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EXISTENCE RESULTS OF NONCONVEX DIFFERENTIAL INCLUSIONS *

Messaoud Bounkhel

Abstract: This paper is devoted to the study of nonconvex differential inclusions by using some concepts of regularity in nonsmooth analysis. In section 2, we prove that the nonconvex sweeping process introduced by J.J. Moreau in 1970's has the same set of solutions of a differential inclusion with convex compact values. Using this result, we deduce, in section 3, some existence results in the finite dimensional setting of the nonconvex sweeping process. In section 4, we introduce a new concept of uniform regularity over sets for functions to prove the existence of viable solutions for another type of nonconvex differential inclusions.

Introduction

In this paper, we study, on one hand, nonconvex sweeping processes (Sections 2 and 3) and, on the other hand, the existence of viable solutions for a class of nonconvex differential inclusions (Section 4).

We consider the following differential inclusion:

$$(P_1) \qquad \begin{cases} \dot{x}(t) \in -N(C(t); x(t)) & \text{a.e. } t \geq 0 \\ x(0) = x_0 \in C(0), \;\; x(t) \in C(t), & \forall \; t \geq 0 \;, \end{cases}$$

where C is an absolutely set-valued mapping (see (1.1) below) taking its values in Hilbert spaces and $N(C(t); x(t))$ denotes a prescribed normal cone to the set $C(t)$ at $x(t)$. The problem (P_1) is the so-called "*sweeping process problem*" (in French, rafle). It was introduced by Moreau in [28, 29] and studied intensively by himself

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in many papers (see for example [28, 29, 30]). This problem is related to the modelization of elasto-plastic materials (for more details see [31, 32]). The existence of solutions of (P_1) was resolved by Moreau in [30] for convex-valued mappings C taking their values in general Hilbert spaces. In [41, 42] Valadier proved for the first time the existence of solutions of (P_1) without convexity assumptions on C for some particular cases in the finite dimensional setting. Since, many authors attacked the study of the existence of solutions for nonconvex sweeping processes (see for instance [2, 9, 12, 16, 21, 37] and the references therein). The first part of the present paper is mainly concerned with the following problem: Under which conditions the solution set of (P_1) can be related to the solution set of the following convex compact differential inclusion (P_2)?

$$(P_2) \qquad \begin{cases} \dot{x}(t) \in -|\dot{v}(t)|\, \partial d_{C(t)}(x(t)), & \text{a.e. } t \geq 0 \\ x(0) = x_0 \in C(0) \ , \end{cases}$$

where v is an absolutely continuous function given as in (1.1) and $\partial d_{C(t)}(\cdot)$ stands for a prescribed subdifferential of the distance function $d_{C(t)}$ associated with the set $C(t)$.

This problem was considered by Thibault in [38] for convex-valued mappings C in the finite dimensional setting. His idea was to use the existence results for differential inclusions with convex compact values which is the case for (P_2) to prove existence results of the sweeping process (P_1). It is interesting to point out that his approach is new and different from those used by the authors who have studied the existence of solutions of the sweeping process (P_1).

In the second part (Section 4) of this paper, we consider the following class of differential inclusion (DI):

$$\dot{x}(t) \in G(x(t)) + F(t, x(t)) \qquad \text{a.e. } \ [0, T] \ ,$$

where $T > 0$ is given, $F : [0, T] \times H \rightrightarrows H$ is a continuous set-valued mapping, $G : H \rightrightarrows H$ is an upper semicontinuous set-valued mapping such that $G(x) \subset \partial^C g(x)$, with $g : H \to \mathbb{R}$ is a locally Lipschitz function (not necessarily convex) and H is a finite dimensional space. Here $\partial^C g(x)$ denotes the Clarke subdifferential of g at x (see the definition given in Section 1). By using some new concepts of regularity in nonsmooth analysis, we prove (Theorem 4.2) under natural additional assumptions the existence of viable solutions for (DI), that is, a solution x of (DI) such that $x(t) \in S$, for all $t \in [0, T]$, where S is a given closed subset in H. Our main existence result in Theorem 4.2 is used to get existence results for a particular type of differential inclusions introduced by Henry [25] for the study of some economic problems.

1 – Preliminaries

Throughout this paper, we will let H denote a Hilbert space and $C : \mathbb{R} \rightrightarrows H$ denote a set-valued mapping satisfying for any $y \in H$ and any $t, t' \in \mathbb{R}$

$$|d(y, C(t)) - d(y, C(t'))| \leq |v(t) - v(t')|, \tag{1.1}$$

where $v : \mathbb{R} \to \mathbb{R}$ is an absolutely continuous function with $|\dot{v}(t)| \neq 0$ a.e. $t \in \mathbb{R}$ and $d(\cdot, S)$ (or $d_S(\cdot)$) stands for the usual distance function to S, i.e., $d(x, S) := \inf\limits_{u \in S} \|x - u\|$. Hereafter, an absolutely continuous mapping means a mapping $x : [0, +\infty[\to H$ such that $x(t) = x(0) + \int_0^t \dot{x}(s)\, ds$, $\forall t \in [0, +\infty[$, with $\dot{x} \in L^1_H([0, +\infty[)$.

Let $f : H \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous (l.s.c.) function and let x be any point where f is finite. We recall that the *Clarke subdifferential* of f at x is defined by (see [34])

$$\partial^C f(x) = \Big\{ \xi \in H : \langle \xi, h \rangle \leq f^{\uparrow}(x; h), J \quad \text{for all} \ \ h \in H \Big\} \ ,$$

where $f^{\uparrow}(x; h)$ is *the generalized Rockafellar directional derivative* given by

$$f^{\uparrow}(x; h) := \limsup_{\substack{x' \to_f x \\ t \downarrow 0}} \inf_{h' \to h} t^{-1} [f(x' + th') - f(x')] \ ,$$

where $x' \longrightarrow^f x$ means $x' \longrightarrow x$ and $f(x') \longrightarrow f(x)$.

If f is Lipschitz around x, then $f^{\uparrow}(x; h)$ coincides with *the Clarke directional derivative* $f^0(x; .)$ defined by $f^0(x; h) = \limsup\limits_{\substack{x' \to x \\ t \downarrow 0}} t^{-1} [f(x' + th) - f(x')]$.

Recall also (see e.g., [26]) that the Fréchet subdifferential $\partial^F f(x)$ is given by the set of all $\xi \in H$ such that for all $\epsilon > 0$ there exists $\delta > 0$ such that

$$\langle \xi, x' - x \rangle \leq f(x') - f(x) + \epsilon \|x' - x\|, \qquad \text{for all} \quad x' \in x + \delta \mathbb{B} \ .$$

Here $\mathbb{B}$ denotes the closed unit ball centered at the origin of H. Note that one always has $\partial^F f(x) \subset \partial^C f(x)$. By convention we set $\partial^F f(x) = \partial^C f(x) = \emptyset$ if $f(x)$ is not finite.

Let S be a nonempty closed subset of H and x be a point in S. Let us recall (see [34, 26]) that *the Clarke normal cone* (resp. *Fréchet normal cone*) of S at x is defined by $N^C(S; x) := \partial^C \psi_S(x)$ (resp. $N^F(S; x) := \partial^F \psi_S(x)$), where ψ_S denotes the indicator function of S, i.e., $\psi_S(x') = 0$ if $x' \in S$ and $+\infty$ otherwise.

We consider now the following notion of regularity for sets.

Definition 1.1 ([7, 11, 35]). Let S be a nonempty closed subset of H and let x be a point in S. We will say that S is *normally Fréchet regular* at x if one has $N^F(S;x) = N^C(S;x)$. □

We summarize, in the following proposition, some results needed in the sequel.

Proposition 1.1 ([7, 11]). *Let S be a nonempty closed subset in H and let $x \in S$. Then*

i) $\partial^F d_S(x) = N^F(S;x) \cap \mathbb{B}$;

ii) *If S is normally Fréchet regular at x, then it is tangentially regular at x in the sense of Clarke [18]. If, in addition, H is a finite dimensional space, then one has the equivalence.*

Note that is the infinite dimensional setting, one can construct subsets that are tangentially regular but not normally Fréchet regular. For more details, we refer the reader to [7, 11].

Let F be a given set-valued mapping from $[0,+\infty[\times H$ to the subsets of H. A *solution* $x(\cdot)$ of the differential inclusion

$$\dot{x}(t) \in F(t,x(t)) \qquad \text{a.e. } t \geq 0 \tag{1.2}$$

is taken to mean an absolutely continuous mapping $x(\cdot) : [0,+\infty[\to H$ which, together with $\dot{x}(\cdot)$, its derivative with respect to t, satisfy (1.2).

