PORTUGALIAE MATHEMATICA
Vol. 59 Fasc. 4 – 2002
Nova Série

SOME STABILITY PROPERTIES FOR MINIMAL SOLUTIONS OF $-\Delta u = \lambda\, g(u)$

Thierry Cazenave, Miguel Escobedo and M. Assunta Pozio

Presented by J.P. Dias

Abstract: We study the stability of the branch of minimal solutions $(u_\lambda)_{0<\lambda<\lambda^*}$ of $-\Delta u = \lambda\, g(u)$ for a nonlinearity g which is neither concave nor convex. We show that it is related to the regularity of the map $\lambda \mapsto u_\lambda$. We then show that in dimensions $N=1$ and $N=2$, discontinuities in the branch of minimal solutions can be produced by arbitrarilly small perturbations of the nonlinearity g. In dimensions $N \geq 3$ the perturbation has to be large enough. We also study in detail a specific one-dimensional example.

1 – Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded, smooth domain. Consider a C^1, positive, increasing function $g\colon [0,\infty) \to (0,\infty)$. It is well-known that there exists $\lambda^* \in (0,\infty]$ such that for $0<\lambda<\lambda^*$ there is a minimal solution of

$$\begin{cases} -\Delta u = \lambda\, g(u) & \text{in } \Omega\,, \\ u = 0 & \text{in } \partial\Omega\,, \end{cases} \tag{1.1}$$

and for $\lambda > \lambda^*$ there is no solution (we consider only positive, smooth solutions). The branch $(u_\lambda)_{0<\lambda<\lambda^*}$ is increasing. (See e.g. Amann [1], Theorem 21.1.) Moreover, $\lambda_1(-\Delta - \lambda g'(u_\lambda)) \geq 0$ for all $0<\lambda<\lambda^*$. (Indeed, if $\lambda_1(-\Delta - \lambda g'(u_\lambda)) < 0$, and if φ_1 is a corresponding positive eigenvector, then $u_\lambda - \varepsilon\, \varphi_1$ is a supersolution for $\varepsilon > 0$ sufficiently small. Since 0 is a subsolution, there exists a solution below the minimal one, which is absurd.)

Received: April 5, 2001.
AMS Subject Classification: 35J65, 35P30, 35B30.
Keywords and Phrases: nonlinear elliptic problem; branch of minimal solutions; stability; first eigenvalue.

It is also well-known that if g is convex, or if it is concave, then the minimal branch is stable in the sense that $\lambda_1(-\Delta - \lambda g'(u_\lambda)) > 0$ for all $0 < \lambda < \lambda^*$. We sketch the proof for completeness. Assume by contradiction that $\lambda_1(-\Delta - \lambda g'(u_\lambda)) = 0$ and fix $\mu \in (0, \lambda^*)$. We have

$$-\Delta\varphi_1 = \lambda\, g'(u_\lambda)\, \varphi_1 \ , \tag{1.2}$$

$$-\Delta u_\lambda = \lambda\, g(u_\lambda) \ , \tag{1.3}$$

$$-\Delta u_\mu = \mu\, g(u_\mu) \ . \tag{1.4}$$

Multiply (1.2) by u_λ, (1.3) by φ_1 and make the difference. Next, multiply (1.2) by u_μ, (1.4) by φ_1 and make the difference. Forming the difference of the two relations thereby obtained, we see that

$$\int_\Omega \Big(g(u_\mu) - g(u_\lambda) - (u_\mu - u_\lambda)\, g'(u_\lambda) \Big)\, \varphi_1 \ = \ \frac{\lambda - \mu}{\lambda} \int_\Omega g(u_\mu)\, \varphi_1 \ . \tag{1.5}$$

If g is convex, the left-hand side of (1.5) is nonnegative and we get a contradiction by choosing $\mu > \lambda$; if g is concave, the left-hand side of (1.5) is nonpositive and we get a contradiction by choosing $\mu < \lambda$.

The property $\lambda_1(-\Delta - \lambda g'(u_\lambda)) > 0$ implies in particular that the solutions $(u_\lambda)_{0<\lambda<\lambda^*}$ on the minimal branch are also stable for the evolution problem

$$\begin{cases} u_t - \Delta u = \lambda\, g(u) & \text{for } \ t > 0, \ x \in \Omega \ , \\ u(x,t) = 0 & \text{for } \ t > 0, \ x \in \partial\Omega \ , \end{cases} \tag{1.6}$$

in the following sense: for every $0 < \lambda < \lambda^*$, there exists $\varepsilon > 0$ such that if $\varphi \in L^\infty(\Omega)$ satisfies $0 \le \varphi \le u_\lambda + \varepsilon$, then the unique positive solution of (1.6) with the initial condition $u(0) = \varphi$ is global and satisfies $u(t) \to u_\lambda$ as $t \to \infty$.

We are interested in investigating under what conditions on the nonlinearity g, the stability property $\lambda_1(-\Delta - \lambda g'(u_\lambda)) > 0$ holds or fails along the minimal branch of solutions of (1.1).

Our first result is a general criteria, established in Section 2. It says that the property $\lambda_1(-\Delta - \lambda g'(u_\lambda)) > 0$ is equivalent to the property that the mapping $\lambda \mapsto u_\lambda$ is C^1. More precisely, we have the following result.

Theorem 1.1. *Let g be a C^2, positive, increasing function $[0, \infty) \to (0, \infty)$ and let $(u_\lambda)_{0<\lambda<\lambda^*}$ be the maximal branch of minimal, positive solutions of* (1.1). *Given $\lambda \in (0, \lambda^*)$, the following properties are equivalent.*

(i) $\lambda_1(-\Delta - \lambda g'(u_\lambda)) > 0$.

(ii) *The mapping* $\mu \mapsto u_\mu$ *is* C^1 *from a neighborhood of* λ *to* $L^\infty(\Omega)$.

(iii) $\displaystyle\int_\Omega |u_\lambda - u_\mu|^2 \, \mathrm{d}_\Omega(x)\, dx = o(|\lambda - \mu|)$, *as* $\mu \to \lambda$, *where* d_Ω *is the distance to* $\partial\Omega$.

Our second observation is that it is quite easy to introduce discontinuities in the branch of minimal solutions by modifying the nonlinearity g. The following results are established in Section 3.

Theorem 1.2. *Suppose* $N = 1$ *or* $N = 2$. *Let* g *be a* C^1, *positive, increasing function* $[0, \infty) \to (0, \infty)$ *and let* $(u_\lambda)_{0<\lambda<\lambda^*}$ *be the maximal branch of minimal, positive solutions of* (1.1). *Let* $\underline{\lambda} \in (0, \lambda^*)$ *and set* $M = \|u_{\underline{\lambda}}\|_{L^\infty}$. *Given* $\varepsilon > 0$, *there exists a* C^1, *increasing function* $\widetilde{g} : [0, \infty) \to (0, \infty)$, *with the following properties.*

(i) *The branch of minimal solutions* $\widetilde{u}_\lambda$ *of* (1.1) *associated with* $\widetilde{g}$ *is defined on the maximal interval* $(0, \widetilde{\lambda}^*)$ *with* $\widetilde{\lambda}^* > \underline{\lambda}$, *and* $\widetilde{u}_\lambda = u_\lambda$ *for* $0 < \lambda \le \underline{\lambda}$.

(ii) $g - \widetilde{g}$ *is supported in* $[M, M + \varepsilon]$ *and* $\|g - \widetilde{g}\|_{L^\infty} \le \varepsilon$.

(iii) *The map* $\lambda \mapsto \widetilde{u}_\lambda$ *has a discontinuity in* $[\underline{\lambda}, \underline{\lambda} + \varepsilon]$.

Theorem 1.3. *Suppose* $N \ge 3$. *Let* g, $\underline{\lambda}$ *and* M *be as in Theorem* 1.2. *Suppose further that* $g(u) \to \infty$ *as* $u \to \infty$. *Given* $\varepsilon > 0$, *there exists a* C^1, *increasing function* $\widetilde{g}$: $[0, \infty) \to (0, \infty)$, *with the following properties.*

(i) *The branch of minimal solutions* $\widetilde{u}_\lambda$ *of* (1.1) *associated with* $\widetilde{g}$ *is defined on the maximal interval* $(0, \widetilde{\lambda}^*)$ *with* $\widetilde{\lambda}^* > \underline{\lambda}$, *and* $\widetilde{u}_\lambda = u_\lambda$ *for* $0 < \lambda \le \underline{\lambda}$.

