PORTUGALIAE MATHEMATICA
Vol. 59 Fasc. 4 – 2002
Nova Série

BOUNDARY LAYERS IN WEAK SOLUTIONS OF HYPERBOLIC CONSERVATION LAWS. III. VANISHING RELAXATION LIMITS

K.T. Joseph and P.G. LeFloch

Recommended by J.P. Dias

Abstract: This is the third part of a series concerned with boundary layers in solutions of nonlinear hyperbolic systems of conservation laws. We consider here self-similar solutions of the Riemann problem, following a pioneering idea by Dafermos. The system under study is strictly hyperbolic but no assumption of genuine nonlinearity is made. The boundary is possibly characteristic, that the sign of the characteristic speed near the boundary is not known a priori. We investigate the effect of vanishing relaxation terms on the solutions of the Riemann problem. We show that the boundary Riemann problem with relaxation admits continuous solutions that remain uniformly bounded in the total variation norm. Following the second part of this series, we derive the necessary uniform estimates near the boundary which allow us to describe the structure of the boundary layer even when the boundary is characteristic. Our analysis provides still a new approach to the existence of Riemann solutions for systems of conservation laws.

1 – Introduction

We continue our investigation [6, 7, 8] of the boundary and initial value problem for nonlinear hyperbolic systems of conservation laws

$$\partial_t u + \partial_x f(u) = 0\,, \qquad u = u(x,t) \in \mathcal{B}(u_*, \delta_0)\;, \tag{1.1}$$

where $\mathcal{B}(_*, \delta_0) \subset \mathbb{R}^N$ is the open ball with center u_* and (small) radius δ_0, and the

Received: February 4, 2002.
Mathematics Subject Classification: Primary 35L65; Secondary 76L05.
Keywords and Phrases: conservation law; shock wave; boundary layer; vanishing relaxation method; self-similar solution.

flux-function $f\colon \mathcal{B}(u_*, \delta_0) \to \mathbb{R}^N$ is a smooth mapping such that $A(u) := Df(u)$ admits N real and distinct eigenvalues denoted by

$$\lambda_1(u) < \dots < \lambda_N(u) \ ,$$

and corresponding basis of left- and right-eigenvectors $l_j(u)$ and $r_j(u)$, $1 \le j \le N$.

It is well-known that weak solutions of (1.1) are not uniquely determined by their boundary and initial data. Parts I and II of this series were concerned with the selection of admissible solutions via the vanishing viscosity method. Here, we aim at constructing weak solutions by the *zero-relaxation method*. Mathematical studies of the effect of relaxation on discontinuous solutions of nonlinear hyperbolic equations go back to the works of Liu [14] and Jin and Xin [9], in particular. See also the review by Natalini [16]. For general properties of systems of conservation laws we refer to the monographs [10, 11, 20].

Given a constant $a > 0$ such that

$$-a < \lambda_1(u) < \dots < \lambda_N(u) < a \, , \qquad u \in \mathcal{B}(u_*, \delta_0) \ , \tag{1.2}$$

we consider the relaxation approximation associated with (1.1)

$$\begin{aligned} &\partial_t u^\varepsilon + \partial_x v^\varepsilon = 0 \ , \\ &\partial_t v^\varepsilon + a^2 \, \partial_x u^\varepsilon = \frac{1}{\varepsilon} \Big(f(u^\varepsilon) - v^\varepsilon \Big) \ , \end{aligned} \tag{1.3}$$

where $u^\varepsilon = u^\varepsilon(x,t)$ and $v^\varepsilon = v^\varepsilon(x,t)$ are the unknowns and $\varepsilon > 0$ (the relaxation) is a parameter tending to zero. As in [8], we restrict attention to self-similar solutions, that is, solutions depending on the variable $\xi = x/t$ only:

$$\begin{aligned} &-\xi \, u^{\varepsilon\prime} + v^{\varepsilon\prime} = 0 \ , \\ &-\xi \, v^{\varepsilon\prime} + a^2 u^{\varepsilon\prime} = \frac{1}{\varepsilon} \Big(f(u^\varepsilon) - v^\varepsilon \Big) \ . \end{aligned} \tag{1.4a}$$

We search for a smooth solution $(u^\varepsilon, v^\varepsilon)$ defined on a bounded interval $[b,c]$ and satisfying the boundary conditions

$$u^\varepsilon(b) = u_L \, , \qquad u^\varepsilon(c) = u_R \ , \tag{1.4b}$$

where u_L and u_R are given in $\mathcal{B}(u_*, \delta_0)$, and b and c are chosen such that

$$\begin{aligned} &-a < b < c < a \ , \\ &\sup_{u \in \mathcal{B}(u_*, \delta_0)} \lambda_N(u) \ < \ c \ . \end{aligned}$$

The first condition is fundamental for the point of view of linear stability of the relaxation approximation. We stress that no inequality is imposed between b and the eigenvalues λ_j, so that the boundary $\xi = b$ may be characteristic. On the other hand, for simplicity in the presentation and without loss of generality, we assume that the boundary $\xi = c$ is not characteristic. Our purpose is to extend the analysis in [8] (concerned with the vanishing viscosity method) to the relaxation approximation, which introduces new technical difficulties.

First of all, we prove in this paper that the boundary-value problem (1.4) admits a smooth solution $(u^\varepsilon, v^\varepsilon)$ which is of uniformly bounded total variation. Our analysis here generalizes previous works on self-similar, vanishing viscosity approximations by Dafermos [1], Dafermos and DiPerna [2], Fan [4], Fan and Slemrod [5], LeFloch and Rohde [12], LeFloch and Tzavaras [13], Slemrod [17, 18], Slemrod and Tzavaras [19], Tzavaras [21], and the authors in [8].

Next, the limiting behavior of u^ε, as ε goes to zero, is investigated by distinguishing between three different regimes:

(i) There is no effect due to the boundary when

$$b < \inf \lambda_1 \ . \tag{1.5a}$$

(ii) There is some effect due to the boundary, and the boundary may be characteristic, when there exits an integer p such that

$$\inf \lambda_p < b < \sup \lambda_p \ . \tag{1.5b}$$

(iii) There is some effect of the boundary but the boundary $\xi = b$ is not characteristic when

$$\sup \lambda_{p-1} < b < \inf \lambda_p(u) \ . \tag{1.5c}$$

We will see that, when b satisfies (1.5a) the limit of u^ε solves the standard Riemann problem associated with the data (1.4b). In the cases (1.5b) and (1.5c), the boundary condition $u(b) = u_L$ *is not satisfied* (in general) by the limit-function u since a *boundary layer* arises near $\xi = b$. In fact, we show that the limit-function

$$u(\xi) = \lim_{\varepsilon \to 0} u^\varepsilon(\xi)$$

satisfies the *boundary Riemann problem* in the interval $[b, c]$

$$-\xi\, u' + f(u)' = 0 \ , \tag{1.6a}$$

$$u(c) = u_R \ , \tag{1.6b}$$

$$u(b+) \in \mathcal{E}(u_L) \ , \tag{1.6c}$$

where the *boundary set* $\mathcal{E}(u_L)$ is determined from the boundary data u_L. We recall that the initial and boundary value problem for the nonlinear hyperbolic equation (1.6a) is usually not well-posed when the boundary data are required in the (strong) sense $u(b+) = u_L$. This latter condition must be weakened, as was pointed out by Dubois and LeFloch [3]. We will also rely here on the technique developed in [7] for vanishing viscosity limits, to rigorously derive the boundary set $\mathcal{E}(u_B)$ and to describe its local structure.

We conclude this introduction with the basic reduction which allows us to reduce the first-order system of $2N$ equations (1.4) to a second-order system of N equations. Taking derivatives with respect to ξ in both equations (1.4a) we find

$$\begin{aligned} -\xi\, u' u'' - u' u' + v'' &= 0 \, , \\ -\xi\, v'' - v' + a^2\, u'' &= \frac{1}{\varepsilon} \Big(Df(u)\, u' - v' \Big) \, . \end{aligned}$$

Eliminating v we obtain a single equation for u

$$-\xi\, (\xi\, u'' + u') - \xi\, u' + a^2\, u'' = \frac{1}{\varepsilon} \Big(Df(u)\, u' - \xi\, u' \Big) \, ,$$

which can be rewritten in the form

$$\varepsilon (a^2 - \xi^2)\, u'' - \Big(Df(u) + (2\,\varepsilon - 1)\, \xi \Big)\, u' = 0 \, . \tag{1.7}$$

We will search for a solution u^ε of (1.7), defined on the interval $[b,c]$ and satisfying the boundary conditions (1.4b). The function v^ε is recovered from u^ε thanks to the relation

$$v^\varepsilon = \varepsilon\, (a^2 - \xi^2)\, u^{\varepsilon\prime} + f(u^\varepsilon) \, , \tag{1.8}$$

which follows from (1.2a) by computing $v^{\varepsilon\prime} = \xi\, u^{\varepsilon\prime}$ from the first equation and substituting in the second one.

2 – Scalar Conservation Laws

In this section we consider the scalar case $f\colon R^1 \to R^1$. The equations (1.7) becomes

$$\varepsilon (a^2 - \xi^2)\, u^{\varepsilon\prime\prime} - \Big(f'(u^\varepsilon) + (2\,\varepsilon - 1)\, \xi \Big)\, u^{\varepsilon\prime} = 0 \tag{2.1}$$

on $[b,c]$ where $-a < b < c < a$ with boundary conditions

$$u^\varepsilon(b) = u_L \, , \qquad u^\varepsilon(c) = u_R \, . \tag{2.2}$$

We assume that u_L and u_R satisfy

$$f'(u_L) \in [b,c]\,, \qquad f'(u_R) \in [b,c]\ . \tag{2.3}$$

To solve (2.1) with the boundary conditions (2.2) we reformulate the problem in an integral form. Precisely, we rewrite (2.1) as

$$\varepsilon\, u^{\varepsilon\prime\prime} = \frac{f'(u^\varepsilon) + (2\,\varepsilon - 1)\,\xi}{(a^2 - \xi^2)}\ u^{\varepsilon\prime}\ .$$

Setting

$$g^\varepsilon(\xi) \ = \int_\alpha^\xi \frac{(1 - 2\,\varepsilon)\,\xi - f'(u^\varepsilon)}{(a^2 - \xi^2)}\ ds \tag{2.4}$$

for some given α in $[b,c]$, we can integrate the above equation and get

$$u^\varepsilon(\xi)' = \ (u_R - u_L)\ \frac{e^{\frac{-g_\varepsilon(\xi)}{\varepsilon}}}{\displaystyle\int_b^c e^{\frac{-g_\varepsilon(s)}{\varepsilon}}\ ds}\ . \tag{2.5}$$

In integrating (2.5) once and using the boundary conditions (2.2), we find

$$u^\varepsilon(\xi) \ = \ u_L + (u_R - u_L)\ \frac{\displaystyle\int_b^\xi e^{\frac{-g_k(s)}{\varepsilon}}\ ds}{\displaystyle\int_b^c e^{\frac{-g_\varepsilon(s)}{\varepsilon}}\ ds}\ . \tag{2.6}$$

Solving the integral equation (2.6) is equivalent to finding a fixed point of the map

$$F(u) \ := \ u_L + (u_R - u_L)\ \frac{\displaystyle\int_b^\xi e^{\frac{-g_k(s)}{\varepsilon}}\ ds}{\displaystyle\int_b^c e^{\frac{-g_\varepsilon(s)}{\varepsilon}}\ ds} \tag{2.7}$$

with g given by (2.4). Let K be the set of all continuous functions on $[b,c]$ which take values in the interval $[\min(u_L,u_R), \max(u_L,u_R)]$. This set is a bounded closed and convex subset of the Banach space $C[b,c]$ of all continuous functions on $[b,c]$ endowed with the uniform topology. It is clear that F maps K into K because the right-hand side of (2.7) is a convex combination of u_L and u_R. Let us also show that the map $F\colon\ K \to K$ is compact. Let u_n be a sequence in K. Let

$$F(u_n^\varepsilon)(\xi) \ := \ u_L + (u_R - u_L)\ \frac{\displaystyle\int_b^\xi e^{\frac{-g_{n-1}(s)}{\varepsilon}}\ ds}{\displaystyle\int_b^c e^{\frac{-g_{n-1}(s)}{\varepsilon}}\ ds} \tag{2.8a}$$

where

$$g_n^\varepsilon(\xi) := \int_\alpha^\xi \frac{(1-2\,\varepsilon)\,s - f'(u_n)}{(a^2-s^2)}\,ds\ . \tag{2.8b}$$

Since

$$F(u_n^\varepsilon)(\xi) \in \Big[\min(u_L,u_R),\max(u_L,u_R)\Big] \tag{2.9}$$

from (2.8b) it follows that

$$|g_n^\varepsilon(\xi)| \,\le\, \frac{(1-2\,\varepsilon)\,a + a}{a^2-a_*^2}\,2\,a \,\le\, \frac{4\,a^2}{a^2-a_*^2}\ ,$$

where $0 < a_* < a$ is a constant such that $f'(u_L), f'(u_R) \in [-a_*, a_*]$. Using the above estimate together with

$$F(u_n^\varepsilon)'(\xi) \,=\, (u_R-u_L)\,\frac{e^{\frac{-g_{n-1}(\xi)}{\varepsilon}}}{\displaystyle\int_b^c e^{\frac{-g_{n-1}(s)}{\varepsilon}}\,ds}$$

we have

$$|F(u_n^\varepsilon)'(\xi)| \,\le\, \frac{|u_B-u_L|}{(b-c)}\,e^{\frac{8\,a^2}{\varepsilon(a^2-a_*^2)}}\ . \tag{2.10}$$

Hence, for each fixed $\varepsilon > 0$ (2.9) and (2.10) provide us with uniform estimates for u_n^ε and its derivatives. By Ascoli Theorem, the sequence $F(u_n)$ is compact. Now, by Schauder's fixed point theorem there must exits $u \in K$ such that $F(u) = u$. This completes the existence of a solution to the equation (2.6). Furthermore, this solution is twice continuously differentiable if $f(u)$ is and u^ε satisfy the estimates

$$u^\varepsilon(\xi) \in \Big[\min(u_L,u_R),\max(u_L,u_R)\Big]\,, \qquad \int_b^c |u^\varepsilon(\xi)'|\,ds \le |u_R-u_L|\ . \tag{2.11a}$$

Using (2.11a) in $v^{\varepsilon\prime} = \xi\,u^{\varepsilon\prime}$ we get

$$\int_b^c |v^\varepsilon(\xi)'|\,ds \,\le\, b\,|u_R-u_L|\ . \tag{2.11b}$$

Additionally, by (1.8) we find

$$v^\varepsilon - f(u^\varepsilon) \,=\, \varepsilon\,(a^2-\xi^2)\,u^{\varepsilon\prime}\ . \tag{2.12}$$

This completes the proof of existence of the solution $(u^\varepsilon, v^\varepsilon)$ to the problem for (1.2), together wih the uniform total variation estimates.