2 – Nonconvex sweeping process

Our main purpose of this section[1] is to show, for a large class of set-valued mappings, that the solution set of the two following differential inclusions are the same:

$$(P_1) \qquad \begin{cases} \dot{x}(t) \in -N^C(C(t);x(t)), \ \text{ a.e. } t \geq 0 & (1) \\ x(0) = x_0 \in C(0) & (2) \\ x(t) \in C(t) \qquad \forall t \geq 0 & (3) \end{cases}$$

and

$$(P_2) \qquad \begin{cases} \dot{x}(t) \in -|\dot{v}(t)|\, \partial^C d_{C(t)}(x(t)), \ \text{ a.e. } t \geq 0 & (4) \\ x(0) = x_0 \in C(0) & (2)\ . \end{cases}$$

[1] While writing the present paper we have received the preprint [37] by Thibault, which contains similar results of this section in the proximal smooth case.

that is, a mapping $x(\cdot) : [0, +\infty[\to H$ is a solution of (P_1) if and only if it is a solution of (P_2).

It is easy to see that one always has $(P_2) + (3) \Rightarrow (P_1)$. Indeed, let $x(\cdot) : [0, +\infty[\to H$ be a solution of (P_2) satisfying (3). Then a.e. $t \geq 0$ we have

$$\dot{x}(t) \in -|\dot{v}(t)|\, \partial^C d_{C(t)}(x(t)) \subset -N^C(C(t); x(t))$$

and hence $x(\cdot)$ is a solution of (P_1).

The use of (P_2) as an intermediate problem to prove existence results of the sweeping process (P_1) is due to Thibault [38]. His idea was to use the existence results for differential inclusions with compact convex values which is the case of the problem (P_2) to prove an existence result of the sweeping process (P_1). Note that all the authors (for example [2, 21]), who have studied the sweeping process (P_1), have attacked it by direct methods for example by proving the convergence of the Moreau catching-up algorithm or by using some measurable arguments and new versions of the well known theorem of Scorza-Dragoni.

Recall that, Thibault [38] showed that, when C has closed convex values in a finite dimensional space H, any solution of (P_2) is also a solution of (P_1) and as (P_2) has always at least one solution by Theorem VI.13 in [17], then he obtained the existence of solutions of the convex sweeping process (P_1) in the finite dimensional setting. His idea is to show the viability of all solutions of (P_2), that is, any solution of (P_2) satisfies (3) and so it is a solution of (P_1) by using the implication $(P_2) + (3) \Rightarrow (P_1)$. Recently, Thibault in [37] used the same idea to extend this result to the proximal smooth case.

In this section we will follow this idea to extend his result in [38] to the nonconvex case by using powerful results by Borwein et al. [5] and recent results by Bounkhel and Thibault [11]. We begin with the following theorem.

Theorem 2.1. *Any solution of* (P_1) *with the Fréchet normal cone satisfies the inequality* $\|\dot{x}(t)\| \leq |\dot{v}(t)|$ *a.e.* $t \geq 0$.

Proof: Let $x(\cdot) : [0, +\infty[\to H$ be an absolutely continuous solution of (P_1) with the Fréchet normal cone, that is, $-\dot{x}(t) \in N^F(C(t); x(t))$ a.e. $t \geq 0$, $x(0) = x_0 \in C(0)$, and $x(t) \in C(t)$ $\forall t \geq 0$. Fix any $t \geq 0$ for which $\dot{x}(t)$ and $\dot{v}(t)$ exist and fix also $\epsilon > 0$. If $\dot{x}(t) = 0$, then we are done, so let suppose that $\dot{x}(t) \neq 0$. By the definition of the Fréchet normal cone, there exists $\delta := \delta(t, \epsilon)$ such that

$$\langle -\dot{x}(t), x - x(t)\rangle \leq \epsilon\|x - x(t)\| \quad \forall x \in (x(t) + \delta\mathbb{B}) \cap C(t)\ . \tag{2.1}$$

On the other hand there exists a mapping $\theta : \mathbb{R}_+ \to H$ such that $\lim_{r\to 0^+} \theta(r) = 0$ and $x(t-r) = x(t) - r\dot{x}(t) - r\theta(r)$, for r small enough. Fix now $r > 0$ small enough such that $0 < r < \min\Big\{1, \frac{\delta}{3\|\dot{x}(t)\|}\Big\}$, $\|\theta(r)\| \le \frac{\delta}{3}$ and $|v(t-r) - v(t)| \le \frac{\delta}{3}$. By (1.1) and (3) one has $x(t-r) \in C(t-r) \subset C(t) + |v(t-r) - v(t)|\mathbb{B}$. So there exists $x_t \in C(t)$ and $b_t \in \mathbb{B}$ such that $x(t-r) = x_t - \xi_t$ where $\xi_t = |v(t-r) - v(t)|b_t$. Therefore $x_t = x(t-r) + \xi_t = x(t) - r\dot{x}(t) - r\theta(r) + \xi_t \in (x(t) + \delta\mathbb{B}) \cap C(t)$, since $\|x_t - x(t)\| = \|-r\dot{x}(t) - r\theta(r) + \xi_t\| \le \|\dot{x}(t)\| + \|\theta(r)\| + \|\xi_t\| \le \frac{\delta}{3} + \frac{\delta}{3} + |v(t-r) - v(t)| \le \delta$. Thus, by (2.1)

$$\langle -\dot{x}(t), -r\dot{x}(t) - r\theta(r) + \xi_t\rangle \le \epsilon\,\|r\dot{x}(t) + r\theta(r) - \xi_t\|$$

and hence

$$r\langle -\dot{x}(t), -\dot{x}(t) - \theta(r) + r^{-1}\xi_t\rangle \le \epsilon\, r\Big[\|\dot{x}(t) + \theta(r)\| + r^{-1}|v(t-r) - v(t)|\Big]$$

and so

$$\begin{aligned}\langle \dot{x}(t), \dot{x}(t)\rangle &\le \langle -\dot{x}(t), \theta(r) - r^{-1}\xi_t\rangle + \epsilon\Big[\|\dot{x}(t) + \theta(r)\| + r^{-1}|v(t-r) - v(t)|\Big]\\ &\le \|\dot{x}(t)\|\Big[\|\theta(r)\| + r^{-1}|v(t-r) - v(t)|\Big]\\ &\quad + \epsilon\Big[\|\dot{x}(t) + \theta(r)\| + r^{-1}|v(t-r) - v(t)|\Big]\ .\end{aligned}$$

By letting $\epsilon, r \to 0^+$, one gets $\|\dot{x}(t)\|^2 \le \|\dot{x}(t)\||\dot{v}(t)|$ and then $\|\dot{x}(t)\| \le |\dot{v}(t)|$. This completes the proof. ∎

The following corollary generalizes Theorem 5.1 of Colombo et al. [21].

Corollary 2.1. *Assume that $C(t)$ is normally Fréchet regular for every $t \ge 0$. Then any solution of (P_1) satisfies the inequality $\|\dot{x}(t)\| \le |\dot{v}(t)|$ a.e. $t \ge 0$.*

Now, we prove that, under the normal Fréchet regularity assumption, any solution of (P_1) must be a solution of (P_2).

Theorem 2.2. *Assume that $C(t)$ is normally Fréchet regular for every $t \ge 0$. Then any solution of (P_1) is also a solution of (P_2).*

Proof: Let $x(\cdot)$ be a solution of (P_1), that is, $x(0) = x_0 \in C(0)$, $x(t) \in C(t)$ $\forall t \ge 0$ and $-\dot{x}(t) \in N^C(C(t); x(t))$ a.e. $t \ge 0$. Then, by the Fréchet normal regularity one has $-\dot{x}(t) \in N^C(C(t); x(t)) = N^F(C(t); x(t))$ a.e. $t \ge 0$. By Theorem 2.1 one has $\|\dot{x}(t)\| \le |\dot{v}(t)|$ a.e. $t \ge 0$. If $\dot{x}(t) = 0$, then $-\dot{x}(t) \in$

$|\dot{v}(t)|\,\partial^C d_{C(t)}(x(t))$, because $x(t) \in C(t)$. So we assume that $\dot{x}(t) \neq 0$ (and hence $\dot{v}(t) \neq 0$). Then, by Proposition 1.1 i), one gets

$$\frac{-\dot{x}(t)}{|\dot{v}(t)|} \in N^F(C(t); x(t)) \cap \mathbb{B} = \partial^F d_{C(t)}(x(t)) \subset \partial^C d_{C(t)}(x(t)) \ .$$

Thus $\dot{x}(t) \in -|\dot{v}(t)|\,\partial^C d_{C(t)}(x(t))$, which ensures that $x(\cdot)$ is a solution of (P_2) and so the proof is finished. ∎

Now we proceed to prove the converse of Theorem 2.2, for a large class of set-valued mappings. We recall (see e.g., [5]) the notion of Gâteaux directional differentiability. A locally Lipschitz function $f : H \to \mathbb{R}$ is *directionally Gâteaux differentiable* at $\bar{x} \in H$ in the direction $v \in H$ if $\lim\limits_{t \to 0} t^{-1}[f(\bar{x} + tv) - f(\bar{x})]$ exists. We call such a limit the Gâteaux directional derivative of f at $\bar{x}$ in the direction v and we denote it by $\nabla_G f(\bar{x}; v)$. When this limit exists for all $v \in H$ and is linear in v we will say that f is Gâteaux differentiable at $\bar{x}$ and the Gâteaux derivative satisfies $\nabla_G f(\bar{x}; v) = \langle \nabla_G f(\bar{x}), v \rangle$ for all $v \in H$. If $\nabla_G f(\cdot)$ is continuous around $\bar{x}$, then f will be called continuously Gâteaux differentiable at $\bar{x}$. We say that f is *directionally regular* at $\bar{x}$ in a direction $v \in H$ provided that $f^{\uparrow}(\bar{x}; v)$ the generalized Rockafellar directional derivative (or $f^0(\bar{x}; v)$ the Clarke directional derivative because f is locally Lipschitz) of f at $\bar{x}$ in the direction v coincides with $f^-(\bar{x}; v)$ the lower Dini directional derivative of f at $\bar{x}$ in the same direction v, where $f^-(\bar{x}; v) := \liminf\limits_{t \to 0^+} t^{-1}[f(\bar{x} + tv) - f(\bar{x})]$.