(ii) $g - \widetilde{g}$ *is compactly supported in* $[M, \infty)$.

(iiii) *The map* $\lambda \mapsto \widetilde{u}_\lambda$ *has a discontinuity in* $[\underline{\lambda}, \underline{\lambda} + \varepsilon]$.

We observe that in Theorem 1.2 (i.e. if $N \le 2$), the perturbation of g can be arbitrarily small, while in Theorem 1.3 the perturbation may be large. This is motivated by the following examples.

A one-dimensional example. Consider the equation

$$\begin{cases} -u'' = \lambda g(u) & \text{in } (0,1) , \\ u(0) = u(1) = 0 . \end{cases} \tag{1.7}$$

In the elementary case $g(u) = a > 0$, the equation (1.7) has the unique nonnegative solution $u_\lambda(x) = \lambda\, a\, x(1 - x)/2$ for every $\lambda > 0$.

Given $0 < a < b$ and $\alpha > 0$, let now g be defined by

$$g(u) = \begin{cases} a & \text{if } 0 \le u \le \alpha \ , \\ b & \text{if } u > \alpha \ . \end{cases} \tag{1.8}$$

Even though the nonlinearity g is not continuous, it displays some interesting properties. One can calculate all solutions of (1.7). They may be of two types: those for which $\max u \le \alpha$, and those for which $\max u > \alpha$. The first ones are obtained by solving the equation $-u'' = \lambda\, a$ and requiring $\max u \le \alpha$. They exist if and only if $\lambda \le 8\,\alpha/a$ and they are given by $u(x) = \lambda\, a\, x(1-x)/2$. The second ones exist whenever there exists $0 < \underline{x} < 1/2$ such that the C^1 function u satisfies $u(0) = u(1) = 0$, $-u'' = \lambda a$ on $(0, \underline{x}) \cup (1-\underline{x}, 1)$ and $-u'' = \lambda b$ on $(\underline{x}, 1-\underline{x})$. It is not difficult to see that this amounts in finding $\underline{x} \in (0, 1/2)$ such that $(2b-a)\,\underline{x}^2 - b\underline{x} + 2\alpha/\lambda = 0$. Therefore, we can draw the following conclusions. If $0 < \lambda < 8\alpha(2b-a)/b^2$, then there is one positive solution of (1.7), which is of the first type. If $\lambda = 8\alpha(2b-a)/b^2$, then there are two positive solutions of (1.7), one of the first type and one of the second type. If $8\alpha(2b-a)/b^2 < \lambda < 8\alpha/a$, then there are three positive solutions of (1.7), one of the first type and two of the second type. If $\lambda = 8\alpha/a$, then there are two positive solutions of (1.7), one of the first type and one of the second type. If $\lambda > 8\alpha/a$, then there is one positive solution of (1.7), of the second type. In other words, the branch of solutions is S-shaped. It is easy to verify that, whenever there are multiple solutions, they are ordered. Moreover, there is a discontinuity of the branch of minimal solutions at $\lambda = 8\alpha/a$. Indeed, at that particular value of λ the minimal solution is $\underline{u}(x) = 4\alpha\, x(1-x)$, while the second one is

$$\overline{u}(x) = \begin{cases} 4\alpha\, x\Big(\dfrac{2b^2 - 2ab + a^2}{a(2b-a)} - x\Big) & \text{for } \ 0 < x < \dfrac{a}{2(2b-a)} \ , \\ \alpha - \dfrac{\alpha\, b(4b-3a)}{(2b-a)^2} + \dfrac{4\alpha\, b}{a} x(1-x) & \text{for } \ \dfrac{a}{2(2b-a)} < x < 1/2 \ . \end{cases}$$

The branch of minimal solutions u_λ is continuous for $\lambda < 8\alpha/a$ and converges to $\underline{u}$ as $\lambda \uparrow 8\alpha/a$; it is continuous for $\lambda > 8\alpha/a$ and converges to $\overline{u}$ as $\lambda \downarrow 8\alpha/a$.

A three-dimensional example. We now consider the problem

$$\begin{cases} -\Delta u = \lambda g(u) & \text{in } \ \Omega \ , \\ u_{|\partial\Omega} = 0 \ , \end{cases} \tag{1.9}$$

where Ω is the unit ball of $\mathbb{R}^3$ and $\lambda > 0$. We consider spherically symmetric solutions, so that, with the usual change of variables $v(r) = r\, u(r)$, the equation (1.9) reduces to

$$\begin{cases} -v'' = \lambda\, r\, g\left(\dfrac{v}{r}\right) & \text{for } 0 < r < 1\,, \\ v(0) = v(1) = 0\,. \end{cases} \tag{1.10}$$

As in the previous example, we consider g defined by (1.8) with $0 < a < b$ and $\alpha > 0$. We easily see that any solution of (1.10) is positive and concave on $(0,1)$ and that $v(r)/r$ is decreasing. Therefore, the solutions of (1.10) are of one of two types. Either $v(r) < \alpha r$ and $-v'' = \lambda a r$ for all $0 < r < 1$; or else, there exists $0 < \underline{r} < 1$ such that $v(r) > \alpha r$ and $-v'' = \lambda b r$ for $0 < r < \underline{r}$ and $v(r) < \alpha r$ and $-v'' = \lambda a r$ for $\underline{r} < r < 1$. Solutions of the first type exist if and only if $0 < \lambda \leq 6\alpha/a$ and they are given by $v(r) = a\lambda r(1-r^2)/6$. Second type solutions exist whenever there exists a solution $\underline{r} \in (0,1)$ of the equation

$$\alpha - \frac{\lambda a}{6} - \lambda\left(\frac{b}{3} - \frac{a}{2}\right)\underline{r}^2 + \frac{\lambda(b-a)}{3}\,\underline{r}^3 = 0\,.$$

Analysing the above equation, we see that the situation depends on the jump in the nonlinearity. If $2b \leq 3a$, i.e. if the jump is not too large, then for all $\lambda > 0$ there is only one radial solution of (1.9). The solution is of the first type if $\lambda \leq 6\alpha/a$ and of the second type otherwise. It is not too difficult to check that the branch of solutions is continuous. If $2b > 3a$, i.e. if the jump is sufficiently large, then the situation is similar to the one-dimensional case. For $\lambda < \frac{6\,\alpha}{a}\left(1 + \frac{(2b-3a)^3}{27\,a(b-a)^2}\right)^{-1}$, there is one radial solution of (1.9), which is of the first type. For $\lambda = \frac{6\alpha}{a}\left(1 + \frac{(2b-3a)^3}{27\,a(b-a)^2}\right)^{-1}$, there are two radial solutions of (1.9), one of the first type, one of the second type, the second one being larger. For $\frac{6\alpha}{a}\left(1 + \frac{(2b-3a)^3}{27\,a(b-a)^2}\right)^{-1} < \lambda < 6\alpha/a$, there are three radial solutions of (1.9), one of the first type, the other two of the second type. For $\lambda = 6\alpha/a$, there are two radial solutions of (1.9), one of the first type, the other of the second type. If $\lambda > 6\alpha/a$, there is one radial solution of (1.9), which is of the second type. Solutions are ordered, the first-type solution being the smallest. The branch of minimal solutions is discontinuous at $\lambda = 6\alpha/a$. As opposed to the one-dimensional case, the jump in the nonlinearity must be large enough to produce multiple solutions and discontinuity of the minimal branch.

On the other hand, one may look for a necessary condition on the nonlinearity g in order that $\lambda_1(-\Delta - \lambda g'(u_\lambda)) > 0$. That problem seems to be more delicate and we do not have a general answer. In Section 4 we consider the example

$$g(u) = u^p + u^q\,, \qquad 0 < q < 1 < p\,,$$

which is neither convex nor concave. Nevertheless we show that in the case $N=1$, the minimal branch of solutions satisfies the stability condition $\lambda_1(-\Delta-\lambda g'(u_\lambda))>0$. (See Proposition 4.1 and Corollary 4.2.) Whether or not this is true in higher dimensions is an open question.