Now, to study the singular limit $\varepsilon \to 0$ we proceed the following way. Because of the estimate (2.11a) the right-hand side of (2.12) tends to 0 in L^1. So, it follows that $(f(u^\varepsilon) - v^\varepsilon) \to 0$ as $\varepsilon \to 0$ in L^1 and, hence, almost everywhere in (b, c) along a subsequence. But, by the estimates (2.11), u^ε is compact and there is a subsequence which converges almost everywhere to a function u. It follows that, along a subsequence, v^ε converges and it limits coincides with $f(u)$. In fact, by (2.11a) and (2.12),

$$|v^\varepsilon - f(u^\varepsilon)|_{L^1[b,c]} \leq C\,\varepsilon\ .$$

Then, from the first equation in (1.4a) we get

$$-\xi\, u' + f(u)' = 0$$

in the sense of distributions in (b, c). Furthermore, the limit u satisfies the entropy condition

$$-\xi\, p(u)_\xi + q(u)_\xi \leq 0$$

for all entropy pairs $(p(u), q(u))$ with $p(u)$ convex. This follows on passing to the limit in

$$(2\,\varepsilon - 1)\,\xi\, p(u^\varepsilon)_\xi + q(u^\varepsilon)_\xi \leq \varepsilon\,(a^2 - x^2)\, p(u^\varepsilon)_{\xi\xi}\ .$$

With regard to the boundary condition for u, we distinguish between sereval cases. When $b < \lambda^m = \min(f'(u_L), f'(u_R))$, u satisfies the boundary conditions (2.2). In fact, we even have the property

$$u(\xi) = \begin{cases} u_L, & \xi < \lambda^m, \\ u_R, & \xi > \lambda^M\ . \end{cases} \tag{2.13}$$

To prove this, consider some small $\delta > 0$. It is easy to see (see Theorem 3.1, estimate (3.16b) below) that, in the region $\xi < \lambda^m - \delta$,

$$\begin{aligned} |u^\varepsilon(\xi) - u_L| &\leq |u_R - u_L|\,\frac{C}{\varepsilon}\int_b^\xi e^{\frac{-(x-\lambda^m)^2}{2\varepsilon a^2}}\,dx \\ &\leq |u_R - u_L|\,\frac{C}{\varepsilon}\,(c-b)\,e^{\frac{-\delta^2}{2\varepsilon a^2}}\ . \end{aligned} \tag{2.14a}$$

Similarly, for $\xi > \lambda^M + \delta$, $\lambda^M = \max(f'(u_L), f'(u_R))$,

$$\begin{aligned} |u^\varepsilon(\xi) - u_R| &\leq |u_R - u_L|\,\frac{C}{\varepsilon}\int_\xi^c e^{\frac{-(x-\lambda^M)^2}{2\varepsilon a^2}}\,dx \\ &\leq |u_R - u_L|\,\frac{C}{\varepsilon}\,(c-b)\,e^{\frac{-\delta^2}{2\varepsilon a^2}}\ . \end{aligned} \tag{2.14b}$$

From the estimate (2.14) it follows that u^ε converges uniformly outside the interval $[\lambda^m - \delta, \lambda^M + \delta]$ to the function given by (2.13), for each $\delta > 0$. This proves (2.13). So, the limit function u can be extended by continuity to the left of λ^m and coincides with u_L and to the right of λ^M with u_R. We arrive at a weak solution to the Riemann problem with left- and right-hand initial data u_L and u_R, respectively.

We now treat the case $\lambda^m < b < \lambda^M$. In this case, the boundary condition $u(b) = u_L$ is generally not satisfied and, in the passage to the limit, a boundary layer is formed and the admissible boundary value belongs to a boundary set defined from the boundary layer. The corresponding ODE will be rigorously derived later in section 4, for general systems. Here, we content ourselves with a leading-order perturbation argument. Introduce the new variable $y = \frac{\xi - b}{\varepsilon}$ and set $V^\varepsilon(y) = u^\varepsilon(b + \varepsilon\, y)$ for $0 \le y \le \frac{c-b}{\varepsilon}$. From (2.1) we get

$$\varepsilon \left(a^2 - (\varepsilon\, y + b)^2\right) V^{\varepsilon\prime\prime} - \left(f'(V^\varepsilon) + (2\,\varepsilon - 1)\,(b + \varepsilon\, y)\right) V^{\varepsilon\prime} = 0\ . \tag{2.15}$$

Expanding in the form $V^\varepsilon = V + o(1)$ and keeping higher-order terms only, we get for $y > 0$

$$(a^2 - b^2)\, V'' = f(V)' - b\, V'\ . \tag{2.16t}$$

Since u^ε is of uniformly bounded variation, so is V. Thus, there exist V_0 and V_∞ such that $V(\infty) = V_\infty$ and $V(0+) = V_0$. It can be seen from (2.6) that $V_0 = u_L$. Integrating (2.16) from y to ∞ we arrive at the equation for the boundary layer:

$$\begin{aligned} &(a^2 - b^2)\, V' = f(V) - f(V_\infty) - b\, V + b\, V_\infty\,, \qquad y > 0\,,\\ &V(0) = u_L\,, \qquad V(\infty) = V_\infty\ . \end{aligned} \tag{2.17}$$

Consider the special case when the flux $f(u)$ is genuinely nonlinear, in other words $f(u)$ is strictly convex. Let $f^*(u)$ be the convex dual of f. Let $u^* = f^{*\prime}(b)$. Given u_L, let u_L^* be the unique solution of

$$f(u) - b\, u = f(u_L) - b\, u_L\ ,$$

which is not equal to u_L itself. A straightforward application of Theorem 4.1 in [7] shows that the set of all states V_∞ for which (2.17) has a solution is the set

$$\tilde{\mathcal{E}}(u_L) = \begin{cases} (-\infty, u_L^*) \cup \{u_L\}, & u_L > u_*,\\ (-\infty, u_*], & u_L \le u_*\ . \end{cases} \tag{2.18}$$

We know from [7] and the references therein that, for convex conservation laws, the problem (1.6) together with the boundary set $\mathcal{E}(u_L) = \tilde{\mathcal{E}}(u_L) \cup \{u_L^*\}$, is well posed. A more careful derivation of the boundary layer (carried out in section 4) would show that the boundary value of the limit namely $u(b+)$ satisfies

$$f(V_\infty) - b\,V_\infty \;=\; f\Big(u(b+)\Big) - b\,u(b+)\;,$$

which shows that indeed the trace $u(b+)$ belongs to the set $\mathcal{E}(u_L)$.

3 – Wave Interaction Estimates

In this section, we study a linearized version of the system of equations (1.7). Given data u_L and $u_R \in \mathcal{B}(u_*, \delta)$ for some $\delta < \delta_0$, the unknown function u^ε takes its values in the ball $\mathcal{B}(u_*, C_*\,\delta)$ with $C_*\,\delta < \delta_0$. For δ_0 sufficiently small the eigenvalues of $Df(u)$ are separated, in the sense that

$$\begin{gathered} -a < \lambda_1^m < \lambda_1(u) < \lambda_1^M < \lambda_2^m < \cdots < \lambda_{N-1}^M < \lambda_N^m < \lambda_N(u) < \lambda_N^M < a\;, \\ u \in \mathcal{B}(u_*, \delta_0)\;. \end{gathered} \tag{3.1}$$

Since $Df(u)$ depends smoothly upon u, one can ensure that $\lambda_k^M - \lambda_k^m = O(\delta_0)$. Given $u_L, u_R \in \mathcal{B}(u_*, \delta)$ for some $\delta < \delta_0$, we are going to construct a solution u^ε of (1.7) having uniformly bounded variation, i.e.,

$$TV(u^\varepsilon) := \int_b^c |u^{\varepsilon\prime}(\xi)|\,d\xi \;\le\; C\;. \tag{3.2}$$

This is done in several steps by dealing, in this section, with a linearized version of (1.7) and, then in Section 4, with the fully nonlinear problem.

The second-order equation for $u = u^\varepsilon\colon [b,c] \to \mathbb{R}^N$ is

$$\varepsilon\,u'' = \frac{Df(u) + (2\,\varepsilon - 1)\,\xi}{a^2 - \xi^2}\;u' \tag{3.3a}$$

and the boundary conditions read

$$u(b) = u_L\,, \qquad u(c) = u_R\;. \tag{3.3b}$$

On the other hand, recall that v^ε (appearing in (1.4)) is recovered from (1.8) and that a uniform bound on $TV(v^\varepsilon)$ will be a direct consequence of (3.2) and

$$v' = \xi\,u'.$$

We aim at proving the existence of the solution u^ε of (3.3), taking values in $\mathcal{B}(u_*, C_*\,\delta)$ (with $C_*\,\delta < \delta_0$) and satisfying the estimate (3.2). Following Tzavaras [21] we set

$$u^{\varepsilon\prime}(\xi) = \sum_{k=1}^{N} a_k^\varepsilon(\xi)\, r_k(u^\varepsilon(\xi)) \ , \tag{3.4}$$

where the "wave strengths" a_k^ε are determined by $a_k^\varepsilon(\xi) = l_k(u^\varepsilon(\xi)) \cdot u^{\varepsilon\prime}(\xi)$.

From (3.3a) and (3.4) we deduce that

$$\begin{aligned} \sum_{k=1}^{N} \frac{\Big(\lambda_k(u^\varepsilon) - (1-2\,\varepsilon)\,\xi\Big)}{a^2-\xi^2}\, a_k^\varepsilon\, r_k(u^\varepsilon) \ &= \ \varepsilon \left(\sum_{k=1}^{N} a_k^\varepsilon\, r_k(u^\varepsilon)\right)' \\ &= \ \varepsilon \sum_{k=1}^{N} a_k^{\varepsilon\prime}\, r_k(u^\varepsilon) + \varepsilon \sum_{j,k=1}^{N} a_j^\varepsilon\, a_k^\varepsilon\, Dr_k(u^\varepsilon)\cdot r_j(u^\varepsilon) \ . \end{aligned} \tag{3.5}$$

Multiplying (3.5) by $l_k(u^\varepsilon)$ $(k = 1, ..., N)$ successively and setting

$$\beta_{ijk}(u^\varepsilon) := \ l_k(u^\varepsilon)\cdot Dr_i(u^\varepsilon)\cdot r_j(u^\varepsilon) \ , \tag{3.6}$$

we find

$$a_k^{\varepsilon\prime} + \frac{(1-2\,\varepsilon)\,\xi - \lambda_k(u^\varepsilon)}{\varepsilon(a^2-\xi^2)}\, a_k^\varepsilon \ = \ \sum_{i,j=1}^{N} \beta_{ijk}(u^\varepsilon)\, a_i^\varepsilon\, a_j^\varepsilon\,, \qquad k = 1, ..., N \ . \tag{3.7}$$

The boundary conditions (3.3b) yield

$$\sum_{k=1}^{N} \int_b^c a_k^\varepsilon\, r_k(u^\varepsilon)\, d\xi \ = \ u_R - u_L \ .$$

The uniform BV bound (3.2) on u^ε is equivalent to the uniform L^1 bound

$$\sum_{k=1}^{N} \int_b^c |a_k^\varepsilon|\, d\xi \ \le \ C \ .$$

The relations (3.6)–(3.7) form a first-order system of coupled, ordinary differential equations. The function u^ε arising in the coefficients $\lambda_k(u^\varepsilon)$ and $\beta_{ijk}(u^\varepsilon)$ is determined implicitly by (3.4) and (3.3b), namely

$$u^\varepsilon(\xi) \ = \ u_L + \sum_{k=1}^{N} \int_b^\xi a_k^\varepsilon(x)\, r_k(u^\varepsilon(x))\, dx \ . \tag{3.8}$$

We start by studying a set of decoupled, linearized homogeneous equations. Consider the equation

$$\varphi_k^{\varepsilon\,\prime} + \frac{(1-2\,\varepsilon)\,\xi - \lambda_k(w)}{\varepsilon(a^2-\xi^2)}\,\varphi_k^\varepsilon \;=\; 0 \tag{3.9}$$

for $k=1,...,N$, where $w\colon\,[0,\infty)\to\mathcal{B}(u_*,\delta_0)$ is a given, continuous function. It admits a unique (positive) solution with "unit mass", i.e.,

$$\int_b^c \varphi_k^\varepsilon(x)\,dx \;=\; 1\;, \tag{3.10}$$

namely

$$\varphi_k^\varepsilon(\xi) = \frac{e^{\frac{-h_k(\xi)}{\varepsilon}}}{\displaystyle\int_b^c e^{\frac{-h_k(x)}{\varepsilon}}\,dx}\,, \qquad h_k(\xi) = \int_{\rho_k}^{\xi} \frac{(1-2\,\varepsilon)\,x - \lambda_k(w(x))}{a^2-x^2}\,dx \tag{3.11}$$

with $\rho_k\in[b,c]$ still to be determined. Now, h_k can be written in a more convenient form:

$$\begin{aligned} h_k(\xi) \;&=\int_{\rho_k}^{\xi}\left(\frac{(1-2\,\varepsilon)\,x - \lambda_k(w(x))}{a^2-\xi^2}\right)dx \\ &= \int_{\rho_k}^{\xi}\frac{-2\,\varepsilon\,x}{a^2-x^2}\,dx \;+\int_{\rho_k}^{\xi}\frac{x-\lambda_k(w(x))}{a^2-x^2}\,dx \\ &= \varepsilon\,\log\frac{a^2-\xi^2}{a^2-\rho_k^2} \;+\int_{\rho_k}^{\xi}\frac{x-\lambda_k(w(x))}{a^2-x^2}\,dx\;. \end{aligned} \tag{3.12}$$

Using (3.12) in (3.11) we get

$$\begin{aligned} &\varphi_k^\varepsilon(\xi) = \frac{(a^2-\xi^2)^{-1}\,e^{\frac{-g_k(\xi)}{\varepsilon}}}{I_{k\varepsilon}}\;, \\ &I_{k\varepsilon} = \int_b^c (a^2-x^2)^{-1}\,e^{\frac{-g_k(x)}{\varepsilon}}\,dx\,, \qquad g_k(\xi) = \int_{\rho_k}^{\xi}\left(\frac{(x-\lambda_k(w(x))}{a^2-x^2}\right)dx\;. \end{aligned} \tag{3.13}$$

When emphasis will be needed, we write explicitly $\varphi_k^\varepsilon = \varphi_k^\varepsilon(\xi;w)$ and $g_k = g_k(\xi;w)$. Observe that φ_k^ε does not depend on the scalar ρ_k. It will be convenient to choose $\rho_k\in[b,c]$ to be any point achieving a *global minimum* of g_k, i.e.,

$$g_k(\rho_k) \;=\; \min_{[b,c]} g_k\;.$$

Since w is continuous, when $\rho_k\in(b,c]$ we have

$$g_k(\xi)\geq 0 \;\text{ for all }\xi\,, \qquad g_k(\rho_k)=0\,, \qquad g_k'(\rho_k)=0\;. \tag{3.14}$$

However, ρ_k may also be the boundary point $\rho_k = b$, but $\rho_k < c$ as can be checked from our non-characteristic assumption $\sup \lambda_N < c$.

