now, we introduce the following notations:

$$\nu(x)=W^{1-p'}(x),\quad \nu(A)=\int_A \nu(x)\,d\mu(x)\ ,$$

$$\gamma_{k,j}=\beta(E_j^k)\left(\frac{\nu(Q_j^k)}{\mu(Q_j^k)}\right)^p,\quad g_{k,j}=\left(\frac{1}{\nu(Q_j^k)}\int_{Q_j^k}\frac{|f(x)|}{\nu(x)}\,\nu(x)\ d\mu(x)\right)^p,$$

$$Y=\Big\{(k,j):\ k\in\mathbb{Z},\ j\in J_k\Big\},\qquad \Gamma(\lambda)=\Big\{(k,j)\in Y:\ g_{k,j}>\lambda\Big\}\ .$$

Let γ be the measure on Y such that $\gamma(\{(k,j)\})=\gamma_{k,j}$ and let g be the function defined on Y by $g((k,j))=g_{k,j}$. We have that

$$\gamma_{k,j}\,g_{k,j}\ =\ \beta(E_j^k)\left(\frac{1}{\mu(Q_j^k)}\int_{Q_j^k}|f(x)|\,d\mu(x)\right)^p$$

and hence it follows by (2.1) that

$$\begin{aligned}\int_{\widetilde{X}}[\mathcal{M}_d^b f(x,r)]^p\,d\beta(x,r) &\le 2^p\sum_{k,j}\gamma_{k,j}\,g_{k,j}\\ &= 2^p\int_0^\infty \gamma(\Gamma(\lambda))\,d\lambda\\ &= 2^p\int_0^\infty\left(\sum_{(k,j)\in\Gamma(\lambda)}\gamma_{k,j}\right)d\lambda\\ &= 2^p\int_0^\infty\sum_{(k,j)\in\Gamma(\lambda)}\int_{E_j^k}\left(\frac{\nu(Q_j^k)}{\mu(Q_j^k)}\right)^p d\beta(x,r)\ d\lambda\ . \qquad (2.2)\end{aligned}$$

For each $\lambda>0$, let $\{Q_i^\lambda : i\in I_\lambda\}$ be the family formed by all maximal elements of the family

$$\Big\{Q_j^k:\ (k,j)\in\Gamma(\lambda)\Big\}\ =\ \left\{Q_j^k:\ \frac{1}{\nu(Q_j^k)}\int_{Q_j^k}\frac{|f(x)|}{\nu(x)}\,\nu(x)\,d\mu(x)>\lambda^{1/p}\right\}\ .$$

If $Q_j^k \subset Q_i^\lambda$ and $(x,r) \in E_j^k$, then $x \in Q_j^k$, $\mu(Q_j^k) \geq \alpha(r)$ and thus

$$\mathcal{M}_d^b(\nu\chi_{Q_i^\lambda})(x,r) \;=\; \sup_{\substack{x\in Q\in\mathcal{A}^b\\ \mu(Q)\geq\alpha(r)}} \frac{\nu(Q\cap Q_i^\lambda)}{\mu(Q)} \;\geq\; \frac{\nu(Q_j^k)}{\mu(Q_j^k)} \;.$$

Therefore, if $Q_j^k \subset Q_i^\lambda$ we obtain

$$\int_{E_j^k} \left(\frac{\nu(Q_j^k)}{\mu(Q_j^k)}\right)^p d\beta(x,r) \;\leq\; \int_{E_j^k} \Big[\mathcal{M}_d^b(\nu\chi_{Q_i^\lambda})(x,r)\Big]^p \, d\beta(x,r) \;. \tag{2.3}$$