Similar problems have previously been considered in the litterature. The most closely related reference is probably the work by K.J. Brown, M.M.A. Ibrahim and R. Shivaji [3]. These authors are interested in determining whether the branch of solutions is “S-shaped”. They consider general elliptic operators but their results are less precise than ours in particular in dimensions $N\geq 2$. Previous examples of discontinuous minimal branches may be found in the work by M.G. Crandall and P.H. Rabinowitz [5].

2 – Proof of Theorem 1.1

We proceed in three steps.

STEP 1. (ii)⇒(iii). This is immediate.

STEP 2. (iii)⇒(i). We already know that $\lambda_1(-\Delta-\lambda g'(u_\lambda))\geq 0$, so we assume by contradiction that $\lambda_1(-\Delta-\lambda g'(u_\lambda))=0$. Fix $\lambda<\overline{\lambda}<\lambda^*$ and let $M=\|u_{\overline{\lambda}}\|_{L^\infty}$. For $0\leq x,y\leq M$, we have $|g(x)-g(y)-(x-y)\,g'(y)|\leq C|x-y|^2$ since g is C^2. Therefore, we deduce from (1.5) that for all $\lambda<\mu<\overline{\lambda}$

$$|\lambda-\mu|\,g(0)\;\leq\;|\lambda-\mu|\int_\Omega g(u_\mu)\,\varphi_1\;\leq\;C\,\lambda\int_\Omega|u_\lambda-u_\mu|^2\,\varphi_1\;, \tag{2.1}$$

where φ_1 is the first eigenfunction of $-\Delta-\lambda g'(u_\lambda)$ normalized by $\int_\Omega\varphi_1=1$. Since $\varphi_1\leq C\,\mathrm{d}_\Omega$, we deduce from (iii) and (2.1) that $|\lambda-\mu|\,g(0)=o(|\lambda-\mu|)$. Letting $\mu\downarrow\lambda$, we obtain that $g(0)=0$, which is absurd.

STEP 3. (i)⇒(ii). We first show that

$$\|u_\mu-u_\lambda\|_{L^\infty}\underset{\mu\to\lambda}{\longrightarrow}0\;. \tag{2.2}$$

Note that the mapping $\mu\to u_\mu$ is increasing on $(0,\lambda^*)$. More precisely, if $\mu>\nu$ then $u_\mu\geq u_\nu$ and $u_\mu\not\equiv u_\nu$, so that by the strong maximum principle $u_\mu\geq u_\nu+\varepsilon\,\mathrm{d}_\Omega$ for some $\varepsilon>0$. Set

$$\underline{u}=\lim_{\mu\uparrow\lambda}u_\mu\quad\text{ and }\quad\overline{u}=\lim_{\mu\downarrow\lambda}u_\mu\;.$$

It is clear that $\underline{u} \le u_\lambda$ and that $\underline{u}$ is a solution of (1.1); and so, $\underline{u} = u_\lambda$. We claim that $\overline{u} = u_\lambda$. Indeed, since (i) holds, there exists a unique solution of

$$\begin{cases} -\Delta\psi - \lambda\, g'(u_\lambda)\, \psi = 1 & \text{in } \Omega\ , \\ \psi = 0 & \text{on } \partial\Omega\ . \end{cases}$$

We set $v = u_\lambda + \delta\psi$ for $\delta > 0$, so that

$$-\Delta v - (\lambda + \theta)\, g(v) \ = \ \delta - \theta\, g(v) - \lambda\Big[g(v) - g(u_\lambda) - (v - u_\lambda)\, g'(u_\lambda)\Big]\ .$$

Since $g(v) \le g(\|u_\lambda\|_{L^\infty} + \delta\|\psi\|_{L^\infty})$ and $|g(v) - g(u_\lambda) - (v - u_\lambda)\, g'(u_\lambda)| = o(\delta)$, we deduce that

$$-\Delta v - (\lambda + \theta)\, g(v) \ \ge \ \delta - \theta\, g(\|u_\lambda\|_{L^\infty} + \delta\|\psi\|_{L^\infty}) - o(\delta)\ .$$

Therefore, we see that for δ sufficiently small, there exists $\theta = \theta(\delta) > 0$ such that $-\Delta v - (\lambda + \theta)\, g(v) \ge 0$. This implies in particular that $u_{\lambda+\theta} \le v$; and so, $\overline{u} \le v$. Letting $\delta \downarrow 0$, we obtain $\overline{u} \le u_\lambda$, thus $\overline{u} = u_\lambda$. So we see that $u_\mu(x) \to u_\lambda(x)$ as $\mu \to \lambda$, for all $x \in \Omega$. Since u_μ is increasing in μ and $u_\mu \in C(\overline{\Omega})$ for all $\mu < \lambda^*$, the convergence is uniform and (2.2) holds. It then follows easily from (2.2) that $\lambda_1(-\Delta - \mu g'(u_\mu)) \to \lambda_1(-\Delta - \lambda g'(u_\lambda))$ as $\mu \to \lambda$. In particular, we deduce from (i) that there exist $\delta, \eta > 0$ such that

$$\lambda_1(-\Delta - \mu\, g'(u_\mu)) \ > \ \eta\ , \tag{2.3}$$

for $|\mu - \lambda| < \delta$. This means that (i) holds with λ replaced by μ such that $|\mu - \lambda| < \delta$; and so we deduce from (2.2) that

$$\text{the mapping } \ \mu \mapsto u_\mu \ \text{ is continuous } \ (\lambda - \delta, \lambda + \delta) \to L^\infty(\Omega)\ . \tag{2.4}$$

We next show that there exists C such that

$$\|u_\mu - u_\nu\|_{L^\infty} \ \le \ C|\mu - \nu|\ , \tag{2.5}$$

for $|\mu - \lambda|, |\nu - \lambda| < \delta$. Indeed, it follows from (2.3) that

$$\begin{aligned} \eta\|u_\mu - u_\nu\|_{L^2}^2 \ &\le \int_\Omega |\nabla(u_\mu - u_\nu)|^2 - \mu\int_\Omega g'(u_\mu)\,(u_\mu - u_\nu)^2 \\ &= \int_\Omega (u_\mu - u_\nu)\,\Big[-\Delta(u_\mu - u_\nu) - \mu\, g'(u_\mu)\,(u_\mu - u_\nu)\Big] \\ &= \int_\Omega (u_\mu - u_\nu)\,\bigg(\mu\Big[g(u_\mu) - g(u_\nu) - g'(u_\mu)\,(u_\mu - u_\nu)\Big] + (\mu - \nu)\, g(u_\nu)\bigg). \end{aligned}$$

Since, by (2.4), $\mu|g(u_\mu) - g(u_\nu) - g'(u_\mu)(u_\mu - u_\nu)| \le \varepsilon(|\mu-\nu|)|u_\mu - u_\nu|$ with $\varepsilon(t) \to 0$ as $t \to 0$, we obtain

$$\eta\|u_\mu - u_\nu\|_{L^2}^2 \le \varepsilon(|\mu-\nu|)\,\|u_\mu - u_\nu\|_{L^2}^2 + C|\mu-\nu|\,\|u_\mu - u_\nu\|_{L^2}\ ,$$

so that $\|u_\mu - u_\nu\|_{L^2} \le C|\mu-\nu|$. Since

$$-\Delta(u_\mu - u_\nu) = \mu\Big(g(u_\mu) - g(u_\nu)\Big) + (\mu-\nu)\,g(u_\nu)\ ,$$

and $|\mu(g(u_\mu)-g(u_\nu))+(\mu-\nu)g(u_\nu)| \le C|u_\mu-u_\nu|+C|\mu-\nu|$, (2.5) now follows from the L^2 estimate and an obvious bootstrap argument. Suppose now $|\mu-\lambda| < \delta$. It follows from (2.3) that there exists a unique solution w_μ of

$$\begin{cases} -\Delta w_\mu - \mu\, g'(u_\mu)\, w_\mu = g(u_\mu) & \text{in } \Omega\ ,\\ w_\mu = 0 & \text{on } \partial\Omega\ . \end{cases}$$

