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LOCAL EXISTENCE OF CLASSICAL SOLUTIONS TO THE WELL-POSED HELE–SHAW PROBLEM

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Abstract: We prove local existence of classical solutions to the well-posed Hele–Shaw problem under general conditions on the fixed boundaries. Our approach consists of a construction of approximate solutions as the solutions to the one-phase Stefan problem with ε- heat capacity and energy estimates in Von Mises variables. These estimates permit us to find some small time interval where norms of approximate solutions in some Sobolev spaces are bounded and pass to the limit when ε goes to zero.

1 – Introduction

The Hele–Shaw problem is a well-known model of liquid filtration in a porous medium. In this model the governing equation for the liquid's pressure is simply the Poisson equation

$$-\Delta p \,=\, f(x) \,\equiv\, \operatorname{div} F \tag{1.1}$$

in the flow region $\Omega \subset \mathbb{R}^n$, $n = 2, 3$. This region is bounded by a multicomponent boundary $\partial\Omega(t)$ which consists of a finite number of connected moving (free) or fixed components without intersection. Let us denote by $S^{(k)}$, $k = 1, ..., m$ the fixed component and by $\Gamma^{(i)}$, $i=1, ..., \ell$ the free component of $\partial\Omega(t)$, so that

$$\partial\Omega(t) \,=\, S \cup \Gamma(t)$$

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with

$$S = \bigcup_{k=1}^{m} S^{(k)} , \quad \Gamma(t) = \bigcup_{i=1}^{\ell} \Gamma^{(i)}(t) \ .$$

On the fixed boundary we assume the following boundary condition of the third type

$$\alpha^{(k)} \cdot \frac{\partial p}{\partial \nu} + (1-\alpha^{(k)}) \cdot \beta(x,t) \cdot p \ = \ p_0(x,t) \, , \qquad x \in S^{(k)} \ , \tag{1.2}$$

where $\frac{\partial}{\partial \nu}$ is the derivative in the outward normal direction, $\alpha^{(k)} = \mathrm{const}$, $0 \leq \alpha^{(k)} \leq 1$, $\beta(x,t) \geq 0$.

Let us denote by S' the part of the fixed boundary where $\alpha^{(k)} = 0$ and, respectively, the Dirichlet boundary condition holds. We put $\beta = 1$ on S'. Note also that $\alpha^{(k)}=1$ corresponds to the Neumann boundary condition.

On the free boundary $\Gamma(t)$ the following boundary conditions hold (in what follows all variables are dimensionless)

$$p = 0 \ , \tag{1.3}$$

$$p_t \ = \ |\nabla p|^2 + F \cdot \nabla p \ . \tag{1.4}$$

The initial condition on the free boundary $\Gamma(t)$

$$\Gamma(0) = \Gamma^0 \, , \qquad \Omega(0) = \Omega^0 \tag{1.5}$$

completes the formulation of the problem.

We call this problem well-posed Hele–Shaw problem (WPHSP) whenever its solution $p(x,t)$ is nonnegative, which corresponds to the case

$$p_0(x,t) > 0 \, , \qquad f(x) \geq 0 \ , \tag{1.6}$$

and ill-posed otherwise.

Note that the problem (1.1)–(1.5) is exactly the one-phase Stefan problem with vanishing heat capacity. It is well-known that the solutions of the one-phase Stefan problem are infinitely smooth for $t > 0$ outside of fixed boundaries independently on the smoothness of given boundary and initial data (supposed $\Gamma(t)$ is Lipshitz continuous). For the Hele–Shaw problem the solution may be irregular with respect to the time variable (see examples in [1]). This peculiarity implies the independent studying of the Hele–Shaw problem. Complete references about this problem one can find in the paper of J.R. Ockendon and his collegues (see [2]).

Weak solutions for WPHSP have been studied by Elliott and Janovsky [3], Gustafsson [4], Louro and Rodrigues [5]. The general case has been considered by Antontsev, Meirmanov, Yurinski in the recent publication [1].

Classical solutions to WPHSP have been investigated by Meirmanov [6], Reissig [7], Escher and Simonett [8].

Meirmanov [6] has studied WPHSP for the case $n = 2$ and strip-like domain $\Omega(t)$ with $\alpha = 1,\ p_0 = \gamma = \text{const},\ F = \gamma\,\nabla x_2$,

$$S:\ x_2 = 0$$
$$\Gamma^0:\ x_2 = 1 + \varepsilon\,R_0(x_1)\ .$$

He announced the local in time existence of the analytical solution such that the position $R(x_1,t)$ of the free boundary $\Gamma(t)$:

$$\Gamma(t):\ x_2 = 1 + \varepsilon\,R(x_1,t)$$

tends to the solution of the Boussinesque equation

$$\frac{\partial h}{\partial t} = \frac{\partial}{\partial x_1}\left(h\,\frac{\partial h}{\partial x_1}\right)$$

when $\varepsilon \to 0$.

The statement follows after the application of the nonlinear abstract Cauchy–Kovalevskaya theorem proved by L. Ovsiannikov in the work [9], where he has studied the free boundary Cauchy–Poisson problem for the Euler equations.

Using the same method Reissig [7] has proved the local in time existence of the analytical solution for the special case of source point function $f(x)$.

The most recent result belongs to Escher and Simonett [8], where the local in time existence of the classical solution has been obtained for the case $\beta = 1$, $p_0 = p_0(x)$ and $f = 0$.