Taking into account that the sets E_j^k are disjoint, it follows from (2.2), (2.3) and by the hypothesis that

$$\begin{aligned}
\int_{\widetilde{X}} [\mathcal{M}_d^b f(x,r)]^p \, d\beta(x,r) \;&\leq\; 2^p \int_0^\infty \sum_{i\in I_\lambda} \sum_{\substack{(k,j)\in\Gamma(\lambda)\\ Q_j^k\subset Q_i^\lambda}} \int_{E_j^k} \Big[\mathcal{M}_d^b(\nu\chi_{Q_i^\lambda})(x,r)\Big]^p \, d\beta(x,r)\\
&\leq\; 2^p \int_0^\infty \sum_{i\in I_\lambda} \int_{\widetilde{Q}_i^\lambda} \Big[\mathcal{M}_d^b(\nu\chi_{Q_i^\lambda})(x,r)\Big]^p \, d\beta(x,r)\\
&\leq\; C\,2^p \int_0^\infty \sum_{i\in I_\lambda} \int_{Q_i^\lambda} \nu(x)\, d\mu(x)\\
&=\; C\,2^p \int_0^\infty \nu\left(\bigcup_{(k,j)\in\Gamma(\lambda)} Q_j^k\right) d\mu(x) \;. \qquad (2.4)
\end{aligned}$$

It follows by the definition of the maximal operator M_ν^b in Section 1 and by the definition of $\Gamma(\lambda)$ that

$$\bigcup_{(k,j)\in\Gamma(\lambda)} Q_j^k \;\subset\; \left\{x\in X\colon\, M_\nu^b\left(\frac{|f|}{\nu}\right)(x) > \lambda^{1/p}\right\}. \tag{2.5}$$

Then, by (2.4), (2.5) and Theorem 1.1,

$$\begin{aligned}
\int_{\widetilde{X}} [\mathcal{M}_d^b f(x,r)]^p \, d\beta(x,r) \;&\leq\; C\,2^p \int_0^\infty \nu\left(\left\{x\colon\, \left(M_\nu^b\left(\frac{|f|}{\nu}\right)(x)\right)^p > \lambda\right\}\right) d\lambda\\
&=\; C\,2^p \int_X \left(M_\nu^b\left(\frac{|f|}{\nu}\right)(x)\right)^p \nu(x)\; d\mu(x)\\
&\leq\; C\,2^p (p')^p \int_X \frac{|f(x)|^p}{(\nu(x))^p}\, \nu(x)\; d\mu(x)\\
&=\; C\,2^p (p')^p \int_X |f(x)|^p \, W(x)\; d\mu(x) \;. \;\blacksquare
\end{aligned}$$

Remark 2.1. Let us fix $g \in G$ and let $g^{-1}\mathcal{A}^b_k = \{g^{-1}Q : Q \in \mathcal{A}^b_k\}$, $g^{-1}\mathcal{A}^b = \{g^{-1}Q : Q \in \mathcal{A}^b\}$. Then for each $-b \le k \le b$, $g^{-1}\mathcal{A}^b_k$ is a partition of X and Lemma 1.1 and Theorem 1.1 in Section 1 also hold, with the same constants, when we change $\mathcal{A}^b_k$ for $g^{-1}\mathcal{A}^b_k$. If f is a real-valued locally integrable function on X, we define

$$\mathcal{M}^{b,g}_d f(x,r) = \sup_{\substack{x\in Q\in g^{-1}\mathcal{A}^b \\ \mu(Q)\ge\alpha(r)}} \frac{1}{\mu(Q)} \int_Q |f(y)|\, d\mu(y) \ .$$

Then

$$\mathcal{M}^b_d(R_g f)(gx,r) = \mathcal{M}^{b,g}_d f(x,r)$$

where $R_g f(x) = f(g^{-1}x)$. The Theorem 2.1 also hold, with the same proof, when we change the operator $\mathcal{M}^b_d$ for $\mathcal{M}^{b,g}_d$ and the family $\mathcal{A}^b$ for $g^{-1}\mathcal{A}^b$. □

3 – The boundedness of the operator $\mathcal{M}$

Given a positive integer b and a real-valued locally integrable function f on X, we define for $(x,r) \in \widetilde{X}$,

$$\mathcal{M}^b f(x,r) = \sup_{\max\{\lambda^{-b-1},r\}\le s\le\lambda^b} \frac{1}{\mu(B(x,s))} \int_{B(x,s)} |f(y)|\, d\mu(y) \ .$$

We define $\mathcal{M}^b f(x,r) = 0$ if $r > \lambda^b$ and we observe that $\mathcal{M}^b f(x,r) \uparrow \mathcal{M} f(x,r)$ if $b \uparrow \infty$ for all $(x,r) \in \widetilde{X}$.