By (2.3), w_μ is bounded in $H^1(\Omega)$, and by standard regularity w_μ is bounded in $C^1(\overline{\Omega})$. Using (2.5), we deduce that w_μ is continuous $(\lambda-\delta,\lambda+\delta) \to L^\infty(\Omega)$. Property (ii) follows if we show that $w_\mu = \frac{d}{d\mu}u_\mu$. This means that

$$\psi = \frac{u_\sigma - u_\mu - (\sigma-\mu)\,w_\mu}{\sigma-\mu} \underset{\sigma\to\mu}{\longrightarrow} 0\ ,$$

in $L^\infty(\Omega)$. We have

$$-\Delta\psi - \mu g'(u_\mu)\,\psi = (u_\sigma - u_\mu)\,g'(u_\mu) + \sigma\,\frac{u_\sigma - u_\mu}{\sigma-\mu}\,\frac{g(u_\sigma) - g(u_\mu) - (u_\sigma - u_\mu)\,g'(u_\mu)}{u_\sigma - u_\mu}\ ,$$

and it follows from (2.5) that the right-hand side converges to 0 in $L^\infty(\Omega)$ as $\sigma \to \mu$. Using (2.3), we conclude that $\|\psi\|_{L^\infty} \to 0$ as $\sigma \to \mu$. ∎

Remark 2.1. Step 2 of the proof of Theorem 1.1 shows that if $\lambda \mapsto u_\lambda$ is any branch of solutions of (1.1) which satisfies property (iii), then $\lambda_1(-\Delta - g'(u_\lambda)) \neq 0$. □

3 – Construction of discontinuities

In this section, we prove Theorems 1.2 and 1.3. We consider g as in the statement of these results, and the minimal branch $(u_\lambda)_{0<\lambda<\lambda^*}$. Fix $\underline{\lambda} \in (0,\lambda^*)$ and set

$$M = \|u_{\underline{\lambda}}\|_{L^\infty} = \sup_{0<\lambda\le\underline{\lambda}} \|u_\lambda\|_{L^\infty}\ .$$

We want to modify $g(u)$ for $u > M$ in order to produce a discontinuity near $\underline{\lambda}$ of the branch corresponding to the modified nonlinearity. The following observation is crucial for our proof. Given $r > 0$, we denote by B_r the ball of $\mathbb{R}^N$ of radius r and center 0. For $0 < \rho < R$, we consider the problem

$$\begin{cases} -\Delta\psi_\rho = 1_{B_\rho} & \text{in } B_R\ , \\ (\psi_\rho)_{|\partial B_R} = 0\ . \end{cases} \tag{3.1}$$

We have the following estimates.

Lemma 3.1. *For $0 < \rho < R/2$,*

$$\inf_{B_{2\rho}} \psi_\rho = \rho^2 K(\rho)\ , \tag{3.2}$$

where the behavior of $K(\rho)$ as $\rho \downarrow 0$ is of the form

$$K(\rho) \approx \begin{cases} R/\rho & \text{if } N = 1\ , \\ |\log\rho|/2 & \text{if } N = 2\ , \\ 2^{2-N}/N(N-2) & \text{if } N \geq 3\ . \end{cases}$$

Proof: If $N = 2$, ψ_ρ is given by

$$\psi_\rho(x) = \begin{cases} \dfrac{\rho^2}{2}(\log R - \log\rho) + \dfrac{\rho^2 - |x|^2}{4} & \text{for } |x| \leq \rho\ , \\ \dfrac{\rho^2}{2}(\log R - \log|x|) & \text{for } \rho \leq |x| \leq R\ . \end{cases}$$

If $N \neq 2$, ψ_ρ is given by

$$\psi_\rho(x) = \begin{cases} \dfrac{\rho^N}{N(N-2)}\,(\rho^{-N+2} - R^{-N+2}) + \dfrac{\rho^2 - |x|^2}{2N} & \text{if } |x| \leq \rho\ , \\ \dfrac{\rho^N}{N(N-2)}\,(|x|^{-N+2} - R^{-N+2}) & \text{if } \rho \leq |x| \leq R\ , \end{cases}$$

and the result follows. ∎

Corollary 3.2. *Let $c, \mu > 0$ and $x_0 \in \Omega$. If $N \geq 3$ suppose, in addition, that $\mu/c < 2^{2-N}/N(N-2)$. There exists $\delta > 0$ such that if*

$$\begin{cases} -\Delta w \geq c\, 1_{\{w>\mu|x-x_0|^2\}}\ , & x \in \Omega\ , \\ w \geq 0\ , & x \in \partial\Omega\ , \end{cases} \tag{3.3}$$

and $w \not\equiv 0$, then $w \geq \delta\, \mathrm{d}_\Omega$.

Proof: We may assume $x_0 = 0$. Consider $R > 0$ such that $B_R \subset \Omega$. Since $w > 0$ by the strong maximum principle, there exists $0 < \rho < R$ such that $\{w > \mu|x|^2\} \supset B_\rho$. Set

$$\overline{\rho} = \sup\Big\{0 < \rho < R;\ \{w > \mu|x|^2\} \supset B_\rho\Big\} > 0 .$$

We deduce from (3.3) that if $\rho < \overline{\rho}$, then $w \ge c\,\psi_\rho$, where ψ_ρ is defined by (3.1). Letting $\rho \uparrow \overline{\rho}$, we obtain, $w \ge c\,\psi_{\overline{\rho}}$. If $\overline{\rho} \ge R/2$, we deduce that $w \ge c\,\psi_{\frac{R}{2}}$. Otherwise, it follows from Lemma 3.1 that $w \ge c\,\overline{\rho}^2 K(\overline{\rho})$ for $|x| < 2\,\overline{\rho}$. In particular, $w > \mu|x|^2$ for $|x| < \overline{\rho}\,\min\{2, \sqrt{cK(\overline{\rho})/\mu}\}$. This implies, by definition of $\overline{\rho}$, that $\overline{\rho} \ge \overline{\rho}\min\{2, \sqrt{cK(\overline{\rho})/\mu}\}$, i.e. $K(\overline{\rho}) \le \mu/c$. By Lemma 3.1, this implies that $\overline{\rho} \ge \rho_1$ for some $\rho_1 > 0$, and we have $w \ge c\,\psi_{\rho_1}$. Setting $\widetilde{\rho} = \min\{\rho_1, R/2\}$, we have $w \ge c\,\psi_{\widetilde{\rho}}$. We observe that $\widetilde{\rho}$ is independent of w, so the result follows from (3.3) and the strong maximum principle. ∎

We now define the modified nonlinearity $\widehat{g}$ by

$$\widehat{g}(u) = \begin{cases} g(u) & \text{if } 0 \le u \le M , \\ g(M) + s & \text{if } u > M , \end{cases} \tag{3.4}$$

where $s > 0$ is to be chosen later. We observe that $\widehat{g}$ is discontinuous at M, but left-continuous.

Lemma 3.3. *For every $\lambda > 0$, there exists a minimal solution $\widehat{u}_\lambda$ of the equation*

$$\begin{cases} -\Delta \widehat{u} = \lambda\,\widehat{g}(\widehat{u}) & \text{in } \Omega , \\ \widehat{u} = 0 & \text{in } \partial\Omega . \end{cases} \tag{3.5}$$

In addition, $\widehat{u}_\lambda = u_\lambda$ for all $0 < \lambda \le \underline{\lambda}$. Furthermore, if $\lambda(g(M)+s)/g(M) < \lambda^*$, then $\widehat{u}_\lambda \le u_\mu$ with $\mu = \lambda(g(M)+s)/g(M)$.