Global existence of the classical solution to WPHSP has been proved by Antontsev, Meirmanov, Yurinski [10] for the case of strip-like domain when $\alpha = 0,\ f = 0$ and $p_0 = p_0(t)$.

The structure of the present article is the following. After the formulation of the main results we consider the simple case of a strip-like domain and show the idea of the method. This method consists of a construction of approximate solutions as the solutions to the one-phase Stefan problem with ε-heat capacity, an introduction of the von Mises variables and construction of corresponding energy estimates in the Sobolev spaces W_2^n. These estimates and the corresponding embedding theorem guarantee $H^{n-1+\alpha}$ smoothness (independently on ε) of the approximate solutions on some small time interval $(0,T_*)$ which doesn't depend of ε. The passage from the simple to the general case is the same as in [11] for the general Stefan problem. Note that this technique is also applicable to the two-phase situation.

All notations of the functional spaces and norms in the present paper are the same as in [12].

2 – Main results

We suppose that the following conditions are fulfilled:

(**A**) $S^{(k)}\in Lip$ if $\alpha^{(k)}=0$ and $S^{(k)}\in C^2$ otherwise;

(**B**) $\Gamma^0\in W_2^n\cap C^{n-1+\gamma_0}$ with some $\gamma_0>0$;

(**C**) $\overline{\operatorname{supp} F}\subset\Omega^0$, $f\in L_\infty(\Omega^0)$;

(**D**) $\beta, p_0, \dfrac{\partial^{n-1}\beta}{\partial t^{n-1}}, \dfrac{\partial^{n-1}p_0}{\partial t^{n-1}}\in L_\infty(S_T)$, $S_T=S\times(0,T)$.

Theorem 1. *Under conditions (A)–(D) there exists at least one classical solution $\{p,\Gamma(t)\}$ to the problem (1.1)–(1.5) on some small time interval $(0,T_*)$ such that $\Gamma(t)$ is infinitely smooth with respect to the spatial variables, p, p_t are infinitely smooth with respect to the spatial variables near $\Gamma(t)$ (outside of* $\operatorname{supp} F$*) for $t>0$ and*

$$p_t\in L_\infty(0,T_*;H^\gamma(\overline{\Omega}(t)))\,,\quad \nabla p\in H^{\gamma,\frac{\gamma}{2}}(\overline{\Omega}_{T_*})\,,\quad \Omega_{T_*}=\{(x,t)\colon\, x\in\Omega(t),\ t\in(0,T_*)\}$$

with some $\gamma>0$.

Our approach is based on a construction of approximate solutions as solutions to the one-phase Stefan problem

$$\varepsilon\,\frac{\partial\theta^\varepsilon}{\partial t}-\Delta\theta^\varepsilon=f\,,\qquad x\in\Omega_\varepsilon(t) \tag{2.1}$$

with additional initial condition

$$\theta^\varepsilon(x,0)=\theta_0^\varepsilon(x)\,,\qquad x\in\Omega^0(t) \tag{2.2}$$

and appropriate energy estimates in von Mises variables.

The special choice of θ_0^ε allows us to evaluate $\frac{\partial^{n-1}\theta^\varepsilon}{\partial t^{n-1}}$ independently on ε.

Lemma 2. *There exists a nonnegative function $\theta_0^\varepsilon \in H^\lambda(\overline{\Omega^0})$ with $\lambda > 4$ such that θ_0^ε satisfies the corresponding compatibility conditions on the boundary Γ^0 up to order $[\lambda]$ and*

$$\Big|\ln|\nabla\theta_0^\varepsilon(x)|\Big| \le M_0\,, \qquad x\in\Gamma^0\,, \tag{2.3}$$

$$\left|\theta_0^\varepsilon,\, \frac{1}{\varepsilon}(\Delta\theta_0^\varepsilon+f),\, \frac{1}{\varepsilon^2}\Delta(\Delta\theta_0^\varepsilon+f)\right|_{\Omega^0}^{(2)} \le M_0\,, \tag{2.4}$$

where M_0 depends only on the given data.

The proof of this lemma is standard if we will look for θ_0^ε as

$$\theta_0^\varepsilon = \theta_0 + \varepsilon\,\overline{\theta}\,.$$

Here θ_0 is a solution of the equation (1.1) in the domain Ω^0 with boundary conditions (1.2) and (1.3) and

$$\overline{\theta}(x) = |\nabla\overline{\theta}(x)| = 0\,, \qquad x\in\Gamma^0\,. \tag{2.5}$$

The last condition and the compatibility condition of the first order determine all second derivatives of $\overline{\theta}$ on the boundary Γ^0. Repeating the procedure we will determine all derivatives of $\overline{\theta}$ up to order $2[\lambda]$ on the boundary Γ^0.

Now, using the usual way we determine $\overline{\theta}$ in Ω^0.

3 – Special case of the strip-like domain

Let

$$S = \Big\{x=(x',x_n)\,|\;x_n=f_0(x'),\; x'\in\Lambda\Big\}\,, \qquad \Lambda=\Big\{x'\,|\;|x'|<1\Big\}\,,$$
$$\Gamma(t) = \Big\{x\,|\;x_n=R(x',t),\; x'\in\Lambda\Big\}\,,$$
$$\Omega(t) = \Big\{x\,|\;f_0(x')<x_n<R(x',t),\; x'\in\Lambda\Big\}$$

and the given data are periodic with respect to the variables x' with period 1.

We suppose also that $\alpha=0$, $\beta=1$, $p_0=1$ and $f=0$.