Let us denote

$$G_b = \Big\{ g \in G\colon\ d(gx,x) \le \lambda^{b+1} \ \text{ for all } x \in X \Big\} \ .$$

If $d(g\mathbb{1},\mathbb{1}) = d(gx,x)$ for all $x \in X$ and $g \in G$, in particular if G is an Abelian group, then

$$G_b = \Big\{ g \in G :\ d(g\mathbb{1},\mathbb{1}) \le \lambda^{b+1} \Big\} \ ,$$

and hence G_b is relatively compact in G and $0 < |G_b| < \infty$ (see [4]).

Lemma 3.1. *Let b be a positive integer, $g \in G$, let $\mathcal{M}^{b,g}_d$ be the maximal operator defined in Remark 2.1, let f be a real-valued locally integrable function on X and let $(x,r) \in \widetilde{X}$. Then*

$$\mathcal{M}^{b,g}_d f(x,r) \le C \mathcal{M} f(x,r) \ . \tag{3.1}$$

If G is a compact or an Abelian group, then

$$\mathcal{M}^b f(x,r) \;\le\; \frac{C}{|G_b|} \int_{G_b} \mathcal{M}_d^{b,g} f(x,r)\, dg \;. \tag{3.2}$$

The constants C in (3.1) and in (3.2) depend only on X and if X is compact we can change G_b for G.

Proof: Let us fix $(x,r) \in \widetilde{X}$ and $g \in G$. If $\mu(Q) < \alpha(r)$ for all $Q \in \mathcal{A}^b$ such that $x \in g^{-1}Q$, we have $\mathcal{M}_d^{b,g} f(x,r) = 0$. Thus to prove (3.1), it is enough to consider $Q \in \mathcal{A}_k^b$, $-b \le k \le b$, such that $x \in g^{-1}Q$ and $\mu(Q) \ge \alpha(r)$. By Lemma 1.1(i) there exist $x_Q \in Q$ such that $Q \subset B(x_Q, \lambda^{k+1})$ and $\mu(B(x_Q,\lambda^{k+1})) \le C\mu(Q)$. For $t = 2K\lambda^{k+1}$ we have $B(g^{-1}x_Q, \lambda^{k+1}) \subset B(x,t)$ and hence

$$\alpha(t) = \mu(B(x,t)) \ge \mu(B(g^{-1}x_Q, \lambda^{k+1})) \ge \mu(Q) \ge \alpha(r) \;.$$

If $2^{a-1} < K \le 2^a$, it follows by Definition 1.1(vi) that

$$\mu(B(x,t)) \le A^{a+1} \mu(B(x_Q, \lambda^{k+1})) \le A^{a+1} C \mu(g^{-1}Q) \;.$$

Therefore

$$\frac{1}{\mu(g^{-1}Q)} \int_{g^{-1}Q} |f(y)|\, d\mu(y) \;\le\; \frac{A^{a+1}C}{\mu(B(x,t))} \int_{B(x,t)} |f(y)|\, d\mu(y) \;\le\; A^{a+1}\, C\mathcal{M} f(x,r)$$

and hence we obtain (3.1).

Let us fix $(x,r) \in \widetilde{X}$. If $r > \lambda^b$ we have $\mathcal{M}^b f(x,r) = 0$ and thus we can suppose $r \le \lambda^b$. Given s such that, $\lambda^{-b-1} \le s \le \lambda^b$ and $s \ge r$, by Lemma 1.1(iii), there exist $Q \in \mathcal{A}_k^b$ for some $-b \le k \le b$ and $g \in G_b$, such that $B(x,s) \subset g^{-1}Q$ and $\mu(Q) \le C\mu(B(x,s))$. Then