Proof: Since $\widehat{g}$ is nondecreasing, the result for $\lambda \le \underline{\lambda}$ is obvious. We now assume $\lambda > \underline{\lambda}$. We apply the usual increasing iteration method, i.e. we solve

$$\begin{cases} -\Delta u^{n+1} = \lambda\,\widehat{g}(u^n)\,, & x \in \Omega , \\ u^{n+1} = 0\,, & x \in \partial\Omega , \end{cases}$$

starting from $u^0 = 0$. It is clear that $u^0 \le u^1 \le \cdots \le u^n \le \cdots \le \lambda(g(M)+s)\,\psi$, where ψ is the solution of the equation $-\Delta\psi = 1$ in Ω with Dirichlet boundary condition. Therefore, $(u^n)_{n\geq 0}$ converges in $C(\overline{\Omega})$ to a function u, which is clearly a solution of (3.5) by left-continuity of $\widehat{g}$.

Next, if w is a nonnegative supersolution of (3.5), then $u^0 \leq w$ and by iteration (since $\widehat{g}$ is nondecreasing), $u^n \leq w$ for all $n \geq 0$. Therefore, $u \leq w$ and in particular, u is the minimal solution. Finally, if $\mu = \lambda(g(M) + s)/g(M)$, then $\lambda\widehat{g} \leq \mu g$; and so, if $\lambda(g(M) + s)/g(M) < \lambda^*$, then u_μ is a supersolution of (3.5). The last statement follows. ∎

Lemma 3.4. *Suppose $s > 0$. If $N \geq 3$ suppose, in addition, that s is sufficiently large. There exists $\delta > 0$ such that if*

$$\begin{cases} -\Delta u \geq \underline{\lambda}\, \widehat{g}(u)\,, & x \in \Omega\,, \\ u \geq 0\,, & x \in \partial\Omega\,, \end{cases}$$

and $u \not\equiv u_{\underline{\lambda}}$, then $u \geq u_{\underline{\lambda}} + \delta\, \mathrm{d}_\Omega$.

Proof: Set $w = u - u_{\underline{\lambda}}$. We have

$$-\Delta w \;\geq\; \underline{\lambda}\Big(\widehat{g}(u_{\underline{\lambda}} + w) - g(u_{\underline{\lambda}})\Big) \;\geq\; \underline{\lambda}\, s\, 1_{\{u_{\underline{\lambda}}+w>M\}}\,.$$

Let $x_0 \in \Omega$ satisfy $u_{\underline{\lambda}}(x_0) = M$. Since $u_{\underline{\lambda}} \in C^2(\overline{\Omega})$, there exists $\mu > 0$ such that $u_{\underline{\lambda}}(x) \geq M - \mu|x - x_0|^2$ for all $x \in \Omega$. Therefore, $1_{\{u_{\underline{\lambda}}+w>M\}} \geq 1_{\{w>\mu|x-x_0|^2\}}$, and the result follows from Corollary 3.2. ∎

Corollary 3.5. *Suppose $s > 0$. If $N \geq 3$ suppose, in addition, that s is sufficiently large. It follows that the mapping $\lambda \mapsto \widehat{u}_\lambda$ is discontinuous at $\underline{\lambda}$. More precisely, there exists $\delta > 0$ such that $\widehat{u}_\lambda \geq u_{\underline{\lambda}} + \delta\, \mathrm{d}_\Omega$ for all $\lambda > \underline{\lambda}$.*

We now consider a local modification of g. Given $s, \ell > 0$, such that

$$g(M) + s \;<\; \lim_{u\to\infty} g(u)\;, \tag{3.6}$$

let $\overline{g}$ satisfy

$$\overline{g}(u) = \begin{cases} g(u) & \text{if } \; u \leq M\;, \\ g(M) + s & \text{if } \; M < u \leq M + \ell\;, \\ g(u) & \text{if } \; u \geq M + 2\,\ell\;, \end{cases} \tag{3.7}$$

and be C^1 and increasing on $[M + \ell, M + 2\ell]$. In other words, $\overline{g}$ is nondecreasing, coincides with g on $[0, M] \cup [M + 2\ell, \infty)$, and has a discontinuity at M. Note also that $\overline{g}$ coincides with $\widehat{g}$ on $[0, M + \ell]$.

Lemma 3.6. *Suppose*

$$\ell > \lim_{\lambda\downarrow\underline{\lambda}} \|\widehat{u}_\lambda\|_{L^\infty} - M \;, \tag{3.8}$$

where $\widehat{u}_\lambda$ is defined in Lemma 3.3. It follows that there exists $\overline{\lambda} > \underline{\lambda}$ such that for every $\lambda \in (0, \overline{\lambda})$, there exists a minimal solution $\overline{u}_\lambda$ of the equation

$$\begin{cases} -\Delta\overline{u} = \lambda\,\overline{g}(\overline{u}) & \text{in } \Omega \;, \\ \overline{u} = 0 & \text{in } \partial\Omega \;. \end{cases} \tag{3.9}$$

In addition, $\overline{u}_\lambda = u_\lambda$ for all $0 < \lambda \le \underline{\lambda}$. Moreover, $\overline{u}_\lambda = \widehat{u}_\lambda$ for all $0 < \lambda < \underline{\lambda} + \varepsilon$ if $\varepsilon > 0$ is small enough.

Proof: The result is a consequence of Lemma 3.3, since $\overline{g} \ge \widehat{g}$ and $\overline{g}$ coincides with $\widehat{g}$ on $[0, M+\ell]$. Note that the assumption (3.8) clearly implies the last statement. ∎

Corollary 3.7. *Let $s > 0$. If $N \ge 3$ suppose, in addition, that s is sufficiently large. If (3.8) holds, then that the mapping $\lambda \mapsto \overline{u}_\lambda$ is discontinuous at $\underline{\lambda}$. More precisely, there exists $\delta > 0$ such that $\overline{u}_\lambda \ge \overline{u}_{\underline{\lambda}} + \delta\,\mathrm{d}_\Omega$ for all $\underline{\lambda} < \lambda < \overline{\lambda}$.*

Proof: The result follows from Lemma 3.6 and Corollary 3.5. ∎

Proof of Theorems 1.2 and 1.3: Let $\overline{g}$ be as in (3.7) and consider a sequence $g_n \in C^1([0,\infty))$ of positive, increasing nonlinearities such that $g_n(u) = g(u)$ for $u \le M$ and $u \ge M + 2\ell$ and such that $g_n(u) \uparrow \overline{g}(u)$ for $M < u < M + 2\ell$. In particular, $g_n \le \overline{g}$ so that the branch of minimal solutions for g_n exists at least for $\lambda < \overline{\lambda}$. If u_λ^n is the corresponding minimal solution, then u_λ^n is nondecreasing in n and $u_\lambda^n = u_\lambda$ if $\lambda \le \underline{\lambda}$. Since $\overline{g}$ is left-continuous, it is not difficult to show that $u_\lambda^n \uparrow \overline{u}_\lambda$ as $n \to \infty$. Suppose that the assumptions of Corollary 3.7 are satisfied. Given $\varepsilon > 0$, we claim that if n is large enough, then the mapping $\lambda \mapsto u_\lambda^n$ has a discontinuity in $[\underline{\lambda}, \underline{\lambda} + \varepsilon]$. Indeed, assume by contradiction that for some sequence $n_k \to \infty$, $u_\lambda^{n_k}$ is continuous on $[\underline{\lambda}, \underline{\lambda} + \varepsilon]$. Since $u_{\underline{\lambda}}^{n_k} = u_{\underline{\lambda}}$ for all k, we have $\|u_{\underline{\lambda}}^{n_k}\|_{L^\infty} = M$. On the other hand, it follows from Corollary 3.7 that there exists $\gamma > 0$ such that $\|\overline{u}_\lambda\|_{L^\infty} \ge M + \gamma$ for $\lambda > \underline{\lambda}$. Therefore, if we consider $\underline{\lambda} < \lambda < \underline{\lambda} + \varepsilon$, we have $\|u_\lambda^{n_k}\|_{L^\infty} \ge M + \gamma/2$ for k large enough. It then follows from the contradiction assumption that if $0 < \nu < \gamma/2$, then there exists a sequence $(\lambda_k)_{k\ge 0}$ such that $\lambda_k \downarrow \underline{\lambda}$ and $\|u_{\lambda_k}^{n_k}\|_{L^\infty} = M + \nu$. Since $g_{n_k}(u_{\lambda_k}^{n_k})$ is bounded in $L^\infty(\Omega)$, we may assume (up to a subsequence) that $u_{\lambda_k}^{n_k} \to w$ in $C(\overline{\Omega})$

for some $w \in C(\overline{\Omega})$. Now we observe that if $w(x) \neq M$, then $g_{n_k}(u^{n_k}_{\lambda_k}(x)) \to \overline{g}(w)$ as $k \to \infty$. If $w(x) = M$, then

$$\liminf_{k\to\infty} g_{n_k}(u^{n_k}_{\lambda_k}(x)) \;\geq\; g(M) \;=\; \overline{g}(w(x)) \;.$$

Therefore, $\liminf\limits_{k\to\infty} g_{n_k}(u^{n_k}_{\lambda_k}) \geq \overline{g}(w)$, so that $-\Delta w \geq \underline{\lambda}\overline{g}(w) \geq \underline{\lambda}\widehat{g}(w)$. Since $\|w\|_{L^\infty} = M + \nu$, this is absurd by Lemma 3.4 if ν is sufficiently small.