3.1. Approximate solution

As approximate solutions $\{\theta^\varepsilon, \Gamma_\varepsilon(t)\}$,

$$\Gamma_\varepsilon(t) = \Big\{x \mid \; x_n = R_\varepsilon(x',t), \; x' \in \Lambda\Big\} \;,$$

to the initial problem (1.1)–(1.5) we consider solutions to the one-phase Stefan problem (2.1), (2.2), (1.2)–(1.5) in the domain

$$\Omega_\varepsilon(t) = \Big\{x \mid \; f_0(x') < x_n < R_\varepsilon(x',t), \; x' \in \Lambda\Big\} \;.$$

Instead of the condition (2.3) we suppose that

$$\left|\ln\left|\frac{\partial\theta_0^\varepsilon}{\partial x_n}(x)\right|\right| \le M_0 \,, \qquad x \in \Omega^0 \;. \tag{3.1.1}$$

Under this condition and the conditions of Lemma 2 there exists some small time interval $(0,T_\varepsilon)$ where the Stefan problem (2.1), (2.2), (1.2)–(1.5) has a unique classical solution $\{\theta^\varepsilon, \Gamma_\varepsilon(t)\}$ ([11]). Our goal is to find some small interval $(0,T_*)$, $0 < T_* \le T_\varepsilon$, which doesn't depend on ε, where $\{\theta^\varepsilon, \Gamma_\varepsilon(t)\}$ converges to the classical solution $\{p, \Gamma(t)\}$ of the initial problem (1.1)–(1.5).

3.2. The von Mises variables

The monotonicity of the initial function $\theta_0^\varepsilon(x)$ with respect to the variable x_n (estimate (3.1.1)) allows us to introduce the von Mises variables

$$t = t\,, \qquad y' = x'\,, \qquad y_n = \theta^\varepsilon(x,t)$$

on the time interval $(0,T_*)$ where

$$\left|\nabla\theta^\varepsilon(x,t),\, \ln\left|\frac{\partial\theta^\varepsilon}{\partial x_n}(x,t)\right|\right| \le 2\,M_0\,, \qquad x \in \Omega_\varepsilon(t) \;. \tag{3.2.1}$$

The new unknown function

$$u(y,t) = x_n$$

satisfies in the known domain Π_{T_*},

$$\Pi_{T_*} = \Big\{(x,t)\colon \; x \in \Pi(t), \; t \in (0,T_*)\Big\}\,, \qquad \Pi(t) = \Big\{y \mid \; 0 < y_n < p_0(y',t), \; y' \in \Lambda\Big\} \;,$$

the following initial boundary-value problem

$$\varepsilon\,\frac{\partial u}{\partial t} - \Delta' u + \frac{\partial}{\partial y_n}\left\{\frac{1+|\nabla' u|^2}{u_n}\right\} = 0\,, \qquad y \in \Pi(t)\ , \tag{3.2.2}$$

$$\frac{\partial u}{\partial t} + \frac{1+|\nabla' u|^2}{u_n} = 0\,, \qquad y \in \Sigma^0\ , \tag{3.2.3}$$

$$u = f_0(y')\,, \qquad y \in \Sigma^1\ , \tag{3.2.4}$$

$$u(y,0) = u_0^{\varepsilon}(y)\,, \qquad y \in \Pi(0)\ , \tag{3.2.5}$$

where

$$\Sigma^0 = \{y \mid\ y_n = 0\}\,, \qquad \Sigma^1 = \{y \mid\ y_n = p_0(y',t)\}\ ,$$

and the function $u_0^{\varepsilon}(y)$ is a solution of the equation

$$y_n = u_0^{\varepsilon}(y', u_0^{\varepsilon}(y))\ .$$

In (3.2.2) and (3.2.3)

$$\Delta' u = \sum_{i=1}^{n-1} \frac{\partial^2 u}{\partial y_i^2}\,, \qquad \nabla' u = (u_1, \ldots, u_{n-1})\,, \qquad u_j = \frac{\partial u}{\partial y_j}, \quad j = 1, \ldots, n\ .$$

Estimates (3.2.1) imply that

$$\Big|\, \nabla u, \ln |u_n| \,\Big| \le M\ . \tag{3.2.6}$$

Here and below we denote by M the constants depending only on M_0 and the given data.

We suppose that on the boundary Σ^1 all derivatives $D^3 u$ (case $n = 2$) and $D^4 u, D^3 u_t, D^2 D_t^2 u$ (case $n = 3$) are bounded by the constant M_0. Such supposition makes sense in view of the local estimates for the solutions of the heat equation ([12]) if the functions p_0 and f_0 are sufficiently smooth.

Note also that the corresponding problem for the derivative $\dfrac{\partial u}{\partial t}$ satisfies the maximum principle:

$$\left|\frac{\partial u}{\partial t}(y,t)\right| \le \max\left\{ \left|\frac{\partial u}{\partial t}\right|^{(0)}_{\Sigma^1_{T_*}}, \left|\frac{\partial u}{\partial t}(.,0)\right|^{(0)}_{\Pi(0)} \right\} \le M\ . \tag{3.2.7}$$

3.3. Energy estimates. Case $n = 2$

The first estimates for the derivatives $u_0 = \frac{\partial u}{\partial t}$ and u_i, $i < n$ are simple and follow from the standard method (multiplication of the equation for u_j, $j = 0, 1, ..., n-1$ by u_j and integration by parts) if we take into account (3.2.6):

$$(3.3.1) \qquad \int_0^{T_*}\int_{\Pi(t)} |\nabla u_j|^2\, dy\, dt \;\leq\; M\,, \qquad j = 0, 1, ..., n-1\ .$$