$$\frac{1}{\mu(B(x,s))} \int_{B(x,s)} |f(y)|\, d\mu(y) \;\le\; \frac{C}{\mu(g^{-1}Q)} \int_{g^{-1}Q} |f(y)|\, d\mu(y) \;\le\; C\, \mathcal{M}_d^{b,g} f(x,r)$$

since $\mu(Q) \ge \mu(B(x,s)) \ge \alpha(r)$. Therefore, integrating both sides of the above inequality on G_b, we have that

$$\frac{1}{\mu(B(x,s))} \int_{B(x,s)} |f(y)|\, d\mu(y) \;\le\; \frac{C}{|G_b|} \int_{G_b} \mathcal{M}_d^{b,g} f(x,r)\; dg$$

and hence we obtain (3.2). ∎

Proof of Theorem 3.1: First we prove the implication (i)⇒(ii). Suppose that there exists $B = B(z,t)$, $0 < t < \infty$ such that

$$\int_B W^{1-p'}(x)\, d\mu(x) \;=\; \infty \;.$$

Then $W^{-1}\chi_B \notin L^{p'}(W)$ and thus, by Lemma 2.1, there exists a positive function $f \in L^p(W)$ such that

$$\int_B f(x)\,d\mu(x) \;=\; \infty\;.$$

Therefore given $(x,r) \in \widetilde{X}$, there exists $s \geq r$ such that $B \subset B(x,s)$ and hence $\mathcal{M}f(x,r) = \infty$. Since β is a positive measure, we have a contradiction of the condition (i). Thus

$$\int_B W^{1-p'}(x)\,d\mu(x) \;<\; \infty\;.$$

To obtain the inequality in (ii) it is sufficient to choose $f(x) = W^{1-p'}(x)\,\chi_B(x)$ in the hypothesis.

Let us prove (ii)$\Rightarrow$(i). We fix a positive integer b, $g \in G$ and $Q \in \mathcal{A}_k^b$, $-b \leq k \leq b$. Then, by Lemma 1.1(i) there exist $x_Q \in Q$ such that $Q \subset B(x_Q, \lambda^{k+1})$ and $\mu(B(x_Q,\lambda^{k+1})) \leq C\mu(Q)$. We write $B = B(g^{-1}x_Q, \lambda^{k+1})$, $Q' = g^{-1}Q$ and $\nu = W^{1-p'}$. Since $\nu \in A_\infty(X)$, there exist positive constants C_ν and δ, depending only on ν, such that

$$\frac{\mu(Q')}{\mu(B)} \;\leq\; C_\nu \left(\frac{\nu(Q')}{\nu(B)}\right)^{\delta}.$$

Therefore

$$\nu(B) \;\leq\; C_\nu^{1/\delta}\left(\frac{\mu(B)}{\mu(Q')}\right)^{1/\delta} \nu(Q') \;\leq\; C_1\,\nu(Q')\;.$$

Then by the hypothesis and (3.1) we obtain

$$\begin{aligned}\int_{\widetilde{Q'}} \Big[\mathcal{M}_d^{b,g}(W^{1-p'}\chi_{Q'})(x,r)\Big]^p\,d\beta(x,r) \;&\leq\; C_2 \int_{\widetilde{B}} \Big[\mathcal{M}(\nu\chi_B)(x,r)\Big]^p\,d\beta(x,r)\\ &\leq\; C_3\,\nu(B)\\ &\leq\; C_4 \int_{Q'} W^{1-p'}(x)\,d\mu(x)\;.\end{aligned}$$