Theorem 2.3 (An abstract formulation). *Let $h : [0, +\infty[\to [0, +\infty[$ be a positive function. Assume that for every absolutely continuous mapping $x(\cdot) : [0, +\infty[\to H$ the following property (A) is satisfied: for a.e. $t \geq 0$ and for any $x(t)$ in the tube $U(h(t)) := \{u \in H : \ 0 < d_{C(t)}(u) < h(t)\}$ one has*

- **i)** *$Proj_{C(t)}(x(t)) \neq \emptyset$ and $d_{C(t)}$ is directionally regular at $x(t)$ in both directions $\dot{x}(t)$ and $p(x(t)) - x(t)$ for some $p(x(t)) \in Proj_{C(t)}(x(t))$.*

Then every solution z of (P_2) in $C(t) + h(t)\mathbb{B}$ for all $t \geq 0$ must lie in $C(t)$ for all $t \geq 0$.

Before giving the proof of Theorem 2.3, we prove the following Lemmas.

Lemma 2.1. *Let S be a closed nonempty subset of H and u is any point outside S such that $Proj_S(u) \neq \emptyset$. Assume that d_S is directionally regular at u in the direction $\bar{u} - u$, for some $\bar{u} \in Proj_S(u)$. Then $\partial^C d_S(u) \subset \{\xi \in H : \ \|\xi\| = 1\}$.*

Proof: Fix any $u \notin S$ with $Proj_S(u) \neq \emptyset$ and any $\xi \in \partial^C d_S(u)$. As the inequality $\|\xi\| \leq 1$ always holds, we will prove the reverse inequality, i.e., $\|\xi\| \geq 1$. Firstly, we fix $\bar{u} \in Proj_S(u) \neq \emptyset$ and we show that

$$(1-\delta)d_S(u) = d_S(u+\delta(\bar{u}-u)), \quad \text{for all} \quad \delta \in [0,1]. \tag{2.2}$$

Observe that one always has

$$d_S(u) \;\leq\; d_S(u+\delta(\bar{u}-u)) + \delta\|\bar{u}-u\| \;=\; d_S(u+\delta(\bar{u}-u)) + \delta\, d_S(u)\ ,$$

and so $(1-\delta)d_S(u) \leq d_S(u+\delta(\bar{u}-u))$. Conversely,

$$d_S(u+\delta(\bar{u}-u)) = d_S(\bar{u}+(1-\delta)(u-\bar{u})) \leq (1-\delta)\|\bar{u}-u\| = (1-\delta)\, d_S(u)\ .$$

Now, let δ_n be a sequence achieving the limit in the definition of $d_S^-(u;\bar{u}-u)$ the lower Dini directional derivative of d_S at u in the direction $\bar{u}-u$. Then, by (2.2), one gets

$$d_S^-(u;\bar{u}-u) = \lim_n \delta_n^{-1}[d_S(u+\delta_n(\bar{u}-u))-d_S(u)] = \lim_n \delta_n^{-1}[(1-\delta_n)d_S(u)-d_S(u)]\ ,$$

and hence $d_S^-(u;\bar{u}-u) = -d_S(u)$. Finally, by the directional regularity of d_S at u in the direction $\bar{u}-u$ and by the definition of the Clarke subdifferential one gets

$$\langle \xi, \bar{u}-u\rangle \leq d_S^0(u;\bar{u}-u) = d_S^-(u;\bar{u}-u) = -d_S(u) = -\|\bar{u}-u\|\ ,$$

and so

$$\left\langle \xi, \frac{u-\bar{u}}{\|\bar{u}-u\|} \right\rangle \geq 1\ ,$$

which ensures that $\|\xi\| \geq 1$. ∎

The following lemma is a direct consequence of Corollary 9 in [5]. We give its proof for the convenience of the reader.

Lemma 2.2. *Let S be a closed nonempty subset of H, $u \notin S$ and $v \in H$. Then the following are equivalent:*

1) $\langle \partial^C d_S(u), v\rangle = \{d_S^0(u;v)\}$;

2) *d_S is directionally regular at u in the direction v;*

3) *d_S is Gâteaux differentiable at u in the direction v.*

Proof: The equivalence between 1) and 3) is given in [5]. The implication 1)$\Rightarrow$2) is obvious. So we proceed to proving the reverse one, i.e., 2)$\Rightarrow$1). By Theorem 8 in [5] one has $-d_S$ is directionally regular at u, hence $(-d_S)^0(u,v) = (-d_S)^-(u,v)$ and hence $d_S^0(u,-v) = -d_S^-(u,v)$. By 1) one has $d_S^0(u,v) = d_S^-(u,v)$. Therefore, one obtains $d_S^0(u,-v) = -d_S^0(u,v)$. Now, as we can easily check that $\langle \partial^C d_S(u), v\rangle = [-d_S^0(u,-v), d_S^0(u,v)]$, then one gets $\langle \partial^C d_S(u), v\rangle = \{d_S^0(u;v)\}$. This completes the proof of the lemma. ∎

Proof of Theorem 2.3: We Prove the theorem for all $t \in [0,1]$ and we can extend the proof to $[0,+\infty[$ in the evident way by considering next the interval $[1,2]$ etc. We follow the proof of Proposition II.18 in Thibault [38]. Let z be a solution of (P_2) satisfying $z(t) \in C(t) + h(t)\mathbb{B}$ for all $t \in [0,1]$. Consider the real function f defined by $f(t) = d_{C(t)}(z(t)$. The function f is absolutely continuous because of (1.1). Put $\Omega := \{t \in [0,1] : z(t) \notin C(t)\}$. Ω is an open subset in $[0,1]$ because $\Omega = \{t \in [0,1] : f(t) > 0\}$. Assume by contradiction that $\Omega \neq \emptyset$. As $0 \notin \Omega$ there exists an interval $]\alpha,\beta[\subset \Omega$ such that $f(\alpha) = 0$ (it suffices to take $]\alpha,\beta[$ any connected component of $]0,1[\cap\Omega)$. Since f, v and z are absolutely continuous, then their derivatives exist a.e. on $[0,1]$. Fix any $t \in]\alpha,\beta[$ such that $\dot f(t), \dot v(t)$ and $\dot z(t)$ exist. Observe that for such t and for every $\delta > 0$ we have

$$\begin{aligned}
\delta^{-1}[f(t+\delta) - f(t)] &= \delta^{-1}[d_{C(t+\delta)}(z(t+\delta)) - d_{C(t)}(z(t))] \\
&= \delta^{-1}[d_{C(t+\delta)}(z(t) + \delta\dot z(t) + \delta\epsilon(\delta)) - d_{C(t+\delta)}(z(t) + \delta\dot z(t))] \\
&\quad + \delta^{-1}[d_{C(t+\delta)}(z(t) + \delta\dot z(t)) - d_{C(t)}(z(t) + \delta\dot z(t))] \\
&\quad + \delta^{-1}[d_{C(t)}(z(t) + \delta\dot z(t)) - d_{C(t)}(z(t))] ,
\end{aligned}$$

where $\epsilon(\delta) \to 0^+$ as $\delta \to 0^+$ and hence

$$\begin{aligned}
\delta^{-1}[f(t+\delta) - f(t)] \le\ & \epsilon(\delta) + \delta^{-1}|v(t+\delta) - v(t)| \\
& + \delta^{-1}[d_{C(t)}(z(t) + \delta\dot z(t)) - d_{C(t)}(z(t))] .
\end{aligned}$$

Thus for such t we have

$$\begin{aligned}
\dot f(t) &\le |\dot v(t)| + \limsup_{\delta\to 0^+} \delta^{-1}[d_{C(t)}(z(t) + \delta\dot z(t)) - d_{C(t)}(z(t))] \\
&\le |\dot v(t)| + d^0_{C(t)}(z(t);\dot z(t)) .
\end{aligned}$$

Now, as z is a solution of (P_2) we have $\dfrac{-\dot z(t)}{|\dot v(t)|} \in \partial^C d_{C(t)}(z(t))$ and hence $\left\langle \dfrac{-\dot z(t)}{|\dot v(t)|}, \dot z(t)\right\rangle \in \langle \partial^C d_{C(t)}(z(t)), \dot z(t)\rangle = \{d^0_{C(t)}(z(t);\dot z(t))\}$ (by Lemma 2.2).