Observe that the behavior at $\xi = b$ depends on the position of b with respect to the eigenvalues λ_j. For instance, if $b < \lambda_1^m$ then we have $\rho_k > b$. In general, we can define

$$p(b) = \min\Big\{ k \,/\, b < \lambda_k^M \Big\} . \tag{3.15}$$

If $p(b) \geq 1$, $\rho_k = b$ for all $k < p(b)$ but ρ_k is bounded away from b for all $k > p(b)$. The characteristic case $k = p(c)$ with $\lambda_{p(c)}^m \leq b < \lambda_{p(c)}^M$, for which we may have $\rho_k = b$ or $\rho_k > b$, will require careful estimates in the forthcoming analysis.

Given b, c it is convenient to choose a_* such that $-a < -a_* < b < \lambda_N^M < c < a_* < a$. This choice of a_* is useful in the proof of the main properties on the functions φ_k^ε and their interactions stated in the following theorem.

Theorem 3.1. *For δ_0 small enough, there exists a constant $C > 0$ independent of ε for which the following estimates hold. Let $d_k = \lambda_k^M - \lambda_k^m > 0$, then for all $k < p(b)$*

$$0 < \varphi_k^\varepsilon(\xi) \leq \frac{C}{\varepsilon}\, e^{-\frac{(\xi-b)}{2\varepsilon a^2}(\xi + b - 2\,\lambda_k^M)}, \qquad b < \xi < c , \tag{3.16a}$$

while for $k = p(b)$

$$0 \;<\; \varphi_p^\varepsilon(\xi) \;\leq\; \begin{cases} \dfrac{C}{\varepsilon}, & b < \xi < \lambda_p^M, \\[2ex] \dfrac{C}{\varepsilon}\, e^{-\frac{(\xi - \lambda_p^M)^2}{2\varepsilon a^2}}, & \lambda_p^M < \xi < c , \end{cases} \tag{3.16b}$$

and for all $k > p(b)$

$$0 \;<\; \varphi_k^\varepsilon(\xi) \;\leq\; \begin{cases} \dfrac{C}{\varepsilon}\, e^{-\frac{(\xi - \lambda_k^m)^2}{2\varepsilon a^2}}, & b < \xi < \lambda_k^m, \\[2ex] \dfrac{C}{\varepsilon}, & \lambda_k^m < \xi < \lambda_k^M, \\[2ex] \dfrac{C}{\varepsilon}\, e^{-\frac{(\xi - \lambda_k^M)^2}{2\varepsilon a^2}}, & \lambda_k^M < \xi < c . \end{cases} \tag{3.16c}$$

Suppose that λ_k is a constant. Then, if $k \leq p(b)$, we have

$$\varphi_k^\varepsilon(\xi) = \frac{C}{\sqrt{\varepsilon}}\, e^{-\frac{(\xi-b)^2}{2\varepsilon a^2}(\xi + b - 2\lambda_k)} \tag{3.17a}$$

and if $p(b) < k$,

$$\varphi_k^\varepsilon(\xi) = \frac{C}{\sqrt{\varepsilon}}\, e^{\frac{-(\xi-\lambda_k)^2}{2\varepsilon a^2}} \; . \tag{3.17b}$$

Set

$$c_k = \begin{cases} \lambda_k^M, & k \geq p(b), \\ 0, & k < p(b), \end{cases}$$

and consider the wave interaction *coefficients* $(k, m, n = 1, 2, ..., N)$

$$F_{kmn}^\varepsilon(\xi) := (a^2 - \xi^2)^{-1}\, e^{-\frac{g_k(\xi)}{\varepsilon}} \int_{c_k}^{\xi} (a^2 - x^2)\, e^{\frac{g_k}{\varepsilon}}\, \varphi_m^\varepsilon\, \varphi_n^\varepsilon \, dx \; . \tag{3.18}$$

Then the following uniform estimates hold

$$|F_{kmn}^\varepsilon| \leq C \sum_{j=1}^{N} \varphi_j^\varepsilon \; . \tag{3.19}$$

The terms F_{kmn}^ε will arise in estimating the coupling terms in the right-hand side of (3.7). Theorem 3.1 implies that, roughly speaking, the limiting measure $\bar\varphi_k := \lim_{\varepsilon\to 0} \varphi_k^\varepsilon$ is supported in the interval spanned by the k-wave speed:

$$\text{supp } \bar\varphi_k \subset \{0\} \quad \text{for all} \;\; k < p(b) \; , \tag{3.20a}$$

$$\text{supp } \bar\varphi_p(c) \subset [0, \lambda_k^M] \quad \text{for} \;\; k = p(b) \; , \tag{3.20b}$$

$$\text{supp } \bar\varphi_k \subset [\lambda_k^m, \lambda_k^M] \quad \text{for all} \;\; k > p(b) \; . \tag{3.20c}$$

In particular, for $k < p(b)$, $\bar\varphi_k$ either is a Dirac measure supported at $\xi = b$, or else vanishes identically.

Proof of Theorem 3.1: For simplicity we omit the explicit dependence in ε throughout this proof. We will first derive (3.16) in the case $k < p(c)$. First we get a lower bound for the integral

$$\begin{aligned} I_k &:= \int_b^c (a^2 - x^2)^{-1}\, e^{-\frac{g_k(x)}{\varepsilon}}\, dx \\ &= \sqrt{\varepsilon} \int_{\frac{b-\rho_k}{\sqrt{\varepsilon}}}^{\frac{c-\rho_k}{\sqrt{\varepsilon}}} \Big(a^2 - (\rho_k + \eta\,\sqrt{\varepsilon})^2\Big)^{-1}\, e^{-\frac{g_k(\rho_k + \eta\sqrt{\varepsilon})}{\varepsilon}}\, d\eta \; . \end{aligned} \tag{3.21}$$

Since $k < p(c)$ we have $\rho_k = b$, using the change of variable $x = \rho_k + \sqrt{\varepsilon}\,\tau$ we get

$$\begin{aligned}\frac{g_k(\rho_k + \eta\sqrt{\varepsilon})}{\varepsilon} &= \frac{1}{\varepsilon}\int_b^{b+\eta\sqrt{\varepsilon}} \Big(\frac{x - \lambda_k(w(x))}{a^2 - x^2}\Big)\,dx \\ &= \int_0^{\eta} \left(\frac{\tau + \frac{1}{\sqrt{\varepsilon}}\Big(b - \lambda_k(w(b + \sqrt{\varepsilon}\,\tau))\Big)}{a^2 - (b + \sqrt{\varepsilon}\,\tau)^2}\right) d\tau \ .\end{aligned}$$

Since $b > \lambda_k^M$ and we are interested in $\eta \geq 0$

$$\begin{aligned} g_k(\rho_k + \eta\sqrt{\varepsilon}) &\leq \int_0^{\eta} \frac{\tau + \frac{1}{\sqrt{\varepsilon}}(b - \lambda_k^m)}{a^2 - a_*^2}\,d\tau \\ &\leq \frac{\eta^2}{2(a^2 - a_*^2)} + \frac{\eta}{\sqrt{\varepsilon}\,a^2 - a_*^2}\,(b - \lambda_k^m) \ .\end{aligned}$$

Using this in (3.21) we get,

$$\begin{aligned} I_k &\geq \frac{\sqrt{\varepsilon}}{(a^2 - a_*^2)}\int_0^{\frac{c-b}{\sqrt{\varepsilon}}} e^{-\frac{\eta^2}{2(a^2-a_*^2)} - \frac{b-\lambda_k^m}{\sqrt{\varepsilon}(a^2-a_*^2)}\eta}\,d\eta \\ &= \frac{\varepsilon}{a^2 - a_*^2}\int_0^{\frac{c-b}{\varepsilon}} e^{-\frac{\varepsilon\eta^2}{2(a^2-a_*^2)} - \frac{b-\lambda_k^m}{(a^2-a_*^2)}\eta}\,d\eta \\ &\geq \varepsilon\int_0^{c-b} e^{-\frac{\eta^2}{2(a^2-a_*^2)} - \frac{b-\lambda_k^m}{a^2-a_*^2}\eta}\,d\eta \\ &= C\,\varepsilon\ ,\end{aligned} \tag{3.22}$$

as ε is small. Since $\xi > b$,

(3.23)
$$g_k(\xi) = \int_b^{\xi}\Big(\frac{x - \lambda_k(w(x))}{a^2 - x^2}\Big)dx \geq \int_b^{\xi}\Big(\frac{x - \lambda_k^M}{a^2}\Big)dx \ = \ \frac{(\xi - b)}{2\,a^2}\,(\xi + b - 2\,\lambda_k^M)\,.$$

The estimate (3.16a) now follows from(3.21) and (3.22).

Consider next the case $k \geq p(c)$, for which either $\rho_k > 0$ if $k > p$ or else $\rho_k \geq 0$ if $k = p$. When $\rho_k = 0$, the same proof as above yields $I_k \geq C\varepsilon$. When $\rho_k > 0$, we have $g_k'(\rho_k) = 0$ and thus $\rho_k - \lambda_k(w(\rho_k)) = 0$. So we obtain

$$\frac{g_k(\rho_k + \eta\sqrt{\varepsilon})}{\varepsilon} = \frac{1}{\varepsilon}\int_{\rho_k}^{\rho_k+\eta\sqrt{\varepsilon}}\Big(\frac{x - \rho_k + \rho_k - \lambda_k(v(x))}{a^2 - x^2}\Big)\,dx\ .$$

Now if $\eta \geq 0$,

$$\begin{aligned}\frac{g_k(\rho_k + \eta\sqrt{\varepsilon})}{\varepsilon} &\leq \frac{1}{\varepsilon(a^2 - a_*^2)}\Big(\int_0^{\eta\sqrt{\varepsilon}} x\,dx + \frac{1}{\varepsilon}\,(\rho_k - \lambda_k^m)\,\eta\sqrt{\varepsilon}\Big) \\ &\leq \frac{\eta^2}{2(a^2 - a_*^2)} + \frac{\eta}{\sqrt{\varepsilon}\,a^2 - a_*^2}\,d_k\ .\end{aligned}$$

Similarly, if $\eta \leq 0$,

$$\frac{g_k(\rho_k + \eta\sqrt{\varepsilon})}{\varepsilon} \leq \frac{\eta^2}{2(a^2 - a_*^2)} - \frac{\eta}{\sqrt{\varepsilon}\, a^2 - a_*^2}\, d_k \; .$$

These lead us to the lower bound:

$$\begin{aligned}
I_k &\geq \sqrt{\varepsilon}\biggl(\int_{\frac{b-\rho_k}{\sqrt{\varepsilon}}}^{0} e^{-\frac{\eta^2}{2(a^2-a_*^2)} - \frac{\eta}{\sqrt{\varepsilon}(a^2-a_*^2)} d_k}\, d\eta + \int_0^{\frac{c-\rho_k}{\sqrt{\varepsilon}}} e^{-\frac{\eta^2}{2(a^2-a_*^2)} + \frac{\eta}{\sqrt{\varepsilon}(a^2-a_*^2)} d_k}\, d\eta\biggr) \\
&= \varepsilon\biggl(\int_{\frac{b-\rho_k}{\varepsilon}}^{0} e^{-\frac{\varepsilon\eta^2}{2(a^2-a_*^2)} - \frac{\eta}{(a^2-a_*^2)} d_k}\, d\eta + \int_0^{\frac{c-\rho_k}{\varepsilon}} e^{-\frac{\varepsilon\eta^2}{2(a^2-a_*^2)} + \frac{\eta}{(a^2-a_*^2)} d_k}\, d\eta\biggr) \\
&\geq \varepsilon\biggl(\int_{b-\rho_k}^{0} e^{-\frac{\eta^2}{2(a^2-a_*^2)} - \frac{\eta}{(a^2-a_*^2)} d_k}\, d\eta + \int_0^{c-\rho_k} e^{-\frac{\eta^2}{2(a^2-a_*^2)} + \frac{\eta}{(a^2-a_*^2)} d_k}\, d\eta\biggr) \\
&= C\,\varepsilon \; .
\end{aligned} \tag{3.24}$$

Since $0 < e^{-\frac{g_k(\xi)}{\varepsilon}} \leq 1$, the estimate $\varphi_k \leq C/\varepsilon$ in (3.16b)–(3.16c) is established.

On the other hand, for $\xi \geq \lambda_k^M$ we have

$$g_k(\xi) = g_k(\lambda_k^M) + \int_{\lambda_k^M}^{\xi} \frac{(x - \lambda_k)}{a^2 - x^2}\, dx \geq \int_{\lambda_k^M}^{\xi} \Bigl(\frac{x - \lambda_k^M}{a^2}\Bigr)\, dx = \frac{(\xi - \lambda_k^M)^2}{2a^2} \; . \tag{3.25a}$$

Combining (3.24) with (3.25a), the estimates (3.16b)–(3.16c) in the region $\xi \geq \lambda_k^M$ are proven. Finally for $\rho_k > b$ and $b < \xi \leq \lambda_k^m$ a similar argument shows that

$$g_k(\xi) \geq \int_{\lambda_k^m}^{\xi} \frac{(x - \lambda_k^m)}{a^2}\, dx = \frac{(\xi - \lambda_k^m)^2}{2\,a^2} \; . \tag{3.25b}$$

This leads us to the estimates (3.16b)–(3.16c) in the region $b < \xi \leq \lambda_k^m$. The proof of (3.16) is completed.