Since the constant C_4 depends only on p, W and β, then by Theorem 2.1 and Remark 2.1, there exists a constant C_5 such that,

$$\text{(3.3)}\qquad \int_{\widetilde{X}} [\mathcal{M}_d^{b,g} f(x,r)]^p\,d\beta(x,r) \;\leq\; C_5 \int_X |f(x)|^p\,W(x)\,d\mu(x)$$

for all $f \in L^p(W)$ and all $g \in G$. Then, it follows by (3.2), (3.3) and by Jensen's inequality that

$$\int_{\widetilde{X}} [\mathcal{M}^b f(x,r)]^p\,d\beta(x,r) \;\leq\; \int_{\widetilde{X}} \left(\frac{C_6}{|G_b|}\int_{G_b} \mathcal{M}_d^{b,g} f(x,r)\,dg\right)^p d\beta(x,r)$$

$$\le C_6^p \int_{G_b} \int_{\widetilde{X}} [\mathcal{M}_d^{b,g} f(x,r)]^p \, d\beta(x,r) \, \frac{dg}{|G_b|}$$
$$\le C_6^p \, C_5 \int_X |f(x)|^p \, W(x) \, d\mu(x) \; .$$

The result follows by the Monotone Convergence Theorem. ∎

Remark 3.1. (**a**) For $W \equiv 1$, the condition (ii) of Theorem 3.1 is given by

$$\int_{\widetilde{B}} \Big[\mathcal{M}(\chi_B)(x,r)\Big]^p \, d\beta(x,r) \;\le\; C\,\mu(B) \tag{3.4}$$

for all balls B. Let us fix $B = B(z,t),\ 0 < t < \infty$. Then, it follows as in the proof of inequality (3.1) of Lemma 3.1 that there exists a constant $C > 0$ such that

$$C \le \mathcal{M}(\chi_B)(x,r) \le 1$$

for all $(x,r) \in \widetilde{B}$. Therefore, from (3.4) we obtain

$$C^p \beta(\widetilde{B}) \;\le\; \int_{\widetilde{B}} [\mathcal{M}(\chi_B)(x,r)]^p \, d\beta(x,r) \;\le\; C\,\mu(B) \; .$$

Then, the condition (3.4) implies the condition:

$$\beta(\widetilde{B}) \le C \mu(B) \tag{3.5}$$

for a constant $C > 0$ and all balls B. But, from the condition (3.5) we obtain

$$\int_{\widetilde{B}} [\mathcal{M}(\chi_B)(x,r)]^p \, d\beta(x,r) \;\le\; \beta(\widetilde{B}) \;\le\; C\,\mu(B) \; ,$$

and therefore the conditions (3.4) and (3.5) are equivalent. The condition (3.5) is the Carleson condition for the homogeneous space X (see [8]).

(**b**) Let $B = B(z,t),\ 0 < t < \infty$ and $\nu = W^{1-p'}$. Then

$$C\,\frac{\nu(B)}{\mu(B)} \;\le\; \mathcal{M}(\nu\chi_B)(x,r)$$

for all $(x,r) \in \widetilde{B}$. Therefore, from the condition (ii) of Theorem 3.1 we obtain

$$\beta(\widetilde{B})^{1/p} \;=\; \left(\frac{\mu(B)}{\nu(B)}\right) \left[\int_{\widetilde{B}} \left(\frac{\nu(B)}{\mu(B)}\right)^p d\beta(x,r)\right]^{1/p}$$
$$\le C \left(\frac{\mu(B)}{\nu(B)}\right) \left[\int_{\widetilde{B}} [\mathcal{M}(\nu\chi_B)(x,r)]^p \, d\beta(x,r)\right]^{1/p}$$
$$\le C' \left(\frac{\mu(B)}{\nu(B)}\right) \nu(B)^{1/p} \; .$$

Then, the condition (ii) of Theorem 3.1 implies the condition:

$$\frac{\beta(\widetilde{B})^{1/p}}{\mu(B)} \left(\int_B W^{1-p'}(x)\, d\mu(x)\right)^{1/p'} \le C \tag{3.6}$$

for a constant $C > 0$ and all balls B. It was proved in Ruiz-Torrea [8] that the condition (3.6) is a necessary and sufficient condition for $\mathcal{M}$ to be a bounded operator from $L^p(X, W(x)\, d\mu(x))$ into weak - $L^p(\widetilde{X}, \beta)$. □