On the other hand as $z(t) \in U(h(t))$ and by the hypothesis (A) and Lemma 2.1 one gets $\frac{\|-\dot{z}(t)\|}{|\dot{v}(t)|} = 1$ and hence $\|\dot{z}(t)\| = |\dot{v}(t)|$. Therefore

$$d^0_{C(t)}(z(t);\dot{z}(t)) = -\left\langle \frac{\dot{z}(t)}{|\dot{v}(t)|}, \dot{z}(t) \right\rangle = -\frac{\|\dot{z}(t)\|^2}{|\dot{v}(t)|} = -|\dot{v}(t)| \ .$$

Now, for such $t \in]\alpha,\beta[$ we have $\dot{f}(t) \leq 0$. So, as f is absolutely continuous we have $f(\theta) = f(\alpha) + \int_\alpha^\theta \dot{f}(t)\,dt \leq 0$ for every $\theta \in]\alpha,\beta[$. But by the definition of f we have $f(\theta) \geq 0$ for every θ. Thus $f(\theta) = 0$ which contradicts that $]\alpha,\beta[\subset \Omega$. Hence $\Omega = \emptyset$. This completes the proof. ∎

Now, we have the following corollary.

Corollary 2.2. *Put $h(t) := 2\int_0^t |\dot{v}(s)|\,ds$ and assume that the hypothesis (A) holds. Then for every solution z of (P_2), one has $z(t) \in C(t)$ for all $t \geq 0$.*

Proof: It is sufficient to show that every solution of (P_2) satisfies the hypothesis (A). Indeed, let z be a solution of (P_2). Then for a.e. $t \geq 0$ one has $\|\dot{z}(t)\| \leq |\dot{v}(t)|$. So, by (1.1) one gets

$$d_{C(t)}(z(t)) \;\leq\; \|z(t)-z(0)\| + |v(t)-v(0)| \;\leq\; \int_0^t |\dot{v}(s)|\,ds + \int_0^t \|\dot{z}(s)\|\,ds \;\leq\; h(t) \ .$$

This ensures that $z(t) \in C(t) + h(t)\mathbb{B}$. ∎

Using Corollary 2.2 one gets the nonemptiness of the set of solutions of both problems (P_1) and (P_2) in the finite dimensional setting and that these two sets of solutions are the same. Note that this result is more strongly than the existence results of the problem (P_1) proved in [2, 21], because it is not necessary that a solution of (P_1) is to be a solution of (P_2). Note also that their existence results for the problem (P_1) have been obtained, respectively, for any Lipschitz set-valued mapping C taking its values in a finite dimensional space, and for any Lipschitz set-valued mapping C having locally compact graph and taking its values in a Hilbert space. Their proofs are strongly based on new versions of Scorza-Dragoni's theorem.

Theorem 2.4. *Assume that $\dim H < +\infty$ and the hypothesis (A) holds with $h(t) := 2\int_0^t |\dot{v}(s)|\,ds$. Then both problems (P_1) and (P_2) have the same set of solutions which is nonempty.*

Proof: By corollary 2.2 and the implication $(P_2) + (3) \Rightarrow (P_1)$, it is sufficient to show that (P_2) admits at least one solution. Indeed, we put $f_t(x) := -|\dot{v}(t)|\, d_{C(t)}(x)$ and we observe that this function satisfies all hypothesis of Lemma II.15 in Thibault [38] (we can apply directly Theorem VI.13 in Castaing and Valadier [17] as in the lemma I.15 in [38]). Then one gets by this lemma that (P_2) admits at least one solution. ∎

In order to give a concrete application of our abstract result in Theorem 2.3, we recall the definition of proximal smoothness for subsets introduced by Clarke et al. [19], which is a generalization of convex subsets. For the importance of this notion of smoothness we refer the reader to [19, 12, 33].

Definition 2.1. *Let S be a closed nonempty subset in H. Following Clarke et al. [19] we will say that S is r-proximally smooth if d_S is continuously Gâteaux differentiable on the tube $U(r) := \{u \in H : \ 0 < d_S(u) < r\}$.* □

Corollary 2.3. *Put $h(t) := 2\int_0^t |\dot{v}(s)|\, ds$ and assume that $C(t)$ is $r(t)$-proximally smooth for all $t \geq 0$ with $h(t) \leq r(t)$. Then for every solution z of (P_2), one has $z(t) \in C(t)$ for all $t \geq 0$.*

Proof: It is easily seen by Lemma 2.2 that under the $r(t)$-proximal smoothness of $C(t)$ for all $t \geq 0$ with $h(t) \leq r(t)$, the hypothesis (A) holds. So, we can directly apply Corollary 2.2. ∎

We close this section by establishing the following result. It is the combination of Theorem 2.2 and Corollary 2.2. It proves the equivalence between (P_1) and (P_2) for any set-valued mapping C satisfying the following hypothesis (A'): given a positive function $h : [0, +\infty[\ \to [0, +\infty[$. For every absolutely continuous mapping $x(\cdot) : [0, +\infty[\ \to H$ and for a.e. $t \geq 0$ the two following assertions hold:

1) $C(t)$ is Fréchet normally regular at $x(t) \in C(t)$;

2) for every $x(t) \in U(h(t)) : Proj_{C(t)}(x(t)) \neq \emptyset$, $d_{C(t)}$ is directionally regular at $x(t)$ in both directions $\dot{x}(t)$ and $p(x(t)) - x(t)$, for some $p(x(t)) \in Proj_{C(t)}(x(t))$.

Theorem 2.5. *Assume that (A') holds with $h(t) := 2\int_0^t |\dot{v}(s)|\, ds$. Then (P_1) is equivalent to (P_2).*

Remark 2.1. Note that under the $r(t)$-proximal smoothness of $C(t)$ for all $t \geq 0$ with $h(t) \leq r(t)$, we can show (see Clarke et al. [19] for the part 1 in (A')) that the hypothesis (A') holds too. So we obtain the following result, also obtained in [38]. □

Theorem 2.6. *Put* $h(t) := 2\int_0^t |\dot{v}(s)|\, ds$ *and assume that* $C(t)$ *is* $r(t)$*-proximally smooth for all* $t \geq 0$ *with* $h(t) \leq r(t)$*. Then* (P_1) *is equivalent to* (P_2)*.*

3 – Existence results of (P_1) and (P_2)

Throughout this section , H will be a finite dimensional space. Our aim here is to prove the existence of solutions to (P_1) and (P_2) by a new and a direct method and under another hypothesis which is incomparable in general with the hypothesis (A) given in the previous section. Note that in the recent preprint by Thibault [37] the same method is used to prove general existence results of (P_1) by using a recent viability result by Frankowska and Plaskacz.

We begin by recalling the following proposition (see e.g. [20])

Proposition 3.1 ([20]). *Let* X *be a finite dimensional space. Let* $F : X \rightrightarrows X$ *be an upper semicontinuous set-valued mapping with compact convex images and let* $S \subset \operatorname{dom} F$ *be a closed subset in* X*. Then the two following assertions are equivalent:*

- **i)** $\forall x \in S$, $\forall p \in \Pi(S;x)$, $\sigma\Big(F(x), -p\Big) \geq 0$;
- **ii)** $\forall x_0 \in S$, $\exists$ *a solution* $x(\cdot) : [0,+\infty[\;\to H$ *of the differential inclusion* $\dot{x}(t) \in F(x(t))$ *a.e.* $t \geq 0$ *such that* $x(0) = x_0$ *and* $x(t) \in S$ *for all* $t \geq 0$*.*

Here $\Pi(S;x)$ *denotes the set of all vectors* $\xi \in H$ *such that* $d_S(x+\xi) = \|\xi\|$*.*

We prove the following result that is the key of the proof of Theorem 3.1.