When λ_k is a constant a direct calculation gives the estimate (3.17). In fact we have a better lower estimate for $I_{k\varepsilon}$, namely

$$I_{k\varepsilon} \geq C\sqrt{\varepsilon} \; .$$

In the rest of this proof we will often use the lower bound $I_k \geq C\,\varepsilon$. We will also need the upper bound for I_k. An easy direct calculation shows that

$$I_k \leq \frac{1}{2\,a} \log\Bigl[\frac{a+c}{a+b} . \frac{a-b}{a-c}\Bigr] \; . \tag{3.26}$$

We now estimate the interaction coefficients F_{kmn} given by (3.18). First, suppose that at least one of m or n coincide with k, for instance $n = k$. Then we find

$$
\begin{aligned}
F_{kmk}(\xi) &= (a^2-\xi^2)^{-1}\, e^{-\frac{g_k(\xi)}{\varepsilon}} \int_{c_k}^{\xi} (a^2-x^2)\, e^{\frac{g_k}{\varepsilon}}\, \varphi_m\, \varphi_k\, dx \\
&= \varphi_k \int_{c_k}^{\xi} \varphi_m\, dx \;\le\; \varphi_k(\xi)\ .
\end{aligned}
\tag{3.27}
$$

To estimate F_{kmn} when both m and n are not equal to k, we observe that

$$
|F_{kmn}| \le \frac{1}{2}(F_{km} + F_{kn})\ , \tag{3.28}
$$

where for all k

$$
F_{kj}(\xi) = \begin{cases} (a^2-\xi^2)^{-1}\, e^{-\frac{g_k(\xi)}{\varepsilon}} \displaystyle\int_{c_k}^{\xi} (a^2-x^2)\, e^{\frac{g_k}{\varepsilon}}\, \varphi_j(x)^2\, dx, & \xi \ge c_k, \\ (a^2-\xi^2)^{-1}\, e^{-\frac{g_k(\xi)}{\varepsilon}} \displaystyle\int_{\xi}^{c_k} (a^2-x^2)\, e^{\frac{g_k}{\varepsilon}}\, \varphi_j(x)^2\, dx, & \xi \le c_k\ , \end{cases} \tag{3.29}
$$

for $j = m, n$. So, it is sufficient to estimate now the coefficients F^{ε}_{km} for $k < m$ and $k > m$.

Case $k < m$**:** In the region $b \le c_k \le \xi$ we have

$$
\begin{aligned}
F_{km}(\xi) &= (a^2-\xi^2)^{-1}\, e^{\frac{-1}{\varepsilon}\int_{\rho_k}^{\xi}(\frac{y-\lambda_k}{a^2-x^2})\,dy} \int_{c_k}^{\xi} (a^2-x^2)\, e^{\frac{1}{\varepsilon}\int_{\rho_k}^{x}(\frac{y-\lambda_k}{a^2-x^2})\,dy}\, \varphi_m(x)^2\ dx \\
&= \frac{1}{I_m^2}\,(a^2-\xi^2)^{-1}\, e^{\frac{-1}{\varepsilon}\int_{\rho_m}^{\xi}(\frac{y-\lambda_m}{a^2-x^2})\,dy} \\
&\quad \cdot \int_{c_k}^{\xi} (a^2-x^2)^{-1}\, e^{\frac{-1}{\varepsilon}\int_{x}^{\xi}(\frac{\lambda_m-\lambda_k}{a^2-x^2})\,dy}\, e^{\frac{-1}{\varepsilon}\int_{\rho_m}^{x}(\frac{y-\lambda_m}{a^2-x^2})\,dy}\ dx \\
&\le \frac{O(1)}{\varepsilon}\, \varphi_m(\xi) \int_{c_k}^{\xi} e^{\frac{-1}{\varepsilon}\int_{x}^{\xi}(\frac{\lambda_m-\lambda_k}{a^2-x^2})\,dy}\ dx \\
&\le \frac{O(1)}{\varepsilon}\, \varphi_m(\xi) \int_{c_k}^{\xi} e^{-\frac{1}{\varepsilon}(\frac{\lambda_m^m-\lambda_k^M}{a^2-a_*^2})(\xi-x)}\ dx \\
&= \frac{O(1)}{(\lambda_m^m-\lambda_k^M)}\, \varphi_m(\xi) \left(1 - e^{-\frac{1}{\varepsilon}\left(\frac{\lambda_m^m-\lambda_k^M}{a^2-a_*^2}\right)(\xi-c_k)}\right)\ ,
\end{aligned}
$$

where we used $c_k \le x \le \xi$, $I_{k\varepsilon} \ge C\varepsilon$ and that due to the choice of ρ_m

$$
\int_{\rho_m}^{x} (y-\lambda_m)\, dy \;\ge\; 0\ .
$$

Since $\lambda_m^m - \lambda_k^M > 0$, it follows that

$$F_{km}(\xi) \le O(1)\,\varphi_m(\xi) \quad \text{for all} \;\; \xi \ge c_k \;. \tag{3.30}$$

Next, consider the region $b < \xi < c_k$. If $c_k = b$, there is nothing to prove. If $k \ge p(b)$ and thus $c_k > b$, we proceed as follows. An easy calculation based on the expression (3.29) of F_{km} gives

$$\begin{aligned} F_{km}(\xi) \;&=\; \frac{I_{k\varepsilon}}{I_{m\varepsilon}^2}\,\varphi_k(\xi) \\ &\quad \cdot \int_\xi^{c_k} (a^2 - x^2)^{-1} e^{\frac{-1}{\varepsilon}\int_{\rho_m}^{\rho_k}\frac{z-\lambda_m}{a^2-z^2}\,dz} . e^{\frac{-1}{\varepsilon}\int_{\rho_m}^{x}\frac{z-\lambda_m}{a^2-z^2}\,dz} . e^{\frac{-1}{\varepsilon}\int_{\rho_k}^{x}\frac{\lambda_m-\lambda_k}{a^2-z^2}\,dz}\, dx \\ &\le\; \frac{O(1)}{\varepsilon^2}\,\varphi_k(\xi)\, e^{\frac{-1}{\varepsilon}\int_{\rho_m}^{\rho_k}\frac{z-\lambda_m}{a^2-z^2}\,dz} . \int_\xi^{c_k} e^{\frac{-1}{\varepsilon}\int_{\rho_k}^{x}\frac{\lambda_m-\lambda_k}{a^2-z^2}\,dz}\, dx \;. \end{aligned} \tag{3.31}$$

Here again we used $I_{k\varepsilon} \ge C\varepsilon$. Now since $b < c_k \le \lambda_k^M < \lambda_m^m \le \rho_m$ and $\rho_k \le x \le c_k$, we have,

$$\int_{\rho_k}^{\rho_m} \frac{(y-\lambda_m)}{a^2-y^2}\,dy \;\le\; -\frac{(\lambda_m^m - \lambda_k^M)^2}{2\,a^2} \;,$$

$$-\int_{\rho_k}^{x} \frac{(\lambda_m-\lambda_k)}{a^2-y^2}\,dy \;\le\; (\lambda_m^M - \lambda_k^m)\,\frac{(\lambda_k^M - \lambda_k^m)}{a^2-a_*^2} \;.$$

Observe finally that

$$\beta_{km} := -\frac{(\lambda_m^m - \lambda_k^M)^2}{2\,a^2} + (\lambda_k^M - \lambda_k^m)\,\frac{(\lambda_k^M - \lambda_k^m)}{a^2-a_*^2} \;=\; -\frac{(\lambda_m^m - \lambda_k^M)^2}{2\,a^2} + O(\delta_0) \;<\; 0 \;.$$

Using this and the fact that $c_k - \xi \le c - b$ in (3.31) it follows that

$$F_{km}(\xi) \;\le\; C\,\frac{O(1)}{\varepsilon^2}\,\varphi_k(\xi)\,e^{\frac{\beta_{km}}{\varepsilon}} \;\le\; o(1)\,\varphi_k(\xi) \quad \text{for all} \;\; \xi \le c_k \;. \tag{3.32}$$

Combining (3.30) and (3.32) we get

$$F_{km}(\xi) \le O(1)\Big[\varphi_k(\xi) + \varphi_m(\xi)\Big] \quad \text{for all} \;\; b \le \xi \le c \;. \tag{3.33}$$

Case $k > m$**:** Suppose first that also $k \geq p(b)$ and thus $c_k = \lambda_k^M$.

First consider the region $\xi > c_k > b$. A simple calculation yields

$$\begin{aligned}
F_{km}(\xi) &= \frac{I_k}{I_m^2}\,\varphi_k(\xi) \\
&\quad \cdot \int_{c_k}^{\xi} (a^2 - x^2)^{-1}\, e^{\frac{1}{\varepsilon}\int_{\rho_k}^{x} \frac{(\lambda_m - \lambda_k)}{a^2 - y^2}\,dy}\, e^{\frac{-1}{\varepsilon}\int_{\rho_m}^{\rho_k} \frac{(y-\lambda_m)}{a^2-y^2}\,dy}\, e^{\frac{-1}{\varepsilon}\int_{\rho_m}^{x} \frac{(y-\lambda_m)}{a^2-y^2}\,dy}\, dx \\
&\leq \frac{O(1)}{\varepsilon^2}\,\varphi_k(\xi)\, e^{\frac{-1}{\varepsilon}\int_{\rho_m}^{\rho_k} \frac{(y-\lambda_m)}{a^2-y^2}\,dy} \int_{c_k}^{\xi} e^{\frac{1}{\varepsilon}\int_{\rho_k}^{x} \frac{(\lambda_m-\lambda_k)}{a^2-y^2}\,dy}\, dx \\
&\leq \frac{O(1)}{\varepsilon^2}\,\varphi_k(\xi)\, e^{\frac{-1}{\varepsilon}\int_{\rho_m}^{\rho_k} \frac{(y-\lambda_m)}{a^2-y^2}\,dy} \int_{c_k}^{\xi} e^{-\frac{1}{\varepsilon a^2}(\lambda_k^m - \lambda_m^M)(x-\rho_k)}\, dx \\
&\leq \frac{O(1)}{\varepsilon(\lambda_k^m - \lambda_m^M)}\,\varphi_k(\xi)\, e^{\frac{-1}{\varepsilon}\int_{\rho_m}^{\rho_k} \frac{(y-\lambda_m)}{a^2-y^2}\,dy}\, e^{-\frac{1}{\varepsilon a^2}(\lambda_k^m - \lambda_m^M)(c_k-\rho_k)}\,,
\end{aligned}$$

where we have used $\rho_k \leq c_k \leq x \leq \xi$. An easy calculation gives

$$-\int_{\rho_m}^{\rho_k} \frac{(y-\lambda_m)}{a^2-y^2}\,dy \;\leq\; \begin{cases} -\dfrac{1}{2\,a^2}\,(\rho_k - \lambda_m^M)^2, & \lambda_m^M > b\,, \\[2mm] -\dfrac{(b-\lambda_m^M)\,(\rho_k - b)}{a^2}, & \lambda_m^M < b\,. \end{cases}$$

This estimate will allow us to conclude in the case $k > p(b)$. When $k = p(b)$ and therefore $m < p(b)$ we also use the second term in the above expression, as follows:

$$-(b-\lambda_m^M)\,(\rho_{p(b)} - b) - (\lambda_{p(b)}^m - \lambda_m^M)\,(\lambda_{p(b)}^M - \rho_{p(b)}) < -(\lambda_{p(b)}^m - \lambda_m^M)\,(\lambda_{p(b)}^M - b) < 0\,.$$

Defining

$$\gamma_{km} = \begin{cases} -\dfrac{1}{2}\,(\lambda_k^m - \lambda_m^M)^2, & k > p(b)\,, \\[2mm] -(\lambda_{p(b)}^m - \lambda_m^M)\,(\lambda_{p(b)}^M - b), & k = p(b)\,, \end{cases}$$

we arrive at

$$F_{km}(\xi) \leq O(1)\, e^{\frac{\gamma_{km}}{\varepsilon a^2}}\,\varphi_k(\xi) \quad \text{for all } \xi \geq c_k\,. \tag{3.34}$$

Now consider the region $b \leq \xi \leq c_k$ under the assumption of course that $c_k > b$. From (3.29) we obtain

$$
\begin{aligned}
F_{km}(\xi) &= \frac{\varphi_m(\xi)}{I_m} \int_\xi^{c_k} (a^2 - x^2)^{-1} \, e^{\frac{1}{\varepsilon}\int_\xi^x \frac{(\lambda_m - \lambda_k)}{a^2 - y^2}\,dy} \, e^{\frac{-1}{\varepsilon}\int_{\rho_m}^x \frac{(y-\lambda_m)}{a^2-x^2}\,dy}\, dx \\
&\leq \frac{O(1)}{\varepsilon}\, \varphi_m(\xi) \int_\xi^{c_k} e^{\frac{1}{\varepsilon a^2}(\lambda_m^M - \lambda_k^m)(x-\xi)}\, dx \qquad (3.35)\\
&\leq \frac{O(1)}{(\lambda_m^M - \lambda_k^m)}\, \varphi_m(\xi) \ .
\end{aligned}
$$

Combining (3.34) and (3.35) we get, if $c_k > b$,

$$
F_{km}(\xi) \leq O(1)\, \varphi_m(\xi) + o(1)\, \varphi_k(\xi) \qquad \text{for} \quad b \leq \xi \leq c\ . \tag{3.36}
$$