4 – The boundedness of the Poisson integral

Let $\xi : [0,\pi]^{n-1} \times [0, 2\pi] \to S^n$ be the function defined by $\xi(\theta) = \xi(\theta_1, \ldots, \theta_n) = (x_1, \ldots, x_{n+1})$, where

$$x_1 = \cos\theta_1; \qquad x_i = \cos\theta_i \prod_{j=1}^{i-1} \sin\theta_j,\ 2 \le i \le n; \qquad x_{n+1} = \prod_{j=1}^{n} \sin\theta_j\ .$$

We identify $S^n \times [0,1]$ with the ball $\mathbb{B} = \{y \in \mathbb{R}^{n+1} :\ |y| \le 1\}$ using the application $(y, r) \mapsto ry$. If f is a real and integrable function on S^n we define $\overline{\mathcal{M}} f(y) = \mathcal{M} f(y', h(|y|))$ for $y \in \mathbb{B},\ y \ne 0,\ y' = y/|y|$.

In Rauch [6] it was proved that

$$u_f^*(y') = \sup_{0 \le r < 1} |u_f(ry')| \le C_n\, \overline{\mathcal{M}} f(y')\,, \qquad y' \in S^n, \quad f \in L^1(S^n)\ .$$

The inequality in the following lemma generalizes the above inequality.

Lemma 4.1. *There exists a constant $C > 0$ such that, for all real-valued integrable function f on S^n and all $y \in \mathbb{B},\ 0 < |y| < 1$, we have*

$$u_f^*(y) \le C\, \overline{\mathcal{M}}\, f(y)\ .$$

Proof: We may assume $y = r\mathbb{1} = r(1, 0, \ldots, 0),\ 0 \le r < 1$. Let us denote $\theta = (\theta_1, \ldots, \theta_n),\ \theta' = (\theta_2, \ldots, \theta_n),\ \omega(\theta') = \sin^{n-2}\theta_2 \cdots \sin\theta_{n-1}$ and

$$p(\theta_1, r)\ =\ P_{r\mathbb{1}}(\xi(\theta))\ =\ \frac{1}{\omega_n}\ \frac{1 - r^2}{(1 - 2r\cos\theta_1 + r^2)^{(n+1)/2}}\ . \tag{4.1}$$

Then

$$u_f(r\mathbb{1})\ = \int_0^{\pi} d\theta_1 \cdots \int_0^{2\pi} p(\theta_1, r)\, f(\xi(\theta))\, \sin^{n-1}\theta_1\ \omega(\theta')\ d\theta_n\ .$$

If $0 \le r \le 1/2$, we have that $p(\theta_1, r) \le 2^{n+1}/\omega_n$ and hence

$$|u_f(r\mathbb{1})| \le \frac{2^{n+1}}{\omega_n} \int_{S^n} |f(x)|\, d\sigma(x) \le 2^{n+1}\, \overline{\mathcal{M}}\, f(r\mathbb{1}) .$$

Now, let us suppose $1/2 \le r < 1$. If $m(r) = \arccos r(2-r)$, then, integrating by parts with respect to θ_1, we obtain

$$\begin{aligned} I_r &= \left| \int_{S^n \setminus B(\mathbb{1}, h(r))} P_{r\mathbb{1}}(x)\, f(x)\; d\sigma(x) \right| \\ &\le p(\pi, r) \int_0^\pi d\theta_1 \cdots \int_0^{2\pi} |f(\xi(\theta))|\, \sin^{n-1}\theta_1\; \omega(\theta')\; d\theta_n \\ &\quad + p(m(r), r) \int_0^{m(r)} d\theta_1 \int_0^\pi d\theta_2 \cdots \int_0^{2\pi} |f(\xi(\theta))|\, \sin^{n-1}\theta_1\; \omega(\theta')\; d\theta_n \\ &\quad + \int_{m(r)}^\pi \left| \frac{\partial p(\theta_1, r)}{\partial \theta_1} \right| \left[\int_0^{\theta_1} \left(\int_0^\pi d\theta_2 \cdots \int_0^{2\pi} |f(\xi(t, \theta'))|\, \sin^{n-1} t\; \omega(\theta')\, d\theta_n \right) dt \right] d\theta_1 \\ &= I_r^1 + I_r^2 + I_r^3 . \end{aligned}$$