Lemma 3.1. *Let* $C : \mathbb{R}_+ \rightrightarrows H$ *be a set-valued mapping satisfying (1.1). For all* $(t,x) \in \operatorname{gph} C$ *and all* $(q,p) \in \mathbb{R}_+ \partial^F \Delta_C(t,x)$ *one has*

$$\sigma(F(t,x), -(q,p)) \geq 0 \ ,$$

for the set-valued mapping $F : \mathbb{R}_+ \times H \rightrightarrows \mathbb{R}_+ \times H$ *defined by* $F(t,x) := \{1\} \times \{-\beta(t)\, \partial^C d_{C(t)}(x)\}$*, where* $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ *is any positive function satisfying* $|\dot{v}(t)| \leq \beta(t)$ *a.e.* $t \geq 0$*. Here* $\Delta_C : \mathbb{R}_+ \times H \to \mathbb{R}_+$ *denotes the distance function to images associated with* C *and defined by* $\Delta_C(t,x) := d_{C(t)}(x)$*.*

Proof: It is sufficient to show the inequality above for only $(q,p) \in \partial^F \Delta_C(t,x)$. Assume the contrary. There exist $(\bar t,\bar x) \in gph\, C$ and $(\bar q,\bar p)(\neq (0,0)) \in \partial^F \Delta_C(\bar t,\bar x)$ such that

$$\sigma(F(\bar t,\bar x), -(\bar q,\bar p)) < 0 \; . \tag{3.1}$$

Fix $\epsilon > 0$. By the definition of the Fréchet subdifferential there exists $\eta > 0$ such that for all $|t-\bar t| \leq \eta$, and all $\|x-\bar x\| \leq \eta$ one has

$$\bar q(t-\bar t) + \langle \bar p, x-\bar x\rangle \;\leq\; d_{C(t)}(x) + \epsilon(|t-\bar t| + \|x-\bar x\|) \; . \tag{3.2}$$

Taking $t = \bar t$ in (3.2) one obtains $\bar p \in \partial^F d_{C(\bar t)}(\bar x)$.

By (1.1) there exists for any $t \in \mathbb{R}_+$, some $x_t \in C(t)$ such that

$$\|x_t - \bar x\| \leq |v(t) - v(\bar t)| \; .$$

Taking now $x = x_t$ in (3.2) for all t sufficiently near to $\bar t$ one gets

$$\begin{aligned} \bar q(t-\bar t) \;&\leq\; \langle -\bar p, x_t - \bar x\rangle + \epsilon(|t-\bar t| + \|x_t - \bar x\|) \\ &\leq\; \|\bar p\|\, |v(t) - v(\bar t)| + \epsilon(|t-\bar t| + |v(t) - v(\bar t)|) \; , \end{aligned}$$

and hence

$$|\bar q| \leq \|\bar p\|\, |\dot v(t)| \leq \|\bar p\|\, \beta(t) \; . \tag{3.3}$$

If $\bar p = 0$, then $\bar q = 0$, which is impossible. Assume that $\bar p \neq 0$, then $\dfrac{\bar p}{\|\bar p\|} \in \partial^F d_{C(\bar t)}(\bar x)$, which ensures that $\left(1, -\beta(\bar t)\dfrac{\bar p}{\|\bar p\|}\right) \in F(\bar t,\bar x)$. Thus by (3.1) one gets $\left\langle \left(1, -\beta(\bar t)\dfrac{\bar p}{\|\bar p\|}\right), -(\bar q,\bar p)\right\rangle < 0$ and hence $\|\bar p\|\beta(\bar t) < |\bar q|$, which contradicts (3.3). This completes the proof. ∎

Now, we are ready to prove our main result of this section.

Theorem 3.1. *Assume that there is a continuous function $\beta : [0,+\infty[\;\to [0,+\infty[$ satisfying $|\dot v(t)| \leq \beta(t)$ a.e. $t \geq 0$ and that the set-valued mapping $G : (t,x) \mapsto \partial^F d_{C(t)}(x)$ is u.s.c. on $\mathbb{R} \times H$. Then there exists a same solution $x(\cdot) : [0,+\infty[\;\to H$ for both problems (P_1) and (P_2), that is, (P_1) and (P_2) have a same nonempty set of solutions.*

Proof: Fix $x_0 \in C(0)$. Put $S := gph\, C$ and $F(t,x) := \{1\} \times \{-\beta(t)\, \partial^F d_{C(t)}(x)\}$. It is well known that $\Pi(S;(t,x))$ is always included in the Fréchet normal cone $N^F(S;(t,x))$ and hence by Proposition 1.1 part i) one gets $\Pi(S;(t,x)) \subset \mathbb{R}_+ \partial^F \Delta_C(t,x)$ for all $(t,x) \in S$. Therefore, Lemma 3.1 yields

$$\sigma(F(t,x), -(q,p)) \geq 0 \ ,$$

for all $(t,x) \in S$ and all $(q,p) \in \Pi(S;(t,x))$. Now, as G is u.s.c. on $\mathbb{R} \times H$ and β is continuous, then F is u.s.c. on $\mathbb{R} \times H$ and hence it satisfies the hypothesis of Proposition 3.1 and then there exists a solution $(s(\cdot), x(\cdot)) : [0, +\infty[\to \mathbb{R} \times H$ of the differential inclusion

$$\begin{cases} (\dot{s}(t), \dot{x}(t)) \in F(s(t), x(t)) \ \text{ a.e. } \ t \geq 0 \\ (s(0), x(0)) = (0, x_0) \in S \\ (s(t), x(t) \in S \qquad \forall t \geq 0 \ . \end{cases}$$

Fix any $t \geq 0$ for which we have $x(t) \in C(s(t))$ and $(\dot{s}(t), \dot{x}(t)) \in F(s(t), x(t)) = \{1\} \times \{-\beta(s(t))\, \partial^F d_{C(s(t))}(x(t))\}$. Then

$$\begin{cases} \dot{s}(t) = 1 \ \text{ and} \\ \dot{x}(t)) \in -\beta(s(t))\, \partial^F d_{C(s(t))}(x(t)) \ . \end{cases}$$

Thus, as $s(0) = 0$ we get $s(t) = t$. Consequently, one concludes that $x(t) \in C(t)$ and $\dot{x}(t) \in -\beta(t)\, \partial^F d_{C(t)}(x(t)) \subset N^F(C(t), x(t))$. This ensures that $x(\cdot)$ is a solution of (P_1). To complete the proof we need by Theorem 2.2 to show that $C(t)$ is normally Fréchet regular for all $t \geq 0$. Indeed, consider any $\bar{t} \geq 0$ and any $\bar{x} \in C(\bar{t})$. Then the u.s.c. of G ensures that $\partial^F d_{C(\bar{t})}(\cdot)$ is closed at $\bar{x}$ in the following sense: for every $x_n \to \bar{x}$ and every $\xi_n \to \bar{\xi}$ with $\xi_n \in \partial^F d_{C(\bar{t})}(x_n)$ one has $\bar{\xi} \in \partial^F d_{C(\bar{t})}(\bar{x})$. Thus, by Theorem 5.1 in [10] and Corollary 3.1 in [11] one concludes that $C(t)$ is normally Fréchet regular. ∎

In order to make clear the importance of this result we give a concrete application. To this end, we need some new results by Bounkhel and Thibault [12] concerning proximally smooth subsets.

Theorem 3.2 ([12]). *Assume that C satisfies (1.1) and $C(t)$ is $r(t)$-proximally smooth for all $t \geq 0$ with $r(t)$ bounded below by a positive number. Then the graph of G is closed and hence G is u.s.c. on $\mathbb{R} \times H$.*

Now another existence result of solutions of proximally smooth case in the finite dimensional setting of both problems (P_1) and (P_2) can be deduced from Theorem 3.1 and Theorem 3.2. We give it in the following theorem.

Theorem 3.3. *Under the hypothesis of Theorem 3.2, there exists a solution of both problems* (P_1) *and* (P_2).

4 – Existence criteria of viable solutions of nonconvex differential inclusions

It is well known that the solution set of the following differential inclusion

$$\begin{cases} \dot{x}(t) \in G(x(t)) \\ x(0) = x_0 \in \mathbb{R}^d \ , \end{cases} \tag{4.1}$$

can be empty when the set-valued mapping G is upper semicontinuous with nonempty nonconvex values. In [14], the authors proved an existence result of (4.1), by assuming that the set-valued mapping G is included in the subdifferential of a convex lower semicontinuous (l.s.c.) function $g : R^d \to \mathbb{R}$. This result has been extended in many ways.

1 – The first one was by [3], where the author replace the convexity assumption of g by its directional regularity in the finite dimensional setting. The infinite dimensional case with the directional regularity assumption on g and some other additional hypothesis has been proved by [4, 3].

2 – The second extension was by [1]. An existence result has been obtained for the following nonconvex differential inclusion

$$\begin{cases} \dot{x}(t) \in G(x(t)) + f(t, x(t)) \ \text{ a.e.} \\ x(0) = x_0 \in \mathbb{R}^d \ , \end{cases} \tag{4.2}$$

under the assumption that G is an upper semicontinuous set-valued mapping with nonempty compact values contained in the subdifferential of a convex lower semicontinuous function, and f is a Caratheodory single-valued mapping.

3 – The third way was to investigate the existence of a viable solution of (4.1) (i.e., a solution $x(\cdot)$ such that $x(t) \in S(t)$, where S is a set-valued mapping). The first existence result of viable solutions of (4.1) has been established by Rossi [36]. Later, Morchadi and Gautier [27] proved an existence result of viable solution of the inclusion (4.2).