The last remaining case is $k < p(b)$. In this case $\lambda_k < b$, $\lambda_m < b$ and $c_k = \rho_k = \rho_m = b$. Hence we find

$$
\begin{aligned}
F_{km}(\xi) &= \varphi_k(\xi)\, \frac{I_k}{I_m^2} \int_b^\xi (a^2 - x^2)^{-1} \, e^{\frac{1}{\varepsilon}\int_b^x \frac{(\lambda_m - \lambda_k)}{a^2 - x^2}\,dy} \, e^{\frac{-1}{\varepsilon}\int_b^x \frac{(y-\lambda_m)}{a^2-y^2}\,dy}\, dx \\
&\leq \frac{\varepsilon\, I_k\, \varphi_k(\xi)}{(\lambda_k^m - \lambda_m^M)\, I_m^2}\, \left(1 - e^{(\lambda_m^M - \lambda_k^m)\frac{\xi - b}{\varepsilon a^2}}\right) \qquad (3.37)\\
&\leq \frac{O(1)}{(\lambda_k^m - \lambda_m^M)}\, \frac{I_k}{I_m}\, \varphi_k(\xi)\ ,
\end{aligned}
$$

using that $I_m \geq C\, \varepsilon$. Since $\lambda_k^M < b$, an easy calculation yields

$$
I_k = \int_b^c e^{\frac{-1}{\varepsilon}\int_b^\xi \frac{(y-\lambda_k)}{a^2-y^2}\,dy}\, d\xi \ \leq\ \sqrt{\varepsilon}\, e^{\frac{\left(b-\lambda_k^M\right)^2}{2\,\varepsilon a^2}} \int_{\frac{b-\lambda_k^M}{\sqrt{\varepsilon} a}}^{\frac{c-\lambda_k^M}{\sqrt{\varepsilon} a}} e^{-\frac{y^2}{2}}\, dy\ . \tag{3.38}
$$

Now, the integral

$$
\begin{aligned}
\int_{\frac{b-\lambda_k^M}{\sqrt{\varepsilon} a}}^{\frac{c-\lambda_k^M}{\sqrt{\varepsilon} a}} e^{-\frac{y^2}{2}}\, dy &= \int_{\frac{b-\lambda_k^M}{\sqrt{\varepsilon} a}}^{\infty} e^{-\frac{y^2}{2}}\, dy - \int_{\frac{c-\lambda_k^M}{\sqrt{\varepsilon} a}}^{\infty} e^{-\frac{y^2}{2}}\, dy \\
&= \frac{\sqrt{\varepsilon}\, a}{b - \lambda_k^M}\, e^{\frac{-(b-\lambda_k^M)^2}{\varepsilon a^2}} \Big(1 + o(\varepsilon)\Big) - \frac{\sqrt{\varepsilon}\, a}{c - \lambda_k^M}\, e^{\frac{-(c-\lambda_k^M)^2}{\varepsilon a^2}} \Big(1 + o(\varepsilon)\Big) \qquad (3.39)\\
&= \frac{\sqrt{\varepsilon}\, a}{b - \lambda_k^M}\, e^{\frac{-(b-\lambda_k^M)^2}{\varepsilon a^2}} \Big(1 + o(\varepsilon)\Big)\ ,
\end{aligned}
$$

since $(b - \lambda_k^M)^2 - (c - \lambda_k^M)^2 < 0$. From (3.37)–(3.39) and using $I_m \geq C\, \varepsilon$ we get

$$
F_{km}(\xi) \leq O(1)\, \varphi_k(\xi) \qquad \text{for} \quad b \leq \xi \leq c\ .
$$

This completes the proof of the interaction estimates (3.19). ∎

4 – Existence Theory and Structure of the Boundary Layer

We now establish the existence of the solution of (1.4). As explained earlier, it is equivalent to construct the solution u^ε of (3.3), since then, by (1.8), v^ε and $TV(v^\varepsilon)$ are deduced from the relation $v' = \xi u'$.

Throughout this section, all the estimates are uniform in the limit $\varepsilon \to 0$. The presentation given here follows [21, 13], and omit all the proofs of the basic existence theorem.

First, we analyze the coupled system (3.7) but still with u^ε replaced with a fixed function $w\colon\ [b,c] \to \mathcal{B}(u_*, C_*\,\delta)$. We are given the boundary value $u_L \in \mathcal{B}(u_*,\delta)$ and, instead of using a right-end state u_R, we first describe the Riemann solutions using a "wave strength" vector $\tau \in \mathbb{R}^N$. The coefficients a_k^ε are sought in the form of an asymptotic expansion in the wave strength:

$$a_k^\varepsilon(\xi; w, \tau) \;=\; \tau_k\,\varphi_k^\varepsilon(\xi; w) + \theta_k^\varepsilon(\xi; w, \tau) \;,$$

where $\tau = (\tau_1, ..., \tau_N) \in \mathcal{B}(0,\delta_1)$, the ball in $\mathbb{R}^N$ having center 0 and radius $\delta_1 > 0$. The remainder $\theta_k^\varepsilon(\xi; w, \tau)$ is sought to be second-order in τ. Next for each u_L and each vector of wave strengths τ, a solution u^ε is constructed for (3.7) with boundary condition (3.3b). Finally for the problem (3.7) with boundary conditions (3.3b) with fixed u_L and u_R, we get the following theorem.

Theorem 4.1. *There exist $\delta, C_*, C > 0$ with the following property. For every $\varepsilon > 0$, $u_L, u_R \in \mathcal{B}(u_*,\delta)$, the problem (3.7) admits a solution $\xi \mapsto u^\varepsilon(\xi)$ connecting $u_L = u^\varepsilon(b)$ to $u_R = u^\varepsilon(c)$ and satisfying $u^\varepsilon(\xi) \in \mathcal{B}(u_*, C_*\,\delta)$ for all $\xi > 0$. It satisfies also the expansion*

$$u^{\varepsilon\prime} = \sum_{k=1}^{N} a_k^\varepsilon\, r_k(u^\varepsilon) \;, \tag{4.1}$$

$$\begin{aligned}
&a_k^\varepsilon(\xi;\tau) = \tau_k\,\varphi_k^\varepsilon(\xi;\tau) + \theta_k^\varepsilon(\xi;\tau)\,, \qquad |\theta_k^\varepsilon(\xi;\tau)| \le C\,|\tau|^2 \sum_{j=1}^{N} \varphi_j^\varepsilon(\xi;\tau) \;,\\
&\varphi_k^\varepsilon(\xi;\tau) = (a^2-\xi^2)^{-1} \frac{e^{-\frac{g_k^\varepsilon(\xi;\tau)}{\varepsilon}}}{\displaystyle\int_b^c (a^2-x^2)^{-1}\, e^{-\frac{g_k^\varepsilon(x;\tau)}{\varepsilon}}\,dx} \;,\\
&g_k^\varepsilon(\xi;\tau) = \int_{\rho_k^\varepsilon}^{\xi} \frac{x-\lambda_k(u^\varepsilon(x;\tau))}{a^2-x^2}\,dx \;,
\end{aligned} \tag{4.2}$$

for some $\tau = \tau^\varepsilon$ *with*

$$\frac{1}{C}\,|\tau^\varepsilon| \;\le\; |u_L - u_B| \;\le\; C\,|\tau^\varepsilon|\ . \tag{4.3}$$

Furthermore, u^ε *is of uniformly bounded total variation and satisfies*

$$|u^{\varepsilon\prime}| \;\le\; O(1)\,|\tau|\,\sum_{j=1}^{N}\varphi_j^\varepsilon \tag{4.4}$$

and, thus, in view of (3.10) and (4.3)

$$TV(u^\varepsilon) \;\le\; O(1)\,|u_L - u_R|\ . \tag{4.5}$$

These results for u^ε together with (1.8) and $v' = \xi u'$ give uniform L^∞ and TV estimates for v^ε. So $(u^\varepsilon, v^\varepsilon)$ is compact family in L^1 and also pointwise almost everywhere. We shall prove the following theorem regarding $(u, v) = \lim_{\varepsilon\to 0}(u^\varepsilon, v^\varepsilon)$.

Theorem 4.2. *There exist* $\delta, c_*, C > 0$ *with the following property. For every* $u_L, u_R \in \mathcal{B}(u_*, \delta)$*, a subsequence of the solution* $(u^\varepsilon, v^\varepsilon)$ *of* (1.4) *constructed in the previous section, converges to* $(u, f(u))$ *and satisfies the equation*

$$-\xi\,u' + f(u)' \,=\, 0 \tag{4.6}$$

and we have the estimate

$$TV(u) \;\le\; C\,|u_R - u_L|\ . \tag{4.7}$$

Furthermore, there exits constant vectors u_k, $k = p(b), \dots, N-1$ *such that*

$$u(\xi) = \begin{cases} u_k, & \lambda_k^M < \xi < \lambda_{k+1}^m\,,\ k = p(b), \dots N-1\ , \\ u_R, & \lambda_N^M < \xi \le c\ , \end{cases} \tag{4.8a}$$

and if $b < \lambda_1^m$*, then*

$$u(\xi) = u_L \quad \text{for}\ \ b \le \xi < \lambda_1^m\ . \tag{4.8b}$$

Finally, u *satisfies the entropy condition*

$$-\xi\,p(u)_\xi + q(u)_\xi \;\le\; 0\ , \tag{4.9}$$

for all entropy pairs $((p(u), q(u))$ *with* $p(u)$ *convex.*

Proof: By (1.8) we have

$$|v^\varepsilon - f(u^\varepsilon)|_{L^1(b,c)} \le \varepsilon\, 2\, a^2\, |u^{\varepsilon\prime}|_{L^1(b,c)} \; . \tag{4.10}$$

By the estimates (4.5) and (1.8) and $v' = \xi u'$, $(u^\varepsilon, v^\varepsilon)$ is compact in L^1 topology and there exists a subsequence which converges pointwise almost everywhere to a function (u, v). This together with (4.10) give v^ε converges to $f(u)$ and the estimate (4.7). Further from the first equation of (1.4a) we get

$$-\xi\, u' + f(u)' = 0 \; .$$

in the sense of distribution.

The limit u satisfies the entropy condition (4.9) for all entropy pairs $((p(u), q(u))$ with $p(u)$ convex follows from after passing to the limit in

$$(2\,\varepsilon - 1)\, \xi\, p(u^\varepsilon)_\xi + q(u^\varepsilon)_\xi \le \varepsilon\, p(u^\varepsilon)_{\xi\xi} \; .$$

To prove (4.8), first consider the case $p(b) \ge 1$, it follows from (3.8b) and (4.1)–(4.2) that $u^{\varepsilon\prime}(\xi)$ converges to 0 as ε goes to 0 uniformly on intervals $[\lambda_k^M + \delta, \lambda_{k+1}^m - \delta]$, $k = p(b), \ldots N-1$ for $\delta > 0$ and small and so u takes constant values on these intervals.

Let us consider the region $\xi > \lambda_N^M + \delta$. By (4.1)–(4.2) and the estimates (3.18),

$$\begin{aligned} |u^\varepsilon(\xi) - u_R| &\le |u_R - u_L|\, \frac{C}{\varepsilon} \int_\xi^c e^{\frac{-(x-\lambda_N^M)^2}{2\varepsilon a^2}}\, dx \\ &\le |u_R - u_L|\, \frac{C}{\varepsilon}\, (b-c)\, e^{\frac{-\delta^2}{2\varepsilon a^2}} \; . \end{aligned} \tag{4.11a}$$

Similarly, if $p(b) < 1$ then $b < \lambda_1^m$, in the region $\xi < \lambda_1^m - \delta$.

$$\begin{aligned} |u^\varepsilon(\xi) - u_L| &\le |u_R - u_L|\, \frac{C}{\varepsilon} \int_b^\xi e^{\frac{-(x-\lambda_1^m)^2}{2\varepsilon a^2}}\, dx \\ &\le |u_R - u_L|\, \frac{C}{\varepsilon}\, (c-b)\, e^{\frac{-\delta^2}{2\varepsilon a^2}} \; . \end{aligned} \tag{4.11b}$$

From (4.11), it follows that u^ε converges uniformly on the interval $[b, \lambda_1^m - \delta]$ to u_L if $b < \lambda_1^m$ and on the interval $[\lambda_N^M + \delta, c]$ to u_R, for each $\delta > 0$. This completes the proof of the theorem. ∎

Note that by (4.8) the condition $u(c) = u_R$ holds and if $b < \lambda_1^m$, then $u(b+) = u_L$. In fact this case solve the standard Riemann problem. However when $b > \lambda_1^m$

the condition at $u^\varepsilon(b) = u_L$ does not pass to the limit in general and must be relaxed and expressed in the weak form (1.6c). The rest of this section is devoted to determine the value $u(0+)$. We will distinguish between the characteristic and the non-characteristic case, the former being comparatively easier to deal with. We start with the derivation of the equation describing the boundary layer near $x = b$.

Theorem 4.3. *The trace $u(0+)$ of the Riemann solution constructed in Theorem 4.2 satisfies the following property. There exist a vector V_∞ and a smooth function $y \geq 0 \mapsto V(y)$ such that*

$$\begin{aligned}&(a^2 - b^2)\, V'(y) \;=\; f(V(y)) - b\, V(y) - f(V_\infty) + b\, V_\infty\ ,\\ &V(0) = u_L, \quad V(\infty) = V_\infty\ ,\end{aligned} \tag{4.12}$$

and

$$f(V_\infty) - b\, V_\infty \;=\; f(u(b+)) - b\, u(b+)\ . \tag{4.13}$$

Proof: Let ξ_ε be a sequence of positive numbers such that

$$\xi_\varepsilon = o(\varepsilon)\ , \tag{4.14}$$

i.e. ξ_ε tends to 0 faster than ε. Define the function

$$V^\varepsilon(y) = u^\varepsilon(b + \xi_\varepsilon + \varepsilon\, y), \qquad \text{for all } \ 0 < y \leq \frac{c-b}{\varepsilon}\ . \tag{4.15}$$

Since u^ε is uniformly bounded and of uniformly bounded total variation (see (4.5)), the functions V^ε are also bounded and of uniformly bounded total variation. So there exists a function $V(y)$ of bounded total variation defined on the interval $[0, \infty)$ and there exist two constants V_0 and V_∞ in R^N such that

$$\lim_{\varepsilon \to 0} V^\varepsilon(y) = V(y) \qquad \text{for all } \ y > 0 \tag{4.16}$$

and

$$\lim_{y \to 0} V(y) = V_0\,, \qquad \lim_{y \to \infty} V(y) = V_\infty\ . \tag{4.17}$$

In fact, $V_0 = u_B$, to see this note that

$$|V(0) - u_L| \;=\; \lim_{y \to 0} |V(y) - u_L| \;\leq\; \lim_{y \to 0} \limsup_{\varepsilon \to 0} |u^\varepsilon(b + \xi_\varepsilon + \varepsilon\, y) - u_L|\ . \tag{4.18}$$

Using the pointwise estimate (3.16)

$$|u^{\varepsilon\prime}(\xi)| \le \frac{C}{\varepsilon} \quad \text{for all} \quad b \le \xi \le c \ ,$$

we have,

$$\begin{aligned} \Big|u^\varepsilon(b+\xi_\varepsilon+\varepsilon\, y) - u_L\Big| \ &\le \ \int_b^{b+\xi_\varepsilon+\varepsilon y} |u^{\varepsilon\prime}(s)|\, ds \\ &\le \ \frac{C}{\varepsilon}\int_b^{b+\xi_\varepsilon+\varepsilon y} ds \ = \ C\Big(y+\xi_\varepsilon/\varepsilon\Big)\ . \end{aligned} \tag{4.19}$$

From (4.14), (4.18), and (4.19), we deduce that $V(0) = u_L$.