We have that

$$I_r^1 = \frac{1}{\omega_n} \frac{1-r}{(1+r)^n} \int_{S^n} |f(x)|\, d\sigma(x) \le \overline{\mathcal{M}}\, f(r\mathbb{1}) .$$

Since

$$\frac{\omega_{n-1}}{n\, 2^{n-1}} (1-r)^n \le \sigma(B(\mathbb{1}, h(r))) \le \frac{2^n\, \omega_{n-1}}{n} (1-r)^n$$

then for $1/2 \le r < 1$, it follows that

$$I_r^2 \le \frac{2}{\omega_n (1-r)^n} \int_{B(\mathbb{1}, h(r))} |f(x)|\, d\sigma(x) \le \frac{2^{n+1}\, \omega_{n-1}}{n\, \omega_n} \overline{\mathcal{M}}\, f(r\mathbb{1}) .$$

Using properties of the Poisson kernel and integration by parts, we obtain

$$\int_0^\pi \left| \frac{\partial p(\theta_1, r)}{\partial \theta_1} \right| \left(\int_0^{\theta_1} \sin^{n-1} t\; dt \right) d\theta_1 = \frac{1}{\omega_{n-1}} \left(1 - \frac{1-r}{(1+r)^n} \right) \le \frac{1}{\omega_{n-1}}$$

and thus

$$I_r^3 \le \frac{1}{\omega_{n-1}} \overline{\mathcal{M}}\, f(r\mathbb{1}) .$$

Therefore, there exists a constant $D > 0$, such that

$$I_r \le I_r^1 + I_r^2 + I_r^3 \le D \overline{\mathcal{M}} f(r\mathbb{1})$$

for all $1/2 \le r < 1$. Consequently

$$\begin{aligned}
|u_f(r\mathbb{1})| &\le \frac{2}{\omega_n}\,\frac{1}{(1-r)^n}\int_{B(\mathbb{1},h(r))}|f(x)|\,d\sigma(x)\;+\;I_r\\
&\le \frac{2^{n+1}\,\omega_{n-1}}{n\,\omega_n}\,\frac{1}{\sigma(B(\mathbb{1},h(r)))}\int_{B(\mathbb{1},h(r))}|f(x)|\,d\sigma(x)\;+\;D\overline{\mathcal{M}}f(r\mathbb{1})\\
&\le \frac{2^{n+1}\,\omega_{n-1}}{n\,\omega_n}\,\overline{\mathcal{M}}f(r\mathbb{1})+D\overline{\mathcal{M}}f(r\mathbb{1}) \;=\; C\overline{\mathcal{M}}f(r\mathbb{1})\;. \blacksquare
\end{aligned}$$

Proof of Theorem 4.1: The Proof of (i)⇒(ii) is exactly as the proof of (i)⇒(ii) in Theorem 2.1 and Theorem 3.1.

Let us prove (ii)⇒(i). Let f be a real-valued positive integrable function on S^n. There exists a constant $C>0$, depending only on n, such that

$$P_{ry'}(x) \;\ge\; \frac{C}{\sigma(B(y',h(r)))}$$

for all $0 \le r < 1$, $y' \in S^n$ and $x \in B(y',h(r))$. Therefore

$$u_f(ry') \;\ge\; \frac{C}{\sigma(B(y',h(r)))}\int_{B(y',h(r))} f(x)\,d\sigma(x)$$

and hence

$$u_f^*(ry') \ge C\overline{\mathcal{M}}f(ry') \tag{4.2}$$

Consider the function $k\colon \mathbb{B} \to \widetilde{S}^n$ defined by $k(x) = (x/|x|, h(|x|))$, $x \ne 0$, $k(0) = (\mathbb{1},0)$, $\mathbb{1} = (1,0,\dots,0)$. Then applying Theorem 3.1 to $X = S^n$ and to the image measure β of ν by k, $\beta(A) = \nu(k^{-1}(A))$ and using the inequalities (4.1) and (4.2), we obtain the wanted proof. $\blacksquare$

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