The last estimate, equation (3.2.2) and estimate (3.2.7) imply

$$(3.3.2) \qquad u \in W_2^{2,1}(\Pi_{T_*})\,, \qquad \|u\|^{(2)}_{2,\Pi_{T_*}} \leq M\ .$$

Let now

$$u_{ij} = \frac{\partial^2 u}{\partial y_i\, \partial y_j}\,, \qquad i, j \leq n\ .$$

For $i, j < n$ the function $v = u_{ij}$ satisfies the problem

$$(3.3.3) \quad \varepsilon\,\frac{\partial v}{\partial t} - \Delta' v + \frac{\partial}{\partial y_n}\left\{\frac{2}{u_n}(\nabla' u.\nabla' v) - \frac{1+|\nabla' u|^2}{u_n^2}\,\frac{\partial v}{\partial y_n} + J\right\} = 0\,, \quad y \in \Pi(t)\ ,$$

$$\frac{\partial v}{\partial t} + \frac{2}{u_n}(\nabla' u.\nabla' v) - \frac{1+|\nabla' u|^2}{u_n^2}\,\frac{\partial v}{\partial y_n} + J \;=\; 0\,, \qquad y \in \Sigma^0\ ,$$

where

$$J \;=\; \frac{2}{u_n}\sum_{k=1}^{n-1} u_{ik}\, u_{jk} - \frac{2}{u_n^2}\sum_{k=1}^{n-1} u_k(u_{ik}\, u_{jn} + u_{jk}\, u_{in}) + \frac{2}{u_n^3}\, u_{in}\, u_{jn}\,(1 + |\nabla' u|^2)\ .$$

Multiplying (3.3.3) by v and integrating by parts we get after some usual evaluations

$$(3.3.4) \quad \frac{d}{dt}\left\{\varepsilon\int_{\Pi(t)} u_{ij}^2\, dy + \int_{\Sigma^0} u_{ij}^2\, dy'\right\} + \int_{\Pi(t)} |\nabla u_{ij}|^2\, dy \;\leq\; M\left\{\sum_{\ell=1}^{n-1}\sum_{r=1}^{n} I_{\ell r} + 1\right\}\ .$$

Here

$$I_{\ell r} = \int_{\Pi(t)} u_{\ell r}^4\, dy\ .$$

To estimate these integrals let us consider new functions

$$z_r \;=\; u_r(y,t) - u_r(y,0)\,, \qquad r = 1, ..., n\ .$$

Using the identity

$$(3.3.5)\qquad 0 \;=\; \int\limits_0^1 \frac{\partial}{\partial y_\ell}\left\{ z_r \left(\frac{\partial z_r}{\partial y_\ell}\right)^3 \right\} dy_\ell \;=\; \int\limits_0^1 \left|\frac{\partial z_r}{\partial y_\ell}\right|^4 dy_\ell \;+\; 3\int\limits_0^1 z_r \left|\frac{\partial z_r}{\partial y_\ell}\right|^2 \frac{\partial^2 z_r}{\partial y_\ell^2}\, dy_\ell$$

we obtain

$$(3.3.6)\qquad I_{\ell r} \;\le\; \delta_1^2 \int\limits_{\Pi(t)} \left|\frac{\partial^2 z_r}{\partial y_\ell^2}\right|^2 dy \;\le\; \delta_1^2 \left\{ \max_{i,j<n} \int\limits_{\Pi(t)} |\nabla u_{ij}|^2\, dy \;+\; M \right\} ,$$

where

$$\delta_1 = 3 \max_{0<r\le n} \left\{ \max_{0\le t\le T_*} \left| u_r(.,t) - u_r(.,0) \right|^{(0)}_{\Pi(t)} \right\} .$$

Now we add to the definition (3.2.1) of the interval $(0, T_*)$ the new restriction

$$(3.3.7)\qquad 4\, M \delta_1^2 < 1 \ .$$

Under this condition, inequalities (3.3.4) and (3.3.6) imply

$$(3.3.8)\qquad \max_{i,j<n} \int\limits_{\Pi_{T_*}} |\nabla u_{ij}|^2\, dy\, dt \;+\; \max_{i,j<n} \left\{ \max_{0\le t\le T_*} \int\limits_{\Sigma^0} u_{ij}^2\, dy' \right\} \;\le\; M \ .$$

Note that in order to evaluate "normal" derivatives u_{inn} and u_{nnn} we have used the equation for the derivatives u_j, $j \le n$ and estimates (3.3.1) for the derivatives u_{tj}.

So,

$$u \;\in\; L_2(0, T_*; W_2^3(\Pi(t)))$$

and

$$(3.3.9)\qquad \int\limits_0^{T_*} \left(\|u(.,t)\|^{(3)}_{2,\Pi(t)} \right)^2 dt \;\le\; M \ .$$

Moreover, the representation of the free boundary $\Gamma_\varepsilon(t)$ in the form

$$x_n = R_\varepsilon(x', t) = u(x', 0, t)$$

and estimates (3.3.8) mean that

$$R_\varepsilon(., t) \in W_2^3(\Lambda)\,, \qquad t \in (0, T_*)$$

and

$$\max_{0\le t\le T_*} \|R_\varepsilon(.,t)\|^{(3)}_{2,\Lambda} \;\le\; M \ .$$

For the case $n = 2$ the last estimate and the well-known imbedding theorem imply

$$R_\varepsilon(.,t) \in H^{1+\beta}(\overline{\Lambda})$$

with any $2\beta \leq 1$ and

$$\max_{0 \leq t \leq T_*} |R_\varepsilon(.,t)|_\Lambda^{(1+\beta)} \leq M \ .$$

Considering now θ^ε as a solution of the Poisson equation with a bounded right-hand side (estimate (3.2.7)) which satisfies a zero Dirichlet boundary condition on the free boundary $\Gamma(t) \in H^{1+\beta}$ we conclude that

$$\theta^\varepsilon(.,t) \in H^{1+\beta}(\overline{\Omega_\varepsilon(t)}) \ .$$