Next we derive the boundary layer equation (4.12). Integrating (1.7) from some point $b+\alpha$ to $b+\xi_\varepsilon+\varepsilon y$, we get

$$\begin{aligned} &\varepsilon\left(a^2-(b+\xi_\varepsilon+\varepsilon\, y)^2\right)u^{\varepsilon\prime}(b+\xi_\varepsilon+\varepsilon\, y) - \varepsilon\left(a^2-(b+\alpha)^2\right)u^{\varepsilon\prime}(b+\alpha) \\ &\qquad + 2\,\varepsilon\int_{b+\alpha}^{b+\xi_\varepsilon+\varepsilon y} x\, u^{\varepsilon\prime}(x)\ dx \\ &\quad = \ (2\,\varepsilon-1)\int_{b+\alpha}^{b+\xi_\varepsilon+\varepsilon y} x\, u^{\varepsilon\prime}(x)\, dx \ + \ f\Big(u^\varepsilon(b+\xi_\varepsilon+\varepsilon\, y)\Big) - f\Big(u^\varepsilon(b+\alpha)\Big)\ . \end{aligned}$$

Simplifying this we get,

$$\begin{aligned} &\varepsilon\left(a^2-(b+\xi_\varepsilon+\varepsilon\, y)^2\right)u^{\varepsilon\prime}(b+\xi_\varepsilon+\varepsilon\, y) - \varepsilon\left(a^2-(b+\alpha)^2\right)u^{\varepsilon\prime}(b+\alpha) \\ &\quad = \ f\Big(u^\varepsilon(b+\xi_\varepsilon+\varepsilon\, y)\Big) - f\Big(u^\varepsilon(b+\alpha)\Big) \\ &\qquad -(b+\xi_\varepsilon+\varepsilon\, y)\, u^\varepsilon(b+\xi_\varepsilon+\varepsilon\, y) + (b+\alpha)\, u^\varepsilon(b+\alpha) + \int_{b+\alpha}^{b+\xi_\varepsilon+\varepsilon y} u^\varepsilon(s)\, ds. \end{aligned}$$

After integration with respect to α varying from 0 to some $\delta > 0$ and then dividing by δ, this identity becomes

$$\begin{aligned} &\Big(a^2-(b+\xi_\varepsilon+\varepsilon\, y)^2\Big)\frac{d}{dy}\Big(u^\varepsilon(b+\xi_\varepsilon+\varepsilon\, y)\Big) \ - \ \frac{\varepsilon}{\delta}\int_0^\delta\Big(a^2-(b+\alpha)^2\Big)u^{\varepsilon\prime}(b+\alpha)\, d\alpha \\ &\qquad = \ -(b+\xi_\varepsilon+\varepsilon\, y)\, u^\varepsilon(b+\xi_\varepsilon+\varepsilon\, y) + f\Big(u^\varepsilon(b+\xi_\varepsilon+\varepsilon\, y)\Big) \\ &\qquad\quad + \frac{1}{\delta}\int_0^\delta \Big((b+\alpha)\, u^\varepsilon(b+\alpha) - f\Big(u^\varepsilon(b+\alpha)\Big)\Big)\, d\alpha \\ &\qquad\quad + \frac{1}{\delta}\int_0^\delta\int_{b+\alpha}^{b+\xi_\varepsilon+\varepsilon y} u^\varepsilon(s)\, ds\, d\alpha. \end{aligned}$$

We now integrate this with respect to y, starting at 0, we get

$$\begin{aligned} &\Big(a^2-(b+\xi_\varepsilon+\varepsilon\, y)^2\Big)u^\varepsilon(b+\xi_\varepsilon+\varepsilon\, y) - \Big(a^2-(b+\xi_\varepsilon)^2\Big)u^\varepsilon(b+\xi_\varepsilon) \\ &+ 2\,\varepsilon\int_0^y (b+\xi_\varepsilon+\varepsilon\, x)\, u(b+\xi_\varepsilon+\varepsilon\, x)\, dx \ - \ \frac{\varepsilon}{\delta}\, y\int_0^\delta\Big(a^2-(b+\alpha)^2\Big)u^{\varepsilon\prime}(b+\alpha)\, d\alpha \ = \end{aligned}$$

$$= \int_0^y \Big(-(b+\xi_\varepsilon+\varepsilon\, x)\, u^\varepsilon(b+\xi_\varepsilon+\varepsilon\, x) + f\Big(u^\varepsilon(b+\xi_\varepsilon+\varepsilon\, x)\Big)\Big)\, dx$$
$$+ \frac{y}{\delta} \int_0^\delta \Big((b+\alpha)\, u^\varepsilon(b+a) - f\Big(u^\varepsilon(b+\alpha)\Big)\Big)\, d\alpha$$
$$+ \frac{1}{\delta} \int_0^y \int_0^\delta \int_{b+\alpha}^{b+\xi_\varepsilon+\varepsilon x} u^\varepsilon(s)\, ds\, d\alpha\, dx\ .$$

Letting $\varepsilon \to 0$ and using $V(0) = u_L$, and the uniform L^∞ and TV estimates on u^ε, we arrive at

$$(a^2 - b^2)\, V(y) - (a^2 - b^2)\, u_L$$
$$= \int_0^y \Big(-b\, V(x) + f(V(x))\Big)\, dx\ +\ \frac{y}{\delta} \int_0^\delta \Big((b+\alpha)\, u(b+\alpha) - f\Big(u(b+\alpha)\Big)\Big)\, d\alpha$$
$$+ \frac{y}{\delta} \int_0^\delta \int_{b+\alpha}^b u(s)\, ds\, d\alpha$$

for all $\delta, y > 0$. Next when $\delta \to 0$ it follows that

$$(a^2 - b^2)\, V(y) - (a^2 - b^2)\, u_L$$
$$= \int_0^y \Big(-b\, V(x) + f(V(x))\Big)\, dx\ +\ y\Big(b\, u(b+) - f(u(b+))\Big)\ .$$

On differentiating this we get the equation

$$(a^2 - b^2)\, V'(y)\ =\ f(V(y)) - b\, V(y) - f(u(b+) + b\, u(b+)\ . \tag{4.20}$$

Next we shall prove (4.13). Integrating the equation in (4.20) from n to $n+1$, it follows that

$$\int_n^{n+1} \Big(f(V(x)) - b\, V(x)\Big)\, dx - f(u(b+)) + b\, u(b+)$$
$$= (a^2 - b^2)\Big(V(n+1) - V(n)\Big)\ . \tag{4.21}$$

Since V has bounded total variation and converges to V_∞ at infinity, we have

$$\int_n^{n+1} |f(V(x)) - f(V_\infty)|\, dx\ \leq\ C \int_n^{n+1} |V(x) - V_\infty|\, dx$$
$$\leq\ C\, TV(V, [n, n+1]) + C\, |V(n) - V_\infty|\ \to\ 0\ .$$

Therefore letting n tend to ∞ in (4.21), we obtain $f(V_\infty) - b\, V_\infty = f(u(b+)) - b\, u(b+)$, which is the condition (4.13). Now using (4.13) in the differential equation (4.20) and $V(0) = u_L$ we get (4.12). The proof of the Theorem is completed. ∎

Our objective now is to characterize $u(0+)$. Here after we use the notation p instead of $p(b)$ as we are dealing with a fixed b. Consider first the *non-characteristic case* that is when

$$\lambda_{p-1}^M < b < \lambda_p^m \ . \tag{4.22}$$

Denote by

$$\begin{array}{c}\mathcal{E}(u_L) \text{ the set of all admissible boundary values } V_\infty \\ \text{determined by the problem (4.12)} \ . \end{array} \tag{4.23}$$

Then we claim that the trace $u(0+)$ of the Riemann solution constructed in Theorem 4.6 belongs to this set. Thus $u(x,t)$ solve (1.6) with boundary condition (1.6c) given by the set $\mathcal{E}(u_L)$. Further we shall show that $\mathcal{E}(u_L)$ has the correct dimension. More precisely we have the following result.

Theorem 4.4. *Suppose that the non-characteristic condition* (4.22) *holds.*

(**1**) *(Existence) There exist $\delta, C > 0$ with the following property. Given $u_L, u_R \in \mathcal{B}(u_*, \delta)$ there exists a weak solution with bounded total variation of the Riemann problem* (1.6), *associated with the set of boundary values* (4.23).

(**2**) *(Uniqueness) Assume that, for $j = p, ..., N$, each j-characteristic field of the matrix Df is genuinely nonlinear. Then the above Riemann solution is unique in the class of piecewise smooth self-similar solutions.*

(**3**) *(Local structure) Then the set $\mathcal{E}(u_L)$ defined in* (4.23) *contains the point u_L and, locally near u_L, is a manifold with dimension $p-1$ whose tangent space at u_L is spanned by the eigenvectors $r_j(u_L)$, $j = 1, 2, ..., p-1$.*

Proof: The property (3) follows from Theorem 3.4 in [7] when the flux $f(u)$ is replaced by $\frac{f(u) - bu}{a^2 - b^2}$.

To prove (1), we note that by implicit function theorem and by the condition, the flux-function $f(u) - bu$ is locally one-to-one and so,

$$u(b+) = V_\infty \ .$$

In other words, the solution satisfies the boundary condition (1.6c), with $\mathcal{E}(u_B)$ being given by (4.23). This concludes the proof that the Riemann problem (1.6) admits a weak solution satisfying the boundary condition in the relaxed sense (1.6c).

To show (2), we observe that the standard Lax wave curves associated with the wave families $j = p+1, ..., N$ generate a smooth manifold with dimension $N-p$, which contains u_R and whose tangent space at u_R is spanned by $r_{p+1}(u_R), ..., r_N(u_R)$. The two manifolds are transverse and by a straight forward generalization of Lax's construction of the Riemann problem give a unique intersection point is the desired trace $u(0+)$. ∎

Theorem 4.4 provides a complete characterization of the boundary layer in the non-characteristic case. Of course in the interior $\xi > 0$, the Riemann solutions also satisfy the Lax and Liu entropy condition, as is the case for the Riemann problem in the whole line.

In the rest of this section we aim at extending the conclusion of Theorem 4.4 to the *characteristic case*. Thus we return to the situation where

$$b \in (\lambda_p^m, \lambda_p^M\Big) \; , \tag{4.24}$$

and we assume that the boundary data is "entering" in the sense that

$$\lambda_p(u_L) > b \; . \tag{4.25}$$

We must introduce a set of admissible traces similar to (4.23). Here again following Joseph and LeFloch [8], we introduce the following set of admissible values:

$$\begin{aligned} \mathcal{E}(u_B) \; = \; & \Big\{ V_\infty \, / \; \text{ there exists a solution to (4.12)} \Big\} \\ & \cup \; \Big\{ \tilde V_\infty \, / \; \text{ there exists a solution to (4.12) for some } V_\infty \\ & \qquad \text{with } f(\tilde V_\infty) - b\tilde V_\infty = f(V_\infty) - bV_\infty \text{ and } \lambda_p(V_\infty) > b \Big\} \; , \end{aligned} \tag{4.26}$$

and a straightforward generalization of Theorem (4.7) of [8] leads us to the following conclusion.

Theorem 4.5. *Suppose that the characteristic condition* (4.24) *holds and that* $\{\lambda_p(u) = b\}$ *is a smooth manifold with dimension* $N-1$*, and the* p*-characteristic field is genuinely nonlinear. Let* u_* *be in this manifold. Then there exist* $\delta > 0$ *with the following property. Given* $u_L, u_R \in \mathcal{B}(u_*, \delta)$ *there exists a weak solution with bounded total variation of the Riemann problem* (1.6)*, associated with the set of boundary values* (4.26)*, i.e.*

$$u(b+) \in \mathcal{E}(u_B) \; . \tag{4.27}$$

5 – Examples from Continuum Physics

The results obtained in this paper are now illustrated with the help of two examples.

The p-system of gas dynamics. We consider the p-system in the case that the flux-function takes the form $f(u_1,u_2) = (-u_2, p(u_1))^t$, with $p'(u_1) < 0$ and $p''(u_1) > 0$. More specifically, werestrict attention to the case $p(u_1) = \frac{k}{u_1^\gamma}$, $\gamma \geq 1$. Here, $u_1 > 0$ is the specific volume and u_2 is the velocity. The equations take the form

$$\partial_t u_1 - \partial_x u_2 = 0\,, \qquad \partial_t u_2 + \partial_x p(u_1) = 0 \,. \tag{5.1}$$

The relaxation approximation (1.4) for the system (5.1) becomes

$$\begin{aligned} &-\xi\, u_1^{\varepsilon\prime} + v_1^{\varepsilon\prime} = 0\,, \qquad -\xi\, u_2^{\varepsilon\prime} + u_2^{\varepsilon\prime} = 0\,,\\ &-\xi\, v_1^{\varepsilon\prime} + a^2 u_1^{\varepsilon\prime} = \frac{1}{\varepsilon}\left(-u_2^\varepsilon - v_1^\varepsilon\right), \qquad -\xi\, v_2^{\varepsilon\prime} + a^2 u_2^{\varepsilon\prime} = \frac{1}{\varepsilon}\left(-p(u_1^\varepsilon) - v_2^\varepsilon\right)\,, \end{aligned} \tag{5.2a}$$

on a bounded interval $[b,c]$ with the boundary conditions

$$u_1^\varepsilon(b) = u_{1L}\,, \quad u_1^\varepsilon(c) = u_{1R}\,, \qquad u_2^\varepsilon(b) = u_{2L}\,, \quad u_2^\varepsilon(c) = u_{2R}\,. \tag{5.2b}$$

After reduction to the second-order equations (see (1.7)) we find

$$\begin{aligned} &\varepsilon(a^2-\xi^2)\, u_1'' = -u_2' + (2\,\varepsilon-1)\,\xi\, u_1'\,,\\ &\varepsilon(a^2-\xi^2)\, u_2'' = p(u_1)' + (2\,\varepsilon-1)\,\xi\, u_2'\,, \end{aligned} \tag{5.3a}$$

on the interval $[b,c]$ with the boundary conditions

$$u_1^\varepsilon(b) = u_{1L}\,, \quad u_1^\varepsilon(c) = u_{1R}\,, \qquad u_2^\varepsilon(b) = u_{2L}\,, \quad u_2^\varepsilon(c) = u_{2R}\,. \tag{5.3b}$$

The function $v^\varepsilon = (v_1^\varepsilon, v_2^\varepsilon)$ is recovered from the relation (1.8):

$$v_1^\varepsilon = \varepsilon\,(a^2-\xi^2)\, u_1^{\varepsilon\prime} - u_2^\varepsilon\,, \qquad v_2^\varepsilon = \varepsilon\,(a^2-\xi^2)\, u_2^{\varepsilon\prime} + p(u_1^\varepsilon) \tag{5.3c}$$

where u_{jL}, $j=1,2$ and u_{jR}, $j=1,2$ are given constants. Our aim is to get a large data result for the p-system (5.1). Let $u_{10} = \min(u_{1L}, u_{2L}) > 0$. First we show the existence of solutions to (5.3).