4 – The recent extension of (4.1) and (4.2) is given by [39]. The author proved an existence result of viable solutions for the following differential inclusion

$$(DI) \qquad \begin{cases} \dot{x}(t) \in G(x(t)) + F(t, x(t)) \ \text{ a.e.} \\ x(t) \in S \ , \end{cases}$$

when $F : [0,T]\times H \rightrightarrows H$ is a continuous set-valued mapping, $G : H \rightrightarrows H$ is an upper semicontinuous set-valued mapping such that $G(x) \subset \partial g(x)$, where $g : H \to \mathbb{R}$ is a convex continuous function and $S(t) \equiv S$ and the set S is locally compact in H, with $dim\, H < +\infty$.

Our aim in this section is to establish an extension of the existence result of (4.1) that cover all the other extensions given in the finite dimensional setting, like the ones proved by [3, 1, 14, 39]. The infinite dimensional case is extremely long and delicate. It will be provided in [6]. We will prove an existence result of viable solutions of (DI) when F is a continuous set-valued mapping, G is an u.s.c. set-valued mapping, g is a uniformly regular locally Lipschitz function over S (see Definition 4.1), and S is a closed subset in H, with $dim\, H < +\infty$.

In all the sequel, we will assume that H is a finite dimensional space.

We begin by recalling the following lemma proved in [39].

Lemma 4.1 ([39]). *Assume that*

i) *S is nonempty subset in H, $x_0 \in S$, and $K_0 := S \cap (x_0 + \rho\mathbb{B})$ is a compact set for some $\rho > 0$;*

ii) *$P : [0,T]\times H \rightrightarrows H$ is an u.s.c. set-valued mapping with nonempty compact values;*

iii) *For any $(t,x) \in I \times S$ the following tangential condition holds*

$$\liminf_{h\downarrow 0} h^{-1} d_S(x + hP(t,x)) = 0\ . \tag{4.3}$$

Let $a \in\,]0, \min\{T, \frac{\rho}{(M+1)}\}[$, where $M := \sup\{\|P(t,x)\| : (t,x) \in [0,T] \times K_0\}$.

Then for any $\epsilon \in]0,1[$, any set $N' = \{t'_i : t'_0 = 0 < \ldots < t'_{\nu'} = a\}$, and any $u_0 \in F(0,x_0)$, There exist a set $N = \{t_i : t_0 = 0 < \ldots < t_\nu = a\}$, step functions f, z, and x defined on $[0,a]$ such that the following conditions holds for every $i \in \{1,\ldots,\nu\}$:

1) *$\{t'_0,\ldots,t'_{k(i)}\} \subset \{t_0,\ldots,t_i\}$, where $k(i)$ is the unique integer such that $k(i) \in \{0,1,\ldots,\nu'-1\}$ and $t'_{k(i)} \le t_i < t'_{k(i)+1}$;*

2) *$0 < t_{j+1} - t_j \le \alpha$, for all $j \in \{0,\ldots,i-1\}$, where*

$$\alpha := \epsilon \min\{1, t'_1 - t'_0, \ldots, t'_{\nu'} - t'_{\nu'-1}\}\ ;$$

3) *$f(0) = u_0$, $f(t) = f(\theta(t)) \in F(\theta(t), x(\theta(t)))$ on $[0,t_i]$ where $\theta(t) = t_j$ if $t \in [t_j, t_{j+1}[$, for all $j \in \{0,1,\ldots,i-1\}$ and $\theta(t_i) = t_i$;*

4) $z(0) = 0$, $z(t) = z(t_{l+1})$ *if* $t \in]t_l, t_{l+1}]$, $l \leq i-1$ *and* $\|z(t)\| \leq 2\epsilon(M+1)T$;

5) $x(t) = x_0 + \int_0^t f(s)\,ds + z(t)$, *for all* $t \in [0, t_i]$, $x(t_j) = x_j \in K_0$ *and*

$$\|x_j - x_{j'}\| \ \leq \ |t_j - t_{j'}|\,(M+1)\ , \tag{4.4}$$

for $j, j' \in \{0, 1, ..., i\}$.

Now, we introduce our concept of regularity that will be used in this last section of the paper.

Definition 4.1. Let $f : H \to \mathbb{R} \cup \{+\infty\}$ be a l.s.c. function and let $O \subset$ *dom* f be a nonempty open subset. We will say that f is *uniformly regular* over O if there exists a positive number $\beta \geq 0$ such that for all $x \in O$ and for all $\xi \in \partial^P f(x)$ one has

$$\langle \xi, x' - x \rangle \geq f(x') - f(x) + \beta \|x' - x\|^2, \quad \text{for all } \ x' \in O\ .$$

We will say that f is uniformly regular over closed set S if there exists an open set O containing S such that f is uniformly regular over O. □

The class of functions that are uniformly regular over sets is so large. We state here some examples.

1 – Any l.s.c. proper convex function f is uniformly regular over any nonempty subset of its domain with $\beta = 0$.

2 – Any lower-C^2 function f is uniformly regular over any nonempty convex compact subset of its domain. Indeed, let f be a lower-C^2 function over a nonempty convex compact set $S \subset$ *dom* f. By Rockafellar's result (see for instance Theorem 10.33 in [35] or Proposition 3.1 in [22]) there exists a positive real number β such that $g := f + \frac{\beta}{2}\|\cdot\|^2$ is a convex function on S. Using the definition of the subdifferential of convex functions and the fact that $\partial^C f(x) = \partial g(x) - \beta x$ for any $x \in S$, we get the inequality in Definition 4.1 and so f is uniformly regular over S.

One could think to deal with the class of lower-C^2 instead of our class of uniformly regular functions. The inconvenience of the class of lower-C^2 functions is the need of the convexity and the compactness of the set S to satisfy the Definition 4.1 which is the exact property needed in our proofs. However, we can find many functions that are uniformly regular over nonconvex noncompact sets. To give an example we need to recall the following result by Bounkhel and Thibault [12] proved for Hilbert spaces H.

Theorem 4.1 ([12]). *Let S be a nonempty closed subset in H and let $r > 0$. Then S is r-proximally smooth if and only if the following holds*

$$(P_r)\quad \begin{cases} \text{for all } x \in H, \ \text{with } d_S(x) < r, \text{ and all } \xi \in \partial^P d_S(x) \text{ one has} \\ \langle \xi, x' - x \rangle \leq \dfrac{8}{r - d_S(x)} \|x' - x\|^2 + d_S(x') - d_S(x), \\ \ \text{for all } x' \in H \ \text{ with } \ d_S(x') \leq r \ . \end{cases}$$

From Theorem 4.1 one deduces that for any r-proximally smooth set S (not necessarily convex nor compact) the distance function d_S is uniformly regular over $S + (r - r')\mathbb{B} := \{x \in H : \ d_S(x) \leq r - r'\}$ for every $r' \in \,]0, r]$.

Some properties for uniformly regular locally Lipschitz functions over sets that will be needed in the next theorem can be stated in the following proposition. Other important properties for l.s.c. uniformly regular functions are obtained in a forthcoming paper by the author.

Proposition 4.1. *Let $f : H \to \mathbb{R}$ be a locally Lipschitz function and let $\emptyset \neq S \subset dom\, f$. If f is uniformly regular over S, then the following hold:*

- **i)** *the proximal subdifferential of f is closed over S, that is, for every $x_n \to x \in S$ with $x_n \in S$ and every $\xi_n \to \xi$ with $\xi_n \in \partial^P f(x_n)$ one has $\xi \in \partial^P f(x)$.*
- **ii)** *$\partial^C f(x) = \partial^P f(x)$ for all $x \in S$;*
- **iii)** *the proximal subdifferential of f is upper hemicontinuous over S, that is, the support function $x \mapsto \langle v, \partial^P f(x) \rangle$ is u.s.c. over S for every $v \in H$.*

Proof: **i)** Let O be an open set containing S as in Definition 4.1. Let $x_n \to x \in S$ with $x_n \in S$ and let $\xi_n \to \xi$ with $\xi_n \in \partial^P f(x_n)$. Then by Definition 4.1 one has

$$\langle \xi_n, x' - x_n \rangle \geq f(x') - f(x_n) + \beta \|x' - x_n\|^2, \quad \text{for all } \ x' \in O \ .$$

Letting n to $+\infty$ we get

$$\langle \xi, x' - x \rangle \geq f(x') - f(x) + \beta \|x' - x\|^2, \quad \text{for all } \ x' \in O \ .$$

This ensures that $\xi \in \partial^P f(x)$ because O is a neighbourhood of x.

ii) Let x be any point in S. By the part i) of the proposition we get $\partial^P f(x) = \partial^{PL} f(x)$, where $\partial^{PL} f(x)$ denotes the limiting proximal subdifferential of f at x (see for instance [20] for a broad discussion of this subdifferential). Now, as f is Lipschitz at x we get by Theorem 6.1 in [20] $\partial^C f(x) = \overline{co}\, \partial^{PL} f(x) = \overline{co}\, \partial^P f(x) = \partial^P f(x)$ ("$\overline{co}$" means "closed convex hull"). The part iii) is a direct consequence of i) and ii) and so the proof is complete. ∎

Now we are in position to prove our main theorem in this section.