In the case $N \geq 3$ and $g(u) \to \infty$ as $u \to \infty$, we just choose s large enough and the ℓ satisfying (3.8), so that the assumptions of Corollary 3.7 are satisfied. Theorem 1.3 follows by choosing $\widetilde{g} = g_n$ with n sufficiently large.

Finally, suppose $N = 1$ or $N = 2$. If the mapping $\lambda \mapsto u_\lambda$ is discontinuous at $\underline{\lambda}$, then we may let $\widetilde{g} = g$. So we now assume that the mapping $\lambda \mapsto u_\lambda$ is continuous at $\underline{\lambda}$. It then follows from the last statement in Lemma 3.3 that

$$\liminf_{\lambda\downarrow\underline{\lambda}} \|\widehat{u}_\lambda\|_{L^\infty} \leq \|u_\mu\|_{L^\infty} \;,$$

where $\mu = \underline{\lambda}\,\frac{g(M)+s}{g(M)} \to \underline{\lambda}$ as $s \downarrow 0$. Thus we may choose ℓ satisfying (3.8) and such that $\ell \downarrow 0$ as $s \downarrow 0$. In particular, we may assume by choosing s sufficiently small that $\|\overline{g} - g\|_{L^\infty} \leq \varepsilon/2$ and that $\overline{g} - g$ is supported in $[M, M+\varepsilon]$. We then let $\widetilde{g} = g_n$ for n sufficiently large, and the conclusions of Theorem 1.2 follow. ∎

4 – A one-dimensional concave-convex nonlinearity

In this section, we consider positive solutions of the equation

$$\tag{4.1} \begin{cases} -\Delta u = \lambda(u^q + u^p)\,, & x \in \Omega\;, \\ u = 0\,, & x \in \partial\Omega\;, \end{cases}$$

where $0 < q < 1 < p$ and $\lambda > 0$. The nonlinearity $g(u) = u^q + u^p$ is not positive at the origin. However, the singularity of g' at the origin allows the existence of a branch of minimal, positive solutions u_λ defined for $0 < \lambda < \lambda^*$ with $0 < \lambda^* < \infty$. For $\lambda = \lambda^*$, there is a (possibly singular) minimal, positive weak solution u_{λ^*}. For $\lambda > \lambda^*$, there is no solution, even in a weak sense. See [2,4](1). We now consider positive solutions of the related heat equation

$$\tag{4.2} \begin{cases} u_t - \Delta u = \lambda(u^q + u^p)\,, & x \in \Omega\;, \\ u = 0\,, & x \in \partial\Omega\;. \end{cases}$$

(1) In the papers [2,4], the nonlinearity is $\lambda u^q + u^p$ rather than $\lambda(u^q + u^p)$. The two problems, however, are equivalent by an obvious scaling.

The initial value problem for (4.2) is studied in [4]. If $0 < \lambda \le \lambda^*$, the minimal solution u_λ is stable from below, in the sense that if $\varphi \in L^\infty(\Omega)$, $\varphi \ge 0$ and $\varphi \le u_\lambda$, then the (unique) positive solution of (4.2) with the initial condition $u(0) = \varphi$ is global and satisfies $u(t) \to u_\lambda$ as $t \to \infty$. The convergence holds in $L^\infty(\Omega)$ if $\lambda < \lambda^*$ and in $L^{p+1}(\Omega)$ if $\lambda = \lambda^*$ (see [4]). The stability from above is related to whether or not $\lambda_1(-\Delta - \lambda g'(u_\lambda)) > 0$. Since g is neither concave nor convex, none of the usual criteria apply. In the one-dimensional case, we have the following result, based on ODE techniques.

Proposition 4.1. *Suppose $N = 1$. Given $0 < \lambda < \lambda^*$, there exist exactly two positive solutions of* (4.1)*, u_λ and $v_\lambda > u_\lambda$. The mapping $\lambda \mapsto u_\lambda$ is C^1 and increasing $(0, \lambda^*) \to L^\infty(\Omega)$. The mapping $\lambda \mapsto v_\lambda$ is C^1: $(0, \lambda^*) \to L^\infty(\Omega)$, and the mapping $\lambda \mapsto \|v_\lambda\|_{L^\infty}$ is decreasing $(0, \lambda^*) \to \mathbb{R}$. Furthermore, $(u_\lambda - u_\mu)/\mathrm{d}_\Omega \to 0$ and $(v_\lambda - v_\mu)/\mathrm{d}_\Omega \to 0$ uniformly in Ω as $\mu \to \lambda \in (0, \lambda^*)$. In addition, $\lambda_1(-\Delta - \lambda g'(u_\lambda)) > 0$ and $\lambda_1(-\Delta - \lambda g'(v_\lambda)) < 0$ for all $\lambda \in (0, \lambda^*)$.*

Proof: We may assume without loss of generality that $\Omega = (-1, 1)$. We proceed in three steps.

Step 1. An auxiliary equation. Given $\mu > 0$, consider the solution $w = w_\mu$ of the equation $-w'' = w^q + w^p$ with the initial conditions $w(0) = \mu$, $w'(0) = 0$. w is even and is given by

$$x = \frac{1}{\sqrt{2}} \int_{w(x)}^{\mu} \frac{d\xi}{\sqrt{\frac{\mu^{q+1} - \xi^{q+1}}{q+1} + \frac{\mu^{p+1} - \xi^{p+1}}{p+1}}}\ ,$$

for $0 \le x \le \theta(\mu)$, with

$$\theta(\mu) = \frac{1}{\sqrt{2}} \int_0^{\mu} \frac{d\xi}{\sqrt{\frac{\mu^{q+1} - \xi^{q+1}}{q+1} + \frac{\mu^{p+1} - \xi^{p+1}}{p+1}}}\ .$$

In particular, w is decreasing on $[0, \theta(\mu)]$ and $w(\theta(\mu)) = 0$. We now study the behavior of $\theta(\mu)$ and, for further convenience, we write $\theta(\mu)$ in three different forms:

$$\theta(\mu) = \frac{1}{\sqrt{2}} \int_0^1 \frac{d\xi}{\sqrt{\mu^{q-1} \frac{1-\xi^{q+1}}{q+1} + \mu^{p-1} \frac{1-\xi^{p+1}}{p+1}}} \tag{4.3}$$

$$= \frac{\mu^{\frac{1-q}{2}}}{\sqrt{2}} \int_0^1 \frac{d\xi}{\sqrt{\frac{1-\xi^{q+1}}{q+1} + \mu^{p-q} \frac{1-\xi^{p+1}}{p+1}}} \tag{4.4}$$

$$= \frac{\mu^{-\frac{p-1}{2}}}{\sqrt{2}} \int_0^1 \frac{d\xi}{\sqrt{\mu^{-(p-q)} \frac{1-\xi^{q+1}}{q+1} + \frac{1-\xi^{p+1}}{p+1}}}\ . \tag{4.5}$$