Applying again the boundness of θ_t^ε and lemma 3.1 (chapter II, [12]) we get

$$\left|\theta_x^\varepsilon(x, t+T) - \theta_x^\varepsilon(x,t)\right| \leq M\tau^{\frac{\gamma}{2}}$$

with some $\gamma = \gamma(M_0) > 0$.

The similar estimates hold for the derivatives $u_k(y,t)$ which permit us to choose the interval $(0, T_*)$ satisfying (3.2.1) and (3.3.7):

$$T_* = \min\{M^{\frac{2}{\gamma}}, M^{\frac{1}{2}}\} \ .$$

Now on the interval $(0, T_*)$ we can pass to the limit when $\varepsilon \to 0$ and get the classical solution $\{p, R\}$ to the initial problem (1.1)–(1.5) such that

$$R(.,t) \ \in \ W_2^2(\Lambda) \cap H^{1+\beta}(\overline{\Lambda}) \ ,$$
$$p_t \in L_\infty(0, T_*; H^\gamma(\overline{\Omega}(t))) \ , \qquad \nabla p \in H^{\gamma, \gamma/2}(\overline{\Omega}_{T_*}) \ .$$

Remark 3. Applying now the Caffarelli's technique [13] we easily get that p, p_t and $\Gamma(t)$ are infinitely smooth with respect to the spatial variables. Note that this technique doesn't allow evaluate corresponding norms on the hole interval $(0, T_*)$. It only permits to evaluate these norms on the interval (t_0, T_*) and the corresponding constants might be unbounded when $t_0 \to 0$. □

3.4. Energy estimates. Case $n = 3$.

For the case $n = 3$,

$$R_\varepsilon(.,t) \in H^{1+\beta}(\overline{\Lambda}) \ ,$$

if

$$R_\varepsilon(.,t) \in W_2^3(\Lambda) \ .$$

To show that, we will use the same method as we have used for the case $n = 2$.

Multiplying the equation for the derivatives

$$u_{ti} = \frac{\partial^2 u}{\partial t \partial y_i}\,, \qquad i < n\,,$$

by u_{ti} and integrating by parts we get

$$\text{(3.4.1)} \qquad \frac{d}{dt}\Big\{\varepsilon \int\limits_{\Pi(t)} u_{ti}^2\,dy + \int\limits_{\Sigma^0} u_{ti}^2\,dy'\Big\} + \int\limits_{\Pi(t)} |\nabla u_{ti}|^2\,dy \;\le\; M\Big\{\sum_{\ell=0}^{n-1}\sum_{r=1}^{n} I_{\ell r} + 1\Big\}\,,$$

where $I_{\ell r}$ are the same as in the previous section for $\ell \ge 1$ and

$$I_{0r} = \int\limits_{\Pi(t)} u_{tr}^4\,dy\;.$$

Let

$$\delta_2(t) = \max\Big\{|u(.,t)|^{(2)}_{\Pi(t)},\ |u_t(.,t)|^{(1)}_{\Pi(t)}\Big\}$$

and

$$\delta_2(0) \le M_0\;.$$

We choose the time interval $(0, T_*)$ from the condition

$$\text{(3.4.2)} \qquad \delta_2(t) \le 2M_0, \quad \text{for } \ 0 \le t \le T_*\;.$$

Then (3.4.1) implies

$$\text{(3.4.3)} \qquad \max_{0\le t\le T_*}\,\varepsilon \int\limits_{\Pi(t)} u_{ti}^2\,dy + \int\limits_{\Pi_{T_*}} |\nabla u_{ti}|^2\,dy\,dt \;\le\; M, \quad \text{for } \ i < n\;.$$

Multiplication the equation for the derivatives

$$v = \frac{\partial^3 u}{\partial y_i\,\partial y_j\,\partial y_\ell}, \quad v = \frac{\partial^3 u}{\partial t\,\partial y_i\,\partial y_j}, \quad v = \frac{\partial^3 u}{\partial t^2\,\partial y_i}, \qquad \text{for } \ i, j, \ell < n$$

by v and integration by parts gives us

$$\text{(3.4.4)} \qquad \frac{d}{dt}\Big\{\varepsilon \int\limits_{\Pi(t)} v^2\,dy + \int\limits_{\Sigma^0} v^2\,dy'\Big\} + \int\limits_{\Pi(t)} |\nabla v|^2\,dy \;\le\; M\Big\{\eta . I_0 + \frac{1}{4\,\eta} \max_{\substack{k<n\\ 1\le s\le n}} I_{ks} + 1\Big\}.$$

Here

$$I_0 \;=\; \max_{\substack{k<n\\ 1\le j<n\\ 1\le s\le n}} \int\limits_{\Pi(t)} |u_{kjs}|^4\,dy\,, \qquad u_{kjs} = \frac{\partial^3 u}{\partial y_k\,\partial y_j\,\partial y_s}$$

and η is any positive number.