Theorem 4.2. *Let $S \subset H$ and let $g : H \to \mathbb{R}$ be a locally Lipschitz function that is uniformly regular over S with a constant $\beta \geq 0$. Assume that*

- **i)** *S is a nonempty closed subset;*
- **ii)** *$G : H \rightrightarrows H$ is an u.s.c. set-valued mapping with compact values satisfying $G(x) \subset \partial^C g(x)$ for all $x \in S$;*
- **iii)** *$F : [0,T] \times H \rightrightarrows H$ is a continuous set-valued mapping with compact values;*
- **iv)** *For any $(t,x) \in I \times S$ the following tangential condition holds*

$$\liminf_{h \downarrow 0} h^{-1}\, d_S(x + h(G(x) + F(t,x))) \;=\; 0\;. \tag{4.5}$$

Then, for any $x_0 \in S$ there exists $a \in \,]0,T[$ such that the differential inclusion (DI) has a viable solution on $[0,a]$.

Proof: Let $L > 0$ and ρ be two positives scalars such that g is L-Lipschitz over $x_0 + \rho\mathbb{B}$. Put $K_0 := S \cap (x_0 + \rho\mathbb{B})$ is a compact set in H. Let M and a be two positives scalars such that $\|F(t,x)\| + \|G(x)\| \leq M$, for all $(t,x) \in [0,T] \times K_0$ and $a \in \,]0, \min\{T, \frac{\rho}{M+1}\}[$. Let $N_0 = \{0,a\}$ and $\epsilon_m = \frac{1}{2^m}$, for $m = 1,2,\cdots$.

First, the uniform continuity of F on the compact K_0 ensures the existence of $\delta_m > 0$ such that

$$\|(t,x) - (t',x')\| \leq (M+2)\,\delta_m \implies \mathcal{H}(F(t,x), F(t',x')) \leq \epsilon_m\;, \tag{4.6}$$

for $t,t' \in [0,a]$, $x,x' \in K_0$, where $\|(t,x)\| = |t| + \|x\|$. Here $\mathcal{H}(A,B)$ stands for the Hausdorff distance between A and B define by

$$\mathcal{H}(A,B) := \max\Big\{\sup_{a\in A} d_B(a),\, \sup_{b\in B} d_A(b)\Big\}\;.$$

Now, applying Lemma 4.1 for the set-valued mapping $P := F + G$, the scalar ϵ_m, $m = 1,2,\cdots$, the set $N_0 = \{0,a\}$, and the set S, one obtains for every

$m = 1, 2, \cdots$ the existence of a set $N_m = \{t_i^m : t_0^m = 0 < \cdots < t_{\nu_m}^m = a\}$, step functions $y_m(\cdot)$, $f_m(\cdot)$, $z_m(\cdot)$, and $x_m(\cdot)$ defined on $[0, a]$ with the following properties:

i) $N_m \subset N_{m+1}$, $m = 0, 1, \cdots$;

ii) $0 < t_{i+1}^m - t_i^m \leq \alpha_m$, for all $i \in \{0, \cdots, \nu_m - 1\}$, where

$$\alpha_m := \epsilon_m \min\{1, \delta_m, t_1^{m-1} - t_0^{m-1}, ..., t_{\nu_{m-1}}^{m-1} - t_{\nu_{m-1}-1}^{m-1}\} ;$$

iii) $f_m(t) = f_m(\theta_m(t)) \in F(\theta_m(t), x_m(\theta_m(t)))$ and $y_m(t) = y_m(\theta_m(t)) \in G(x_m(\theta_m(t)))$ on $[0, a]$ where $\theta_m(t) = t_i^m$ if $t \in [t_i^m, t_{i+1}^m[$, for all $i \in \{0, 1, \cdots, \nu_m - 1\}$ and $\theta_m(a) = a$;

iv) $z_m(0) = 0$, $z_m(t) = z_m(t_{i+1})$ if $t \in]t_i, t_{i+1}]$, $0 \leq i \leq \nu_m - 1$ and

$$\|z_m(t)\| \leq 2\,\epsilon_m (M + 1)T \; ; \tag{4.7}$$

v) $x_m(t) = x_0 + \int_0^t (y_m(s) + f_m(s))ds + z_m(t)$ and $x_m(\theta_m(t)) \in K_0$, for all $t \in [0, a]$, and for $i, j \in \{0, 1, \cdots, \nu_m\}$

$$\|x_m(t_i^m) - x_m(t_j^m)\| \leq |t_i^m - t_j^m| \, (M + 1) \; . \tag{4.8}$$

Observe that (4.8) ensures that for $i, j \in \{0, 1, \cdots, \nu_m\}$

$$\|(t_i^m, x_m(t_i^m)) - (t_j^m, x_m(t_j^m))\| \leq |t_i^m - t_j^m| \, (M + 2) \; . \tag{4.9}$$

We will prove that the sequence $x_m(\cdot)$ converges to a viable solution of (DI). First, we note that the sequence f_m can be constructed with the relative compactness property in the space of bounded functions. We don't give the proof of this part here. It can be found in [39, 40, 24]. Therefore, without loss of generality we can suppose that there is a bounded function f such that

$$\lim_{m \to \infty} \sup_{t \in [0,a]} \|f_m(t) - f(t)\| = 0 \; . \tag{4.10}$$

Now, we use our characterizations of the uniform regularity proved in Proposition 4.1 and some techniques of [12, 1, 14] to prove that the approximate solutions $x_m(\cdot)$ converges to a function that is a viable solution of (DI).

Put $q_m(t) = x_0 + \int_0^t (y_m(s) + f_m(s))ds$. By the property iv), one has $\|\dot{z}_m(t)\| = 0$ a.e. on $[0, a]$. Then $\|\dot{q}_m(t)\| = \|\dot{x}_m(t)\| \leq M$ a.e. on $[0, a]$ and the sequence

q_m is equicontinuous and the sequence of their derivatives $\dot{q}_m$ is equibounded. Hence, a subsequence of q_m may be extracted (without loss of generality we may suppose that this subsequence is q_m) that converges in the sup-norm topology to an absolutely continuous mapping $x : [0, a] \to H$ and such that the sequence of their derivatives $\dot{q}_m$ converges to $\dot{x}(\cdot)$ in the weak topology of $L^2([0, a], H)$. Since $\|q_m(t) - x_m(t)\| = \|z_m(t)\|$ and $\|\dot{z}_m(t)\| = 0$ a.e. on $[0, a]$ one gets by (4.7)

$$\begin{cases} \lim\limits_{m\to\infty} \max\limits_{t\in[0,a]} \|x_m(t) - x(t)\| = 0 \\ \dot{x}_m(\cdot) \rightharpoonup \dot{x}(\cdot) \ \text{ in the weak topology of } L^2([0, a], H) \ . \end{cases} \tag{4.11}$$

Recall now that the sequence f_m converges pointwisely a.e. on $[0, a]$ to f. Then, the continuity of F and the closedness of $F(t, x(t))$ entail $f(t) \in F(t, x(t))$. Further, by the properties of the sequence x_m and the closedness of K_0, we get $x(t) \in K_0 \subset S$.

Put $y(t) = -f(t) + \dot{x}(t)$. It remains to prove that $y(t) \in G(x(t))$ a.e. $[0, a]$. By construction and the hypothesis on G and g we have $y_m(t) = \dot{x}_m(t) - f_m(t)$ and

$$y_m(t) \in G(x_m(\theta_m(t))) \subset \partial^C g(x_m(\theta_m(t))) = \partial^P g(x_m(\theta_m(t))) \ , \tag{4.12}$$

for a.e. on $[0, a]$, where the last equality follows from the uniform regularity of g over S and the part ii) in Proposition 4.1.

We can thus apply Castaing techniques (see for example [16]). The weak convergence (by (4.11)) in $L^2([0, a], H)$ of $\dot{x}_m(\cdot)$ to $\dot{x}(\cdot)$ and Mazur's Lemma entail

$$\dot{x}(t) \in \bigcap_m \overline{co}\{\dot{x}_k(t) : k \geq m\}, \ \text{ for a.e. on } [0, a] \ .$$

Fix any such t and consider any $\xi \in H$. Then, the last relation above yields

$$\langle \xi, \dot{x}(t) \rangle \leq \inf_m \sup_{k\geq m} \langle \xi, \dot{x}_m(t) \rangle$$

and hence according to (4.12)

$$\langle \xi, \dot{x}(t) \rangle \leq \limsup_m \sigma(\xi, \partial^P g(x_m(\theta_m(t))) + f_m(t)) \leq \sigma(\xi, \partial^P g(x(t)) + f(t)) \ ,$$

where the second inequality follows from the upper hemicontinuity of the proximal subdifferential of uniformly regular functions (see part ii) in Proposition 4.1) and the convergence pointwisely a.e. on $[0, a]$ of f_m to f, and the fact that

$x_m(\theta_m(t)) \to x(t)$ in K_0 a.e. on $[0,a]$. Thus, by the convexity and the closedness of the proximal subdifferential of uniformly regular functions (part ii) in Proposition 4.1) we obtain

$$y(t) = \dot{x}(t) - f(t) \in \partial^P g(x(t)) \ . \tag{4.13}$$

To complete the proof we need to show that $y(t) \in G(x(t))$.