We deduce from (4.4) and from (4.5), respectively, that

$$\theta(\mu) \underset{\mu\downarrow 0}{\longrightarrow} 0\,, \qquad \theta(\mu) \underset{\mu\to\infty}{\longrightarrow} 0\ . \tag{4.6}$$

We now claim that there exists $\mu^* > 0$ such that

$$\theta'(\mu) > 0 \quad \text{for } 0 < \mu < \mu^*\ ; \qquad \theta'(\mu) < 0 \quad \text{for } \mu > \mu^*\ . \tag{4.7}$$

Indeed, we deduce from (4.3) that

$$\theta'(\mu) \;=\; \frac{1}{\mu^{\frac{q+1}{2}}\, 2^{\frac{3}{2}}} \int_0^1 \frac{\frac{1-q}{q+1}(1-\xi^{q+1}) \;-\; \frac{p-1}{p+1}(1-\xi^{p+1})\,\mu^{p-q}}{\left[\frac{1-\xi^{q+1}}{q+1} + \mu^{p-q}\,\frac{1-\xi^{p+1}}{p+1}\right]^{\frac{3}{2}}}\, d\xi\ . \tag{4.8}$$

We now observe that for all $0 < \xi < 1$,

$$1 > \frac{1-\xi^{q+1}}{1-\xi^{p+1}} > \frac{q+1}{p+1}\ , \tag{4.9}$$

so that we deduce from (4.8) that

$$\begin{aligned}\int_0^1 \frac{\left[\frac{1-q}{q+1} - \frac{p-1}{p+1}\,\mu^{p-q}\right](1-\xi^{p+1})}{\left[\frac{1-\xi^{q+1}}{q+1} + \mu^{p-q}\,\frac{1-\xi^{p+1}}{p+1}\right]^{\frac{3}{2}}}\, d\xi \;&>\; \mu^{\frac{q+1}{2}}\, 2^{\frac{3}{2}}\, \theta'(\mu)\\ &> \int_0^1 \frac{\left[(1-q) - (p-1)\,\mu^{p-q}\right](1-\xi^{p+1})}{(p+1)\left[\frac{1-\xi^{q+1}}{q+1} + \mu^{p-q}\,\frac{1-\xi^{p+1}}{p+1}\right]^{\frac{3}{2}}}\, d\xi\ .\end{aligned}$$

It follows in particular that if

$$\mu < \mu_1 = \Big(\frac{1-q}{p-1}\Big)^{\frac{1}{p-q}}\ ,$$

then $\theta'(\mu) > 0$ and if

$$\mu > \mu_2 = \left(\frac{(1-q)\,(1+p)}{(1+q)\,(p-1)}\right)^{\frac{1}{p-q}}\ ,$$

then $\theta'(\mu) < 0$. The claim will now be proved if we show that for every $\xi \in (0,1)$, the integrand in (4.8) is a decreasing function of $\mu \in (\mu_1, \mu_2)$. Letting $\tau = \mu^{p-q}$, we set

$$h(\tau) \;=\; \frac{\frac{1-q}{q+1}(1-\xi^{q+1}) \;-\; \frac{p-1}{p+1}(1-\xi^{p+1})\tau}{\left[\frac{1-\xi^{q+1}}{q+1} + \tau\,\frac{1-\xi^{p+1}}{p+1}\right]^{\frac{3}{2}}}\ ,$$

so that

$$h'(\tau) = \frac{(p-1)\,(1-\xi^{p+1})^2}{2\,(p+1)^2}\,\frac{\left[\tau - \frac{1-\xi^{q+1}}{1-\xi^{p+1}}\left(2\,\frac{p+1}{q+1} + 3\,\frac{(1-q)\,(p+1)}{(q+1)\,(p-1)}\right)\right]}{\left[\frac{1-\xi^{q+1}}{q+1} + \tau\,\frac{1-\xi^{p+1}}{p+1}\right]^{\frac{5}{2}}}\ .$$

It follows from (4.9) that

$$\begin{aligned}\frac{1-\xi^{q+1}}{1-\xi^{p+1}}\left(2\,\frac{p+1}{q+1} + 3\,\frac{(1-q)\,(p+1)}{(q+1)\,(p-1)}\right) &> \frac{(p+1)\left[(1-q)+2\,(p-q)\right]}{(q+1)\,(p-1)}\\ &> \frac{(p+1)\,(1-q)}{(q+1)\,(p-1)} = \sup_{\mu_1<\mu<\mu_2}\tau\ .\end{aligned}$$

Thus $h'(\tau) < 0$, which proves the claim (4.7).

STEP 2. The solutions u_λ and v_λ. Given $\mu > 0$ and w as in Step 1, set

$$u(x) = w(x\,\theta(\mu))\ . \tag{4.10}$$

We see that u is a positive solution of (4.1) if and only if

$$\lambda = \theta(\mu)^2\ . \tag{4.11}$$

In this case, we have

$$\|u\|_{L^\infty(\Omega)} = u(0) = \mu\ . \tag{4.12}$$

Setting

$$\lambda^* = \theta(\mu^*)^2\ ,$$

it follows from (4.6)–(4.7) that (4.1) has a positive solution if and only if $0<\lambda\le\lambda^*$. Given $0<\lambda<\lambda^*$, let $0<\mu_-<\mu_+$ be the two solutions of $\lambda=\theta(\mu)^2$, and let u_λ and v_λ be the corresponding solutions of (4.1) given by (4.10). It follows that u_λ and v_λ are the only positive solutions of (4.1). Moreover, $\|u_\lambda\|_{L^\infty} < \|v_\lambda\|_{L^\infty}$ by (4.12). Thus u_λ must be the minimal solution; and so $v_\lambda > u_\lambda$. Since θ is C^1 on $(0,\infty)$, the mappings $\lambda \mapsto \mu_\pm$ are C^1 on $(0,\lambda^*)$, and one deduces easily that the mappings $\lambda \mapsto u_\lambda$ and $\lambda \mapsto v_\lambda$ are C^1 $(0,\lambda^*) \to L^\infty(\Omega)$. It follows easily that $(u_\lambda - u_\mu)/\mathrm{d}_\Omega \to 0$ and $(v_\lambda - v_\mu)/\mathrm{d}_\Omega \to 0$ uniformly in Ω as $\mu \to \lambda \neq 0$. $(u_\lambda)_{0<\lambda<\lambda^*}$ being the minimal branch is increasing. Finally, by (4.7) and (4.12), $v_\lambda(0)$ is decreasing.

STEP 3. $\lambda_1(-\Delta - \lambda g'(u_\lambda)) > 0$ and $\lambda_1(-\Delta - \lambda g'(v_\lambda)) < 0$ for all $\lambda \in (0,\lambda^*)$. We first show that $\lambda_1(-\Delta - \lambda g'(u_\lambda)) > 0$. (Note that we may not apply

Theorem 1.1 because g is not smooth at the origin.) We already know that $\lambda_1(-\Delta - \lambda g'(u_\lambda)) \geq 0$ by Remark 3.2 in [2], so we assume by contradiction that $\lambda_1(-\Delta - \lambda g'(u_\lambda)) = 0$ and fix $\lambda < \overline{\lambda} < \lambda^*$. Given $\lambda < \mu < \overline{\lambda}$, we deduce from (1.5) that

$$(\mu - \lambda) \int_\Omega g(u_\lambda)\, \varphi_1 \;\leq\; \lambda \int_\Omega |g(u_\mu) - g(u_\lambda) - g'(u_\lambda)\,(u_\mu - u_\lambda)|\varphi_1 \;.$$