Using the identity (3.3.5) for the functions u_{ks} we evaluate I_0 as

$$I_0 \leq \delta_1^2 \int\limits_{\Pi(t)} |\nabla v|^2 \, dy \, . \tag{3.4.5}$$

Choosing η sufficiently small we get from (3.4.4) and (3.4.5)

$$\max_{0<t<T_*} \Big\{ \varepsilon \max_{\substack{j,k<n \\ 1\leq s<n}} \|u_{jks}(.,t)\|^2_{2,\Pi(t)} + \|u(.,t)\|^2_{W_2^3(\Sigma^0)} \Big\} + \\ + \max_{\substack{j,k<n \\ 1\leq s<n}} \|\nabla u_{jks}\|^2_{2,\Pi_{T_*}} \leq M \, . \tag{3.4.6}$$

Estimates (3.4.6) for "tangential" derivatives and corresponding equations for "normal" derivatives permit us to evaluate all derivatives D^4u, D^3u_t and D^2u_{tt}. For example, the estimate for $D_iD_n^3u$ follows from (3.4.6) and equation (3.2.2) if we differentiate it with respect to the variables y_i and y_n.

Thus,

$$u(.,t) \in W_2^3(\Sigma^0) \, , \qquad t \in (0,T_*) \ ,$$
$$D_t^k u \in L_2(0,T_*; W_2^{4-k}(\Pi(t)))$$

and

$$\max_{0\leq t\leq T_*} \|u(.,t)\|_{W_2^3(\Sigma^0)} + \int\limits_0^{T_*} \Big(\|D_t^k u(.,t)\|^{(4-k)}_{2,\Pi(t)} \Big)^2 dt \leq M \, , \qquad k = 0,1,2 \, . \tag{3.4.7}$$

Coming back to the original variables and using the representation of the free boundary $\Gamma_\varepsilon(t)$ in the form

$$\Gamma_\varepsilon(t): \quad x_n = u(x',0,t) \ ,$$

we get

$$R_\varepsilon(.,t) \in W_2^3(\Lambda) \, , \qquad t \in (0,T_*) \ ,$$
$$D_t^2\theta^\varepsilon \in L_2(0,T_*; W_2^2(\Omega_\varepsilon(t))) \ ,$$
$$D^2\theta^\varepsilon, DD_t\theta^\varepsilon \in W_2^{2,1}(\Omega_{\varepsilon,T_*})$$

and

$$\max_{0\leq t\leq T_*} \|R_\varepsilon(.,t)\|^{(3)}_{2,\Lambda} \leq M \ , \tag{3.4.8}$$

$$\|D^2\theta^\varepsilon, DD_t\theta^\varepsilon\|^{(2)}_{2,\Omega_{\varepsilon,T_*}} + \int\limits_0^{T_*} \Big(\|D_t^2\theta^\varepsilon\|^{(2)}_{2,\Omega_\varepsilon(t)} \Big)^2 dt \leq M \, . \tag{3.4.9}$$

Here $Dv(D^2v)$ means all first (second) derivatives of the function v with respect to spatial variables and

$$\Omega_{\varepsilon,T_*} = \Big\{(x,t)\colon\ x\in\Omega_\varepsilon(t),\ t\in(0,T_*)\Big\}\ .$$

The estimate (3.4.8) and the corresponding embedding theorem imply

$$R_\varepsilon(.,t) \in H^{1+\beta}(\overline{\Lambda})\,, \qquad \max_{0\le t\le T_*} \|R_\varepsilon(.,t)\|_\Lambda^{(1+\beta)} \le M \tag{3.4.10}$$

with any $\beta,\ 0<\beta<1$.

So, as we have proved before

$$\frac{\partial\theta^\varepsilon}{\partial t} \in L_\infty(0,T_*;H^{\gamma_o}(\overline{\Omega}(t)))\,, \qquad \nabla\theta^\varepsilon \in H^{\gamma_o,\frac{\gamma_o}{2}}(\overline{\Omega}_{T_*}) \tag{3.4.11}$$

with some positive $\gamma_o=\gamma_o(M_0)$.

The last inclusion permits us to choose some small interval $(0,T_*)$ (independently on ε) on which the condition (3.2.1) is satisfied.

To satisfy the condition (3.4.2) we have to prove the Hölder continuity of the derivatives $D^2\theta^\varepsilon$ and $DD_t\theta^\varepsilon$. For the derivatives $v=D^2\theta^\varepsilon$ we have

$$\Delta v = \varepsilon\, D^2 D_t\, \theta^\varepsilon \equiv F$$

with $F\in L_2(\Omega(t))$ and $v\in W_2^2(\Gamma(t))$.

So,

$$v\in W_2^2(\Omega(t))$$

and

$$\max_{0\le t\le T_*} \|D^2\theta^\varepsilon(.,t)\|_{2,\Omega(t)}^{(2)} \le M\ .$$

Thus,

$$D^2\theta^\varepsilon(.,t)\in H^{\beta_1}(\overline{\Omega(t)})$$

with some $\beta_1\in(0,\frac12)$ (lemma 3.3, [12]).