We largely follow the ideas of Slemrod and Tzavaras [19] for their study of Riemann problem for isentropic gas dynamics equation in Eulerian co-ordinates. Consider the one-parameter family of problems:

$$\begin{aligned} &\varepsilon(a^2-\xi^2)\, u_1'' = -\mu\, u_2' + (2\,\varepsilon-1)\,\xi\, u_1'\,,\\ &\varepsilon(a^2-\xi^2)\, u_2'' = \mu\, p(u_1)' + (2\,\varepsilon-1)\,\xi\, u_2'\,, \end{aligned} \tag{5.4a}$$

on the interval $[b,c]$ with the boundary conditions

$$\begin{aligned} &u_1^\varepsilon(b) = u_{10} + \mu\,(u_{1L} - u_{10})\,, \qquad u_1^\varepsilon(c) = u_{10} + \mu\,(u_{1R} - u_{10})\,\varphi \\ &u_2^\varepsilon(b) = \mu\, u_{2L}\,, \qquad u_2^\varepsilon(c) = \mu\, u_{2R}\,. \end{aligned} \tag{5.4b}$$

Here, the parameter μ lies in $[0,1]$. We note that the equation (5.4a) can be written as

$$\begin{aligned} &\left(u_1'(a^2-\xi^2)^{1-\frac{1}{2\varepsilon}}\right)' = -\mu(a^2-\xi^2)^{\frac{-1}{2\varepsilon}}\, u_2'\,, \\ &\left(u_2'(a^2-\xi^2)^{1-\frac{1}{2\varepsilon}}\right)' = \mu(a^2-\xi^2)^{\frac{-1}{2\varepsilon}}\, p(u_1)'\,. \end{aligned} \tag{5.5}$$

An easy computation shows that solving (5.4) is equivalent to solving the integral equation:

$$\begin{aligned} u(\xi) \;=&\; u(b) + A\int_b^\xi (a^2-y^2)^{\frac{1}{2\varepsilon}-1}\,dy \\ &+ \frac{\mu}{\varepsilon}\int_b^\xi \frac{f(u(y))}{(a^2-y^2)}\,dy \;-\; \frac{\mu}{\varepsilon^2}\int_b^\xi\int_b^y \frac{(a^2-y^2)^{1-\frac{1}{2\varepsilon}}}{(a^2-z^2)^{1+\frac{1}{2\varepsilon}}}\; z\, f(u(z))\; dz\, dy \end{aligned} \tag{5.6a}$$

with the constant A given by

$$\begin{aligned} u(c) \;=&\; u(b) + A\int_b^c (a^2-y^2)^{\frac{1}{2\varepsilon}-1}\,dy \\ &+ \frac{\mu}{\varepsilon}\int_b^c \frac{f(u(y))}{(a^2-y^2)}\,dy \;-\; \frac{\mu}{\varepsilon^2}\int_b^c\int_b^y \frac{(a^2-y^2)^{1-\frac{1}{2\varepsilon}}}{(a^2-z^2)^{1+\frac{1}{2\varepsilon}}}\; z\, f(u(z))\; dz\, dy \end{aligned} \tag{5.6b}$$

and $u(b)$ and $u(c)$ given by (5.4a). Let $X = C^0([b,c],R^2)$ be the Banach space of continuous functions endowed with sup-norm, and

$$\Omega = \left\{u\in X\colon \min_{b\le\xi\le c} u_1(\xi) > \delta > 0,\ \max_{b\le\xi\le c} |u_2(\xi)| + u_1(\xi) < M+1\right\}. \tag{5.7}$$

Consider the map $T\colon\, \Omega\to X$ by

$$\begin{aligned} Tu(\xi) \;=&\; u_L + A\int_b^\xi (a^2-y^2)^{\frac{1}{2\varepsilon}-1}\,dy \\ &+ \frac{1}{\varepsilon}\int_b^\xi \frac{f(u(y))}{(a^2-y^2)}\,dy \;-\; \frac{1}{\varepsilon^2}\int_b^\xi\int_b^y \frac{(a^2-y^2)^{1-\frac{1}{2\varepsilon}}}{(a^2-z^2)^{1+\frac{1}{2\varepsilon}}}\; z\, f(u(z))\; dz\, dy \end{aligned} \tag{5.8}$$

with A determined by (5.6b) with $\mu = 1$. It is easy to see that $T:\ \Omega\to X$ is a compact map. Fixed point of T gives solution of (5.3). Using a fixed point theorem we prove the following existence theorem for the system (5.4) and hence in particular for (5.3).

Theorem 5.1. *There exists a solution* $(u_1^\varepsilon, u_2^\varepsilon)$ *of* (5.4) *such that, for all* $0 \le \mu \le 1$,

$$\sup_{b\le\xi\le c}\Big\{u_1(\xi) + |u_2(\xi)|\Big\} \le M\,, \qquad \inf_{b\le\xi\le c} u_1(\xi) \ge \delta \tag{5.9}$$

for some constants $M > 0$ *and* $\delta > 0$ *(which may depend on* ε*). Furthermore,* u_1^ε *and* u_2^ε *belong to one of the following class:*

(**a**) u_1 *and* u_2 *are constant functions,*

(**b**) u_1 *and* u_2 *are monotone functions,*

(**c**) u_1 *is monotone increasing (decreasing) and then* u_2 *has exactly one critical point which is a maximum (minimum),*

(**d**) u_2 *is monotone increasing (decreasing) and then* u_1 *has exactly one critical point which is a maximum (minimum).*

To prove this theorem, we will rely on several a-priori estimates for the solutions of (5.4). We start with a technical lemma.

Lemma 5.2. *Let* $(u_1^\varepsilon, u_2^\varepsilon)$ *be a solution of* (5.4). *Then, there exist* M *and* $\delta > 0$ *such that, for all* $0 \le \mu \le 1$,

$$\sup_{b\le\xi\le c}\Big\{u_1^\varepsilon(\xi) + |u_2^\varepsilon(\xi)|\Big\} \le M\,, \qquad \inf_{b\le\xi\le c} u_1^\varepsilon(\xi) \ge \delta\,. \tag{5.10}$$

Proof: We start by showing that any solution $(u_1^\varepsilon, u_2^\varepsilon)$ of (5.4a) belongs one of the classes (a)–(d) stated in Theorem 5.1. For simplicity in the notation we supress the dependence in ε, that is, $(u_1, u_2) = (u_1^\varepsilon, u_2^\varepsilon)$. Clearly, (5.4) admits a solution for $\mu = 0$ namely $(u_1, u_2) = (u_{10}, 0)$. We assume that $\mu \ne 0$. We note that, by uniqueness of the solution of the initial value problem for (5.4a) with initial conditions $(u_1(\xi_0), u_2(\xi_0))$ and $(u_1'(\xi_0), u_2'(\xi_0))$, $(u_1(\xi), u_2(\xi))$ is uniquely determined on its interval of existence. From this, it follows that either u_1 and u_2 are constant functions or both must be non-constant functions with at no point derivatives which vanish simultaneously.

Now, consider non-constant solutions u_1, u_2 of (5.4a) and suppose that one of them is not monotone, say u_2 has a critical point at ξ_0. Then, $u_2''(\xi_0) \ne 0$, otherwise at that point u_1' would also vanish by (5.4a), which is impossible for a non-constant solution by our earlier observation. Assume that $u_2''(\xi_0) < 0$. If (c) is true, then we are done. If not, there are two possibilities:

- **(i)** u_2 has two consecutive critical points with a maximum at ξ_0 and a minimum at ξ_1,
- **(ii)** u_2 has exactly one critical point with maximum at ξ_0 and u_1 has atleast one critical point in (b,c).

In case (i), assume for definiteness $\xi_0 > \xi_1$ (the case $\xi_0 < \xi_1$ is similar), then $u_2'(\xi) > 0$ on (ξ_1, ξ_0). From (5.4a) it follows that $p(u_1)'(\xi_0) < 0$ and, since $p'(u_1) < 0$, it also follows that $u_1'(\xi_0) > 0$. By a similar reasoning we have $u_1'(\xi_1) < 0$. So, u_1 has a critical point at some point ξ_2 satisfying $\xi_1 < \xi_2 < \xi_0$, which is a minimum and so $u_1'(\xi_2) = 0$ and $u_1''(\xi_2) > 0$. Using this result in (5.4a), we get $u_2'(\xi_2) < 0$ which contradicts the property $u_2'(\xi) > 0$ on (ξ_1, ξ_0).

For the case (ii), $u_2'(\xi) > 0$ on (b, ξ_0) and $u_2'(\xi) < 0$ on (ξ_0, c). As before, using $u_2'(\xi_0) = 0$ and $u_2''(\xi_0) < 0$ in (5.4a), we get $u_1'(\xi_0) > 0$. This, together with our assumption that u_1 has a critical point, shows that u_1 has a minimum at a point ξ_3 with $b < \xi_3 < \xi_0$ or a maximum at ξ_4 with $\xi_0 < \xi_4 < c$. From (5.4a) we get $u_2'(\xi_3) < 0$ or $u_2'(\xi_4) > 0$ which contradicts $u_2'(\xi) > 0$ on (b, ξ_0) and $u_2'(\xi) < 0$ on (ξ_0, c).

The analysis for the case $u_2''(\xi_0) > 0$ is similar. In that case we can conclude that u_1 is monotone decreasing. Thus, we have proved that u_2 can have only (at most) one critical point ξ_0 and that (c) holds. Proceeding similarly with u_1 we get (d). This completes the structure of the solutions of (5.4a).

Using this structure we get the a priori estimates (5.8). Integrating (5.5) from any point ξ_1 to ξ we get

$$\text{(5.11a)} \qquad u_1'(\xi)\,(a^2 - \xi^2)^{1-\frac{1}{2\varepsilon}} = u_1'(\xi_1)\,(a^2 - \xi_1^2)^{1-\frac{1}{2\varepsilon}} - \mu \int_{\xi_1}^{\xi} (a^2 - y^2)^{\frac{-1}{2\varepsilon}}\, u_2'(y)\, dy$$

and

$$\text{(5.11b)} \quad u_2'(\xi)\,(a^2 - \xi^2)^{1-\frac{1}{2\varepsilon}} = u_2'(\xi_1)\,(a^2 - \xi_1^2)^{1-\frac{1}{2\varepsilon}} + \mu \int_{\xi_1}^{\xi} (a^2 - y^2)^{\frac{-1}{2\varepsilon}}\, p(u_1)'(y)\, dy\,.$$

The only cases in which the estimate (5.10) is not obvious is on non-monotone component. Thus, suppose that u_2 is monotone but that u_1 has a critical point at ξ_0. Taking $\xi_1 = \xi_0$ in (5.11a) we get

$$u_1'(\xi)\,(a^2 - \xi^2)^{1-\frac{1}{2\varepsilon}} = -\mu \int_{\xi_1}^{\xi} (a^2 - y^2)^{\frac{-1}{2\varepsilon}}\, u_2'(y)\, dy\ .$$

Using the monotonicity of u_2 we get

$$|u_2'(\xi)| \;\leq\; \mu \,\frac{a^{\frac{1}{\varepsilon-1}}}{(a^2-a_*^2)^{\frac{-1}{2\varepsilon}}}\;|u_{2R}-u_{2L}|\;.$$

From this we get

$$\text{(5.12a)}\quad |u_1(\xi)| = |u_{1L}| + \int_b^\xi |u_1(y)|' dy \;\leq\; |u_{1L}| + (c-b)\,\mu \frac{a^{\frac{1}{\varepsilon-1}}}{(a^2-a_*^2)^{\frac{-1}{2\varepsilon}}}\,|u_{2R}-u_{2L}|\,.$$

Similarly, from (5.11b), if u_1 is monotone and u_2 has a critical point we obtain

$$\text{(5.12b)}\qquad |u_2(\xi)| \;\leq\; |u_{2L}| + (c-b)\,\mu\, \frac{a^{\frac{1}{\varepsilon-1}}}{(a^2-a_*^2)^{\frac{-1}{2\varepsilon}}}\,|p(u_{1R})-p(u_{1L})|\;.$$

The estimates (5.12) and the monotonicity properties yield precisely (5.10).