As $x(\cdot)$ is an absolutely continuous mapping and g is a uniformly regular locally Lipschitz function over S (hence directionally regular over S (see [10])), one gets by Theorem 2 in Valadier [41, 42] (see also [8, 9]) for a.e. on $[0,a]$

$$\frac{d}{dt}(g \circ x)(t) = \langle \partial^P g(x(t)), \dot{x}(t)\rangle = \langle \dot{x}(t) - f(t), \dot{x}(t)\rangle = \|\dot{x}(t)\|^2 - \langle f(t), \dot{x}(t)\rangle \ .$$

Consequently,

$$g(x(a)) - g(x_0) = \int_0^a \|\dot{x}(s)\|^2 ds - \int_0^a \langle f(s), \dot{x}(s)\rangle ds \ . \tag{4.14}$$

On the other hand, we have by construction $\dot{x}_m(t) = y_i^m + f_i^m$ with $y_i^m \in G(x_i^m) \subset \partial^C g(x_i^m) = \partial^P g(x_i^m)$ for $t \in]t_i^m, t_{i+1}[$, $i = 0, \cdots, \nu_m - 1$. Then, by Definition 4.1 one has

$$\begin{aligned}
g(x_{i+1}^m) - g(x_i^m) &\geq \langle y_i^m, x_{i+1}^m - x_i^m\rangle - \beta\|x_{i+1}^m - x_i^m\|^2 \\
&= \left\langle \dot{x}_m(t) - f_m(t), \int_{t_i^m}^{t_{i+1}^m} \dot{x}_m(s)\, ds \right\rangle - \beta\, \|x_{i+1}^m - x_i^m\|^2 \\
&\geq \int_{t_i^m}^{t_{i+1}^m} \|\dot{x}_m(s)\|^2\, ds - \int_{t_i^m}^{t_{i+1}^m} \langle \dot{x}_m(s), f_m(s)\rangle\, ds \\
&\quad - \beta\, (M+1)^2\, (t_{i+1}^m - t_i^m)^2 \\
&\geq \int_{t_i^m}^{t_{i+1}^m} \|\dot{x}_m(s)\|^2\, ds - \int_{t_i^m}^{t_{i+1}^m} \langle \dot{x}_m(s), f_m(s)\rangle\, ds \\
&\quad - \beta\, (M+1)^2\, \epsilon_m (t_{i+1}^m - t_i^m) \ .
\end{aligned}$$

By adding, we obtain

$$g(x_m(a)) - g(x_0) \geq \int_0^a \|\dot{x}_m(s)\|^2 ds - \int_0^a \langle \dot{x}_m(s), f_m(s)\rangle ds - \epsilon_m (M+1)^2 a \,. \tag{4.15}$$

According to (4.10) and (4.11) one gets

$$\lim_m \int_0^a \langle \dot{x}_m(s), f_m(s)\rangle\, ds = \int_0^a \langle \dot{x}(s), f(s)\rangle\, ds \ .$$

Passing to the limit superior for $m \to \infty$ in (4.15) and the continuity of g yield

$$g(x(a)) - g(x_0) \ \geq \ \limsup_m \int_0^a \|\dot{x}_m(s)\|^2 \, ds \ - \int_0^a \langle \dot{x}(s), f(s) \rangle \, ds \ ,$$

and hence a comparison with (4.14) gives

$$\int_0^a \|\dot{x}(s)\|^2 ds \geq \limsup_m \int_0^a \|\dot{x}_m(s)\|^2 ds \ ,$$

that is

$$\|\dot{x}\|^2_{L^2([0,a],H)} \geq \limsup_m \|\dot{x}_m\|^2_{L^2([0,a],H)} \ .$$

On the other hand the weak lower semicontinuity of the norm ensures

$$\|\dot{x}\|_{L^2([0,a],H)} \leq \liminf_m \|\dot{x}\|_{L^2([0,a],H)} \ .$$

Consequently, we get

$$\|\dot{x}\|_{L^2([0,a],H)} = \lim_m \|\dot{x}_m\|_{L^2([0,a],H)} \ .$$

This means that the sequence $\dot{x}_m(\cdot)$ converges to $\dot{x}(\cdot)$ strongly in $L^2([0,a],H)$. Hence there exists a subsequence of $\dot{x}_m(\cdot)$ still denoted $\dot{x}_m(\cdot)$ converges pointwisely a.e. on $[0,a]$ to $\dot{x}(\cdot)$. Finally, by the construction, one has $(x_m(t), \dot{x}_m(t) - f_m(t)) \in gph\, G$ a.e. on $[0,a]$ and so the closedness of the graph ensures that $(x(t), \dot{x}(t) - f(t)) \in gph\, G$ a.e. on $[0,a]$. This completes the proof of the theorem. ∎

Remark 4.1.

1 – An inspection of our proof in Theorem 4.2 shows that the uniformity of the constant β was needed only over the set K_0 and so it was not necessary over all the set S. Indeed, it suffices to take the uniform regularity of g locally over S, that is, for every point $\bar{x} \in S$ there exist $\beta \geq 0$ and a neighbourhood V of x_0 such that g is uniformly regular over $S \cap V$.

2 – As we can see from the proof of Theorem 4.2, the assumption needed on the set S is the local compactness which holds in the finite dimensional setting for nonempty closed sets.

3 – As observed by the author in [39], under the assumptions i)–iv) of Theorem 4.2, if we assume that $F([0,T] \times S) + G(S)$ is bounded, then for any $a \in\,]0,T[$, the differential inclusion (DI) has a viable solution on $[0,a]$. □

We close the paper with two corollaries of our main result in Theorem 4.2.

Corollary 4.1. *Let $K \subset H$ be a nonempty proximally smooth closed subset and $F : [0,T] \times H \rightrightarrows H$ be a continuous set-valued mapping with compact values. Then, for any $x_0 \in K$ there exists $a \in]0,T[$ such that the following differential inclusion*

$$\begin{cases} \dot{x}(t) \in -\partial^C d_K(x(t)) + F(t,x(t)) & \text{a.e. on } [0,a] \\ x(0) = x_0 \in K \ , \end{cases}$$

has at least one absolutely continuous solution on $[0,a]$.

Proof: Theorem 4.1 shows that the function $g := d_K$ is uniformly regular over K and so it is uniformly regular over some neighbourhood V of $x_0 \in K$. Thus, by Remark 4.1 part 1, we apply Theorem 4.2 with $S = H$ (hence the tangential condition (4.5) is satisfied), $K_0 := V \cap S = V$, and the set-valued mapping $G := \partial^C d_K$ which satisfies the hypothesis of Theorem 4.2. ∎

Our second corollary concerns the following differential inclusion

$$\begin{cases} \dot{x}(t) \in -N^C(S;x(t)) + F(t,x(t)) \ \text{ a.e.} \\ x(t) \in S, \ \text{ for all } t, \ \text{ and } x(0) = x_0 \in S \ . \end{cases} \tag{4.16}$$

First, we recall that this type of differential inclusion has been introduced by Henry [25] for studying some economic problems. In the case when F is an u.s.c set-valued mapping and is autonomous (that is F is independent of t), he proved an existence result of (4.16) under the convexity assumption on the set S and on the images of the set-valued mapping F. In the autonomous case, this result has been extended by Cornet [23] by assuming the tangential regularity assumption on the set S and the convexity on the images of F with the u.s.c of F. Recently, Thibault in [38], proved in the nonautonomous case, an existence result of (4.16) for any closed subset S (without any assumption on S), which also required the convexity of the images of F and the u.s.c. of F. The question arises whether we can drop the assumption of convexity of the images of F. Our corollary here establishes an existence result in this vein, but we will pay a heavy price for the absence of the convexity. We will assume that F is continuous, and above all, that the following tangential condition holds.

$$\liminf_{h \downarrow 0} h^{-1} d_S(x + h(\partial^C d_S(x) + F(t,x))) = 0 \ , \tag{4.17}$$

for any $(t,x) \in I \times S$.

Corollary 4.2. *Assume that*

i) $F : [0,T] \times H \rightrightarrows H$ *is a continuous set-valued mapping with compact values;*

ii) S *is a nonempty proximally smooth closed subset in* H*;*

iii) *For any* $(t,x) \in I \times S$ *the tangential condition (4.17) holds. Then, for any* $x_0 \in S$*, there exists* $a \in]0,T[$ *such that the differential inclusion* (4.16) *has at lease one absolutely continuous solution on* $[0,a]$.

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Messaoud Bounkhel,
Centro de Modelamiento Matemático, UMR, CNRS-UCHILE,
Blanco Encalada 2120, 7^0 Piso, C.C.: 170-3, Santiago – CHILE
E-mail: bounkhel@dim.uchile.cl