Taking into account the inclusion (3.4.11) and applying lemma 3.1 ([12]) we get

$$D^2\theta^\varepsilon \in H^{\gamma,\frac{\gamma}{2}}(\overline{\Omega}_{T_*})\ . \tag{3.4.12}$$

To prove the inclusion

$$DD_t\theta^\varepsilon \in H^{\gamma,\frac{\gamma}{2}}(\overline{\Omega}_{T_*}) \tag{3.4.13}$$

note that from the maximum principle for the solution $D_t^2\theta^\varepsilon$ to the heat equation and estimates (3.4.2) follows the bound

$$|D_t^2\theta^\varepsilon(x,t)| \le M\,, \qquad (x,t) \in \overline{\Omega}_{T_*}\ .$$

For the function $v = D_t\theta^\varepsilon$ we have

$$\Delta v = \varepsilon\, D_t^2\theta^\varepsilon \equiv F \in L_\infty(\Omega_\varepsilon(t))\ ,$$

$$v\,|_{\Gamma(t)} = |\nabla\theta^\varepsilon|^2 \in H^{1+\beta}(\overline{\Gamma_\varepsilon(t)})\ .$$

So,

$$v = D_t\theta^\varepsilon \in H^{1+\beta}(\overline{\Omega_\varepsilon(t)})\ .$$

Taking into account the inclusion (3.4.11) and applying again lemma 3.1 ([12]) we finally get the inclusion (3.4.9).

The rest of the proof is the same as in the previous section.

4 – Case of arbitrary domain

As we have mentioned above, the approximate solutions to the initial problem (1.1)–(1.5) are the solutions of the one-phase Stefan problem (2.1), (2.2), (1.2)–(1.5). The existence of the classical solutions for this last problem for $\varepsilon > 0$ follows from [11]. This solution exists on some small time interval $(0, T_\varepsilon)$, and our goal is to prove that there exists some $T_* > 0$ such that $T_\varepsilon \ge T_*$ for any $\varepsilon > 0$ and

$$|D\theta^\varepsilon, D_t\theta^\varepsilon|^{(\gamma)}_{\Omega_{\varepsilon,T_*}} \le M\ .$$

It is obvious that we cannot introduce the von Mises variables in the hole domain $\Omega_{\varepsilon,T}$, as we have done it in the special case of the srip-like domain, but we do it locally near the initial position Γ^0 of the free boundary $\Gamma(t)$.

Let us consider the system of open sets $\{\pi^{(\ell)}\}$ and $\{\Pi^{(\ell)}\}$ such that

$$\pi^{(\ell)} \subset \Pi^{(\ell)}\,, \qquad \bigcup_\ell \pi^{(\ell)} = \bigcup_\ell \Pi^{(\ell)} = \Gamma^0$$

and in the local coordinates on the surface Γ^0 the set $\Pi^{(\ell)}$ is represented as

$$\Pi^{(\ell)} = \Big\{\xi \mid\ \xi_n = R_0^{(\ell)}(\xi'),\ \xi' \in \Lambda\Big\}\,, \qquad \Lambda = \Big\{|\xi'| < 1\Big\}\ .$$

Moreover, there exists N_0 such that the intersection of any (N_0+1) different $\Pi^{(\ell)}$ is empty.

Now we have to construct domains $\Omega^{(\ell)}(t)$ where we can introduce the von Mises variables. If $\nu(x_0)$ is a normal vector to the surface Γ^0 at the point $x_0 \in \Gamma^0$, then we put

$$\widetilde{\Omega}^{(\ell)} = \Big\{ x \mid \ x = x_0 + \tau\,\nu(x_0),\ |\tau| < h,\ x_0 \in \Pi^{(\ell)} \Big\}\ ,$$

$$\widetilde{\omega}^{(\ell)} = \Big\{ x \mid \ x = x_0 + \tau\,\nu(x_0),\ |\tau| < h,\ x_0 \in \pi^{(\ell)} \Big\}\ .$$

Considering $\theta_0^\varepsilon(x)$ in the local coordinates ξ

$$\theta_0^\varepsilon(x) = \widetilde{\theta}_0^\varepsilon(\xi)$$

we choose sufficiently small h_0 such that for $|h| \le h_0$

$$\left| \ln \left| \frac{\partial \widetilde{\theta}_0^\varepsilon}{\partial \xi_n}(\xi) \right| \right| < 2\,M_0\,, \qquad \xi \in \widetilde{\Omega}^{(\ell)} \tag{4.1}$$

and

$$\widetilde{\Omega}^{(\ell)} \cap \overline{\operatorname{supp} f} \ = \ \emptyset\ . \tag{4.2}$$

Next, we choose the time interval $(0, T_*)$ where

$$\left| \ln \left| \frac{\partial \widetilde{\theta}^\varepsilon}{\partial \xi_n}(\xi,t) \right| \right|,\ \ |D^2 \widetilde{\theta}^\varepsilon(\xi,t)|,\ \ |D D_t \widetilde{\theta}^\varepsilon(\xi,t)| \ \le\ 3\,M_0 \tag{4.3}$$

for

$$\xi\ \in\ \Omega_\varepsilon(t) \cap \widetilde{\Omega}^{(\ell)}\ .$$

These conditions imply that

$$|\nabla\theta^\varepsilon(x,t)|,\ |D^2\theta^\varepsilon(x,t)|,\ |DD_t\theta^\varepsilon(x,t)|\ <\ 3\,M_0$$

for $x \in \Gamma_\varepsilon(t),\ t \in (0, T_*)$.

Applying the maximum principle for the derivatives $D_t\theta^\varepsilon$, $D_t^2\theta^\varepsilon$ we get

$$|D_t\theta^\varepsilon(x,t)|,\ |D_t^2\theta^\varepsilon(x,t)|\ <\ 3\,M_0 \tag{4.4}$$

for $x \in \Omega^\varepsilon(t),\ t \in (0, T_*)$.