Since $u_\lambda < u_\mu < u_{\overline{\lambda}}$, we see that there exists C such that

$$\begin{aligned}|g(u_\mu) - g(u_\lambda) - g'(u_\lambda)\,(u_\mu - u_\lambda)| \;&\leq\; C\,|u_\mu - u_\lambda|^2 + C\,\frac{|u_\mu - u_\lambda|^2}{u_\lambda^{2-q}}\\ &\leq C\,\frac{|u_\mu - u_\lambda|^2}{\varphi_1^{2-q}}\;;\end{aligned}$$

and so,

$$(\mu - \lambda) \int_\Omega g(u_\lambda)\, \varphi_1 \;\leq\; C \int_\Omega \frac{|u_\mu - u_\lambda|^2}{\varphi_1^{1-q}} \;\leq\; C\,\|u_\mu - u_\lambda\|^2_{L^\infty} \int_\Omega \varphi_1^{-1+q} \;\leq\; C\,|\mu - \lambda|^2\;,$$

since u_λ is C^1. A contradiction follows by letting $\mu \downarrow \lambda$. We finally show that $\lambda_1(-\Delta - \lambda g'(v_\lambda)) < 0$. We observe that an obvious modification of the above argument (taking $\mu < \lambda$) shows that $\lambda_1(-\Delta - \lambda g'(v_\lambda)) \neq 0$. We then assume by contradiction that $\lambda_1(-\Delta - \lambda g'(v_\lambda)) > 0$. Given $0 < \theta < 1$, set $\varphi = (1-\theta)u_\lambda + \theta\, v_\lambda$ and let u be the positive solution of (4.2) with the initial value $u(0) = \varphi$ (see [4]). It follows from the maximum principle that $u_\lambda \leq u(t) \leq v_\lambda$. In particular, the ω-limit set of φ is well-defined and is either $\{u_\lambda\}$ or $\{v_\lambda\}$. On the other hand, since $\lambda_1(-\Delta - \lambda g'(u_\lambda)) > 0$, it follows from standard techniques that $\omega(\varphi) = \{u_\lambda\}$ if θ is small enough; and since $\lambda_1(-\Delta - \lambda g'(v_\lambda)) > 0$, $\omega(\varphi) = \{v_\lambda\}$ if θ is sufficiently close to 1. Also the set of θ such that $\omega(\varphi) = \{u_\lambda\}$ is open and so is the set of θ such that $\omega(\varphi) = \{v_\lambda\}$. It follows that there exists $\theta \in (0,1)$ such that $\omega(\varphi) \neq \{u_\lambda\}$ and $\omega(\varphi) \neq \{v_\lambda\}$, which is absurd. ∎

Corollary 4.2. *Suppose $N = 1$. Given $0 < \lambda < \lambda^*$, let $\varphi \in L^\infty(\Omega)$, $\varphi \geq 0$ and let u be the positive solution of* (4.2) *with the initial condition $u(0) = \varphi$. The following properties hold.*

- **(i)** *There exists $\varepsilon > 0$ such that if $\varphi \leq u_\lambda + \varepsilon$ or if $\varphi \leq v_\lambda$, $\varphi \not\equiv v_\lambda$, then u is globally defined and $u(t) \to u_\lambda$ uniformly as $t \to \infty$.*
- **(ii)** *If $\varphi \geq v_\lambda$, $\varphi \not\equiv v_\lambda$, then u blows up in finite time.*

Proof: (**i**) Since $\lambda_1(-\Delta - \lambda g'(u_\lambda)) > 0$, it follows easily that there exists $\varepsilon > 0$ such that if $\|\varphi - u_\lambda\|_{L^\infty} \leq \varepsilon$, then $u(t) \to u_\lambda$ in $L^\infty(\Omega)$ as $t \to \infty$. Since

u_λ is stable from below, we see that if $\varphi \le u_\lambda + \varepsilon$, then $u(t) \to u_\lambda$ in $L^\infty(\Omega)$ as $t \to \infty$. Suppose now $\varphi \le v_\lambda$, $\varphi \not\equiv v_\lambda$. It follows from the strong maximum principle that there exists $\delta > 0$ such that $u(1) \le v_\lambda - \delta\, \mathrm{d}_\Omega$; and so, there exists $\lambda < \mu < \lambda^*$ such that $u(1) \le v_\mu$. Since v_μ is clearly a supersolution of (4.2), we have $u(t) \le v_\mu$ for all $t \ge 1$. Now the ω-limit set $\omega(\varphi)$ of φ is either $\{u_\lambda\}$ or $\{v_\lambda\}$. Since $u(t,0) \le v_\mu(0) < v_\lambda(0)$, we deduce that $\omega(\varphi) = \{u_\lambda\}$.

(**ii**) It follows from the strong maximum principle that there exist $\delta, \varepsilon > 0$ such that $u(\varepsilon) \ge v_\lambda + \delta\, \mathrm{d}_\Omega$; and so, there exists $0 < \mu < \lambda$ such that $u(\varepsilon) \ge v_\mu$. It thus remains to show that the positive solution z of (4.2) with the initial condition $z(0) = v_\mu$ blows up in finite time. We assume by contradiction that z is globally defined. Since v_μ is a subsolution of (4.2), $z(t)$ is nondecreasing. Using the technique of [4] (see in particular the proof of Lemma 3.1), it follows that $z(t)$ converges as $t \to \infty$ to a positive weak solution of (4.1), which is either u_λ or v_λ. This is absurd since $z(t,0) \ge v_\mu(0) > v_\lambda(0) > u_\lambda(0)$. ∎

Remark 4.3. If $\lambda = \lambda^*$, then the stability of u_λ can be studied in any dimension. Note first that u_{λ^*} is stable from below, see [4]. Using the techniques of Martel [6], one then shows that u_{λ^*} is the unique, positive weak solution of (4.1) for $\lambda = \lambda^*$. (The nonlinearity is not convex, but it is convex for u large, and one can proceed as in [4] to construct the appropriate supersolutions.) If $u_{\lambda^*} \in L^\infty(\Omega)$ (which is the case in particular if p is not too large), then u_{λ^*} is unstable in the sense that if $\varphi \in L^\infty(\Omega)$, $\varphi \ge u_{\lambda^*}$, $\varphi \ne u_{\lambda^*}$, then the corresponding solution u of (4.2) blows up in finite time. (See the proof of Corollary 4.2 (ii).) If $u_{\lambda^*} \notin L^\infty(\Omega)$, then u_{λ^*} is unstable (by "instantaneous blow up") in the sense that if $\varphi \ge u_{\lambda^*}$, $\varphi \ne u_{\lambda^*}$, then there does not exist any positive weak solution of (4.2) satisfying $u(0) = u_{\lambda^*}$. This follows from the techniques of Martel [7]. □

REFERENCES

[**1**] Amann, H. – Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces, *SIAM Rev.*, 18 (1976), 620–709.

[**2**] Ambrosetti, A.; Brezis, H. and Cerami, G. – Combined effects of concave and convex nonlinearities in some elliptic problems, *J. Funct. Anal.*, 122 (1994), 519–453.

[**3**] Brown, K.J.; Ibrahim, M.M.A. and Shivaji, R. – S-shaped bifurcation curves, *Nonlinear Anal.*, 5 (1981), 475–486.

[**4**] Cazenave, T.; Dickstein, F. and Escobedo, M. – A semilinear heat equation with concave-convex nonlinearity, *Rend. Mat. Appl. (7)*, 19 (1999), 211–242.

[5] Crandall, M.G. and Rabinowitz, P.H. – Bifurcation, perturbation of simple eigenvalues, and linearized stability, *Arch. Ration. Mech. Anal.,* 52 (1973), 161–180.

[6] Martel, Y. – Uniqueness of weak extremal solutions of nonlinear elliptic problems, *Houston J. Math.,* 23 (1997), 161–168.

[7] Martel, Y. – Dynamical instability of singular extremal solutions of nonlinear elliptic equations, *Adv. Math. Sci. Appl.,* 9 (1999), 163–181.

Thierry Cazenave,
Laboratoire Jacques-Louis Lions, UMR CNRS 7598, B.C. 187,
Université Pierre et Marie Curie,
4, place Jussieu, 75252 Paris Cedex 05 – FRANCE
E-mail: `cazenave@ccr.jussieu.fr`

and

Miguel Escobedo,
Departamento de Matemáticas, Universidad del País Vasco,
Apartado 644, 48080 Bilbao – SPAIN
E-mail: `mtpesmam@lg.ehu.es`

and

M. Assunta Pozio,
Dipartimento di Matematica, Universitá di Roma "La Sapienza",
P. le A. Moro, 2, 00185 Roma – ITALY
E-mail: `pozio@mat.uniroma1.it`