Now let us choose the level set

$$\Sigma(t) = \Big\{ x \in \bigcup_\ell \widetilde{\Omega}^{(\ell)} \mid\ \theta^\varepsilon(x,t) = a = \text{const} > 0 \Big\}\ .$$

It is always possible to do this for sufficiently small a due to conditions (4.3).

As a last step we consider the set $\omega^{(\ell)}(t)(\Omega^{(\ell)}(t))$ which is the set of all points $\widetilde{\omega}^{(\ell)}(\widetilde{\Omega}^{(\ell)})$ laying between the surfaces $\Gamma_\varepsilon(t)$ and Σ.

Note that near the surface Σ the functions θ^ε, $D_t\theta^\varepsilon$ and $D_t^2\theta^\varepsilon$ are infinitely smooth with respect to the spatial variables. This fact follows from the local estimates for the solution $v = D_t^k\theta^\varepsilon$, $k = 0, 1, 2$, of the heat equation if we consider a new variable $t' = \frac{t}{\varepsilon}$.

Now we are ready to repeat the same procedure, as we have done before, and find the lower bound T_* for the intervals $(0, T_\varepsilon)$.

Let us consider equation (2.1) and boundary conditions (1.3), (1.4) for the approximate solutions θ^ε in the local coordinates ξ in the domain $\Omega^{(\ell)}(t)$. These local coordinates are just the orthogonal transformation of the initial ones. So, in the local coordinates we have the same heat equation and the same boundary conditions (1.3), (1.4). The condition (4.3) permits us to introduce the von Mises variables in the domain $\Omega^{(\ell)}(t)$. We denote as $G^{(\ell)}$ the image of the domain $\Omega^{(\ell)}(t)$ in the von Mises variables and, correspondingly, as $g^{(\ell)}$ the image of the domain $\omega^{(\ell)}(t)$.

Let $\eta(y') \in C^\infty$, $\eta(y') = 1$ for $y \in g^{(\ell)}$ and $\eta(y) = 0$ outside of some small neighborhood of $g^{(\ell)}$ (which still contains in $G^{(\ell)}$).

Repeating all what we have done before with an evident correction (this is we multiply the equation not by v but by $\eta\, v$) we get

$$\max_{0<t<T_*}\|\Gamma_\varepsilon(t)\|^{(2)}_{2,\pi^{(\ell)}} + \int_0^{T_*}\left(\|\theta^\varepsilon(.,t)\|^{(3)}_{2,\omega^{(\ell)}(t)}\right)^2 dt \;\le\; \le\; M\left\{\delta_1^2\int_0^{T_*}\left(\|\theta^\varepsilon(.,t)\|^{(3)}_{2,\Omega^{(\ell)}(t)}\right)^2 dt + 1\right\}. \tag{4.5}$$

Here

$$\delta_1 = \max_{0\le t\le T_*}|\nabla\theta^\varepsilon(.,t) - \nabla\theta_0^\varepsilon|^{(0)}_{\Omega_\varepsilon(t)}\ .$$

Let

$$G(t) = \bigcup_\ell \Omega^{(\ell)}(t) = \bigcup_\ell \omega^{(\ell)}(t)\,, \qquad G_{T_*} = \bigcup_{t=0}^{T_*} G(t)\ .$$

We define the norm in the Sobolev space $W_2^m(G(t))$ as

$$\|v(.,t)\|^{(m)}_{2,G(t)} = \max_\ell \|v(.,t)\|^{(m)}_{2,\Omega^{(\ell)}(t)}\ .$$

It is obvious that

$$\|v(.,t)\|^{(m)}_{2,G(t)} \le C_1 \max_\ell \|v(.,t)\|^{(m)}_{2,\omega^{(\ell)}(t)} \le C_2\, \|v(.,t)\|^{(2)}_{2,G(t)}\ .$$

Thus (4.5) implies

$$\max_{0\le t\le T_*} \|\Gamma_\varepsilon(t)\|^{(2)}_{2,\Gamma^0} \le M \,, \tag{4.6}$$

if

$$M\, C_1\, \delta_1^2 < \frac{1}{2} \tag{4.7}$$

and the rest of the proof for the case $n=2$ is the same as for the special case of the strip-like domain.

For the case $n=3$ we get

$$\begin{aligned} \max_{0\le t\le T_*} \|\Gamma_\varepsilon(t)\|^{(3)}_{2,\pi^{(\ell)}} + \sum_{k=0}^{2} \int_0^{T_*} \Big(\|D_t^k \theta^\varepsilon(.,t)\|^{(4-k)}_{2,\omega^{(\ell)}(t)}\Big)^2 dt \;\le \\ \le\; M\Big\{\eta \sum_{k=0}^{2} \int_0^{T_*} \Big(\|D_t^k \theta^\varepsilon(.,t)\|^{(4-k)}_{2,\Omega^{(\ell)}(t)}\Big)^2 dt + C(\eta)\Big\} \,. \end{aligned} \tag{4.8}$$

Taking maximum over all domains $\omega^{(\ell)}(t)$ and choosing η sufficiently small, we get

$$\max_{0\le t\le T_*} \|\Gamma_\varepsilon(t)\|^{(3)}_{2,\Gamma^0} + \sum_{k=0}^{2} \|D_t^k \theta^\varepsilon\|^{(4-k)}_{2,G_{T_*}} \le M \,. \tag{4.9}$$

These estimates permit us to satisfy the conditions (4.3) on some small interval $(0,T_*)$ which doesn't depend on ε and pass to the limit when $\varepsilon\to 0$.

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