Now, we need to show $u_1(\xi) > \delta$ for some $\delta > 0$. This is not obious only in the case that u_2 is decreasing and that u_1 has a minimum. Assume that $\mu > 0$. Integrating the second equation in (5.4a) we get, for any $\xi_1 < \xi_2$,

$$\mu(p(u_1(\xi_2)) - p(u_1(\xi_1))) \;=\; \varepsilon(a^2-\xi_2^2)\,u_2'(\xi_2) - \varepsilon(a^2-\xi_1^2)\,u_2'(\xi_1) + \int_{\xi_1}^{\xi_2} y\,u_2'(y)\,dy\;.$$

Since the above right-hand side is finite, the left-hand side also must be finite. Since $p(u_1) = ku_1^{-\gamma}$, $\gamma \geq 1$ it follows that u_1 cannot be 0. Since we have given positive boundary conditions, we get $u_1 > 0$ for $\mu > 0$. For $\mu = 0$ clearly $u_1 = u_{10} > 0$ as observed earlier. Now using continuous dependence of solution of (5.4) on the parameter μ existence of $\delta > 0$ (which may depend on ϵ) bounding u_1 below follows. The proof of the lemma is complete. ∎

Proof of Theorem 5.1: Let $u^* = (u_{10}, 0)$, $u_{10} = \min(u_{1L}, u_{1R})$. Consider the map $F\colon cl(\Omega)\times[0,1] \to X$ defined by $F(u,\mu) = u - \mu\,Tu - (1-\mu)u^*$. $F(u,\mu)$ can be written as $u + G(u,\mu)$ with G compact as T is compact. The esimate (5.10) shows that for suitable choice of M and δ in the definition of Ω $F(u,\mu \neq 0$ for all $u \in \partial\Omega$, for all $0 \leq \mu \leq 1$. Also $F(u^*,0) = 0$ and the constant function $u = u^*$ is in Ω. Then, by Theorem (4.1) of [15], the desired result follows. ∎

Next, we are interested in the limit $\lim_{\varepsilon\to 0}(u_1^\varepsilon, u_2^\varepsilon)$. We start by obtaining a total variation bound independent of ε. From Theorem 5.1 the solutions has to satisfy (a),(b),(c) or (d). It is enough to consider only two cases, namely (c) and (d).

Lemma 5.3.

(i) *Let u_1^ε be strictly increasing and suppose that u_2^ε has exactly one critical point which is a maximum at ξ_ε. Then*

$$u_2^\varepsilon(\xi_\varepsilon) < u_{2L} + \int_{u_{1L}}^{u_{1R}} \sqrt{-p'(s)}\, ds \ . \tag{5.13}$$

(ii) *Let u_1^ε be strictly decreasing and suppose that u_2^ε has exactly one critical point which is a minimum at ξ_ε. Then*

$$u_2^\varepsilon(\xi_\varepsilon) > u_{2R} - \int_{u_{1L}}^{u_{1R}} \sqrt{-p'(s)}\, ds \ . \tag{5.14}$$

(iii) *Let u_2^ε be strictly increasing and suppose that u_1^ε has exactly one critical point which is a maximum at ξ_ε. Then*

$$u_{2R} - u_2^\varepsilon(\xi_\varepsilon) > \int_{u_{1R}}^{u_1(\xi_\varepsilon)} \sqrt{-p'(s)} ds \ , \tag{5.15a}$$

$$u_2^\varepsilon(\xi_\varepsilon) - u_{2L} > \int_{u_{1L}}^{u_1(\xi_\varepsilon)} \sqrt{-p'(s)} ds \ . \tag{5.15b}$$

(iv) *Let u_2^ε be strictly decreasing and suppose that u_1^ε has exactly one critical point which is a minimum at ξ_ε. Then*

$$u_2^\varepsilon(\xi_\varepsilon) - u_{2L} > -\int_{u_{1L}}^{u_1(\xi_\varepsilon)} \sqrt{-p'(s)} ds \ , \tag{5.16a}$$

$$u_{2R} - u_2^\varepsilon(\xi_\varepsilon) > -\int_{u_{1R}}^{u_1(\xi_\varepsilon)} \sqrt{-p'(s)} ds \ . \tag{5.16b}$$

Proof: For simplicity we supress the dependence of ε and call $(u_1^\varepsilon, u_2^\varepsilon)$ as (u_1, u_2). In the case of (i) and (ii) $u_1(\xi)$ is invertible in $[b, c]$. Call this inverse $\xi_\varepsilon(u_1)$. Differentiating

$$\frac{du_2}{du_1} = \frac{u_2'}{u_1'} \tag{5.17}$$

with respect to ξ and using (5.3a) in the resulting expression we get

$$\varepsilon(a^2 - \xi^2)\left(\frac{du_2}{du_1}\right)' = p'(u_1) + \left(\frac{du_2}{du_1}\right)^2 \tag{5.18}$$

and

$$\varepsilon(a^2-\xi^2)\Big(\frac{du_2}{du_1}\Big)'' = p''(u_1)\,u_1' \;+\; 2\Big(\varepsilon\,\xi + \frac{du_2}{du_1}\Big)\,\Big(\frac{du_2}{du_1}(u_1)\Big)' \,. \tag{5.19}$$

We prove (i), the proof of (ii) being similar. Since $u_2'(\xi_\varepsilon)=0$, we have $\frac{du_2}{du_1}(u_2(\xi_\varepsilon))=0$ and from (5.18) $(\frac{du_2}{du_1})'(\xi_\varepsilon)=\frac{p'(u_1)(\xi_\varepsilon)}{(a^2-\xi^2)}<0$. Define

$$\xi_1 \;=\; \inf\Bigg[\xi_0\in[b,\xi_\varepsilon]:\; p'(u_1)(\xi)+\Big(\frac{du_2}{du_1}\Big)^2(\xi)<0,\; \xi_0<\xi\le\xi_\varepsilon\Bigg].$$

If $\xi_1>b$ then we have $(\frac{du_2}{du_1})'(\xi_1)=0$ and $(\frac{du_2}{du_1})''(\xi_1)\le 0$. But by (5.19) we have $\varepsilon(a^2-\xi^2)(\frac{du_2}{du_1})''(\xi_1)=p''(u_1)(u_1)'(\xi_1)>0$. So $\xi_1=b$ and $p'(u_1)+(\frac{du_2}{du_1})^2<0$ for all $\xi\le\xi_\varepsilon$. Factorizing this we have

$$\Bigg(\frac{du_2}{du_1}+\sqrt{-p'(u_1)}\Bigg)\,\Bigg(\frac{du_2}{du_1}-\sqrt{-p'(u_1)}\Bigg)\;<\;0\;.$$

Using the monotonicity of u_1 and u_2 on $[b,\xi_\varepsilon]$, we have the first factor is positive and so the second factor

$$\frac{du_2}{du_1}-\sqrt{-p'(u_1)}<0\,,\qquad \xi\in[b,\xi_\varepsilon)\;.$$

Multiplying this with u_1' and integrating from b to ξ_ε gives (5.13). The proof for (5.14) is similar.

Now, we take up the case (iii). The proof for (iv) also is similar. Here u_2 is strictly increasing and u_1 is strictly increasing on $[b,\xi_\varepsilon)$ and strictly decreasing on $(\xi_\varepsilon,c]$ with a maximum at ξ_ε. We treat the intervals $[b,\xi)$ and $(\xi,c]$ seperately and denote the corresponding inverse of u_1 by $\xi(u_1)$. On each of these intervals the derivatives $(\frac{du_2}{du_1})'$ and $(\frac{du_2}{du_1})''$ are given by (5.18) and (5.19). Because of the monotonicity properties of u_1 and u_2 we have $\frac{du_2}{du_1}>0$ on $[b,\xi_\varepsilon)$ and $\frac{du_2}{du_1}<0$ on $(\xi_\varepsilon,c])$. Furthermore, $\frac{du_2}{du_1}\to\infty$ as $\xi\to\xi_\varepsilon$ from the left. Define

$$\xi_1 \;=\; \inf\Bigg\{\xi_0\in[b,\xi_\varepsilon]:\; p'(u_1)(\xi)+\Big(\frac{du_2}{du_1}\Big)^2(\xi)>0,\; \xi_0<\xi\le\xi_\varepsilon\Bigg\}$$

and

$$\xi_2 \;=\; \max\Bigg\{\xi_0\in[\xi_\varepsilon,c]:\; p'(u_1)(\xi)+\Big(\frac{du_2}{du_1}\Big)^2(xi)>0,\; \xi_\varepsilon<\xi\le\xi_0\Bigg\}.$$

As before it follows that $\xi_1 = b$ and so

$$p'(u_1)(\xi) + \left(\frac{du_2}{du_1}\right)^2(\xi) > 0\,, \qquad b < \xi \le \xi_\varepsilon$$

and $\xi_2 = c$ and so

$$p'(u_1)(\xi) + \left(\frac{du_2}{du_1}\right)^2(\xi) > 0\,, \qquad \xi_\varepsilon < \xi \le c$$

from which it follows that

$$\frac{du_2}{du_1} > \sqrt{-p'(u_1)}\,, \qquad b \le \xi < \xi_\varepsilon$$

and

$$\frac{du_2}{du_1} < -\sqrt{-p'(u_1)}\,, \qquad \xi_\varepsilon < \xi \le c\ .$$

Using $u_1' > 0$ on $[b, \xi_\varepsilon)$ and $u_1' < 0$ on $(\xi_\varepsilon, c]$, as before multiplying the first inequality by u_1' and the second by $-u_1'$ and then integrating, (5.15) follows. The proof for (5.16) is completely similar. ∎

We know that the Riemann problem for the p-system does not have solution for arbitrary data (Smoller [20], Chapter 17, for instance). and that a non-vacuum condition is necessary on the initial data for the existence of a solution. However, for $\gamma = 1$ no conditions are required.

Theorem 5.4. *Assume that the data is such that*

$$u_{2R} - u_{2L} < \int_{u_{10}}^{\infty} \sqrt{-p'(s)}\; ds\ . \tag{5.20}$$

Let $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ be a solution of (5.3) and let v^ε be given by (5.3c). Then, these functions are of uniformly bounded variation. The limits $u(\xi) := \lim_{\varepsilon\to 0} u^\varepsilon(\xi)$ and $v(\xi) := \lim_{\varepsilon\to 0} v^\varepsilon(\xi)$ exist pointwise along a subsequence and $v_1 = -u_2$ and $v_2 = p(u_1)$. Furthermore, u satisfies $\xi\, u' + f(u)' = 0$ in the interior.

Proof: Since u_1^ε and u_2^ε has at most one maximum or one minimum L^∞ estimate gives uniform BV bound for u^ε. It is enough to consider non-monotone cases and so we consider the four cases in Lemma (5.3). Cases (i) and (ii) the bound follows from (5.13) and (5.14). For the case (iv), adding (5.16a) and (5.16b) we get

$$u_{2R} - u_{2L} > -\int_{u_{1R}}^{u_1(\xi_\varepsilon)} \sqrt{-p'(s)}\; ds - \int_{u_{1L}}^{u_1(\xi_\varepsilon)} \sqrt{-p'(s)}\; ds\ .$$

Since $u_{2R} - u_{2L} < 0$ and $u_{1L} > 0$, $u_{2L} > 0$, it follows that there exists $c > 0$ such that $u_1^\varepsilon(\xi_\varepsilon) \geq c$. The remaining case is (iii), where we use (5.20). From (5.15) by adding we get

$$u_{2R} - u_{2L} > \int_{u_{1R}}^{u_1(\xi_\varepsilon)} \sqrt{-p'(s)}\, ds + \int_{u_{1L}}^{u_1(\xi_\varepsilon)} \sqrt{-p'(s)}\, ds .$$

In this case $u_{2R} - u_{2L} > 0$. We want to show $u_1^\varepsilon(\xi_\varepsilon) \leq c$, for some $c > 0$. If not we have

$$u_{2R} - u_{2L} > \int_{u_{1R}}^{\infty} \sqrt{-p'(s)}\, ds + \int_{u_{1L}}^{\infty} \sqrt{-p'(s)}\, ds .$$

which contradicts (5.20). The rest of the proof is as of Theorem (4.2). ∎

As observed earlier boundary conditions may not be satisfied by the limit. We consider the interesting case where the boundary is characteristic. Let λ_1,λ_2 are the eigenvalues of the system (5.1). Then $\lambda_1 = -(-p'(v))^{1/2}$ and $\lambda_2 = (-p'(v))^{1/2}$. Assume the characteristic condition

$$\lambda_1^m < b < \lambda_1^M . \tag{5.21}$$

The boundary layer equations (4.12)–(4.13) after a scaling of the independent variable y to $\frac{y}{a^2-b^2}$ becomes

$$u_1' = -u_2 - b\,u_1 + u_{2\infty} + b\,u_{1\infty}\,, \quad u_2' = p(u_1) - b\,u_2 - p(u_{1\infty}) + b\,u_{2\infty}\ , \tag{5.22a}$$

with boundary conditions

$$u_1(0) = u_{1L}, \quad u_1(\infty) = u_{1\infty}, \quad u_2(0) = u_{2L}, \quad u_2(\infty) = u_{2\infty}\ , \tag{5.22b}$$

where $(u_1(b+), u_2(b+))$ are related to $(u_{1\infty}, u_{2\infty})$ by the equations

$$u_2(b+) + b\,u_1(b+) = u_{2\infty} + b\,u_{1\infty}, \quad p(u_1(b+)) - b\,u_2(b+) = p(u_{1\infty}) - b\,u_{2\infty}\ . \tag{5.23}$$

Eliminating u_2 from (5.23), we get $b^2 u_1(b+) + p(u_1(b+)) = b^2 u_{1\infty} + p(u_{1\infty})$.

Note that the function $k(u_1) = b^2 u_1 + p(u_1)$ is convex since $p''(u_1) > 0$. This function is bounded below and has a minimum by (5.21). Let u_1^{**} is defined by the minimum of this function that is $b^2 = -p'(u_1^{**})$. For any $u_{10} \neq u_1^{**}$ define u_{10}^* as the unique solution of the equation $b^2 u_1 + p(u_1) = b^2 u_{10} + p(u_{10})$ other than u_{10} itself. Eliminating u_2 from (5.22), we get the equation

$$u_1'' = -2\,b\,u_1' + k(u_{1\infty}) - k(u_1) \tag{5.24a}$$

with boundary condition

$$u_1(b) = u_{1L}\,, \qquad u_1(\infty) = u_{1\infty}\ . \tag{5.24b}$$

For a given (u_{1L}, u_{2L}) define the sets

$$A(u_{1L}, u_{2L}) = \Big\{(u_1, u_2) :\ b^2(u_{1L} - u_1) < -b(u_{2L} - u_2) < -(p(u_{1L}) - p(u_1))\Big\} \tag{5.25a}$$

and

$$B(u_{1L}, u_{2L}) = \Big\{(u_1, u_2) :\ b^2(u_{1L} - u_1) > -b(u_{2L} - u_2) > -(p(u_{1L}) - p(u_1))\Big\}\ . \tag{5.25b}$$

Differentiating (5.22a) we get its solutions (u_1, u_2) satisfy the equation

$$u_1'' = -u_2' - bu_1'\,, \qquad u_2'' = p(u_1)' - bu_2'\ . \tag{5.26}$$

As in the proof of the structure of solutions of (5.4a) in Lemma(5.2) we have a solution (u_1, u_2) of (5.26) must be of one of the following type:

- (**a**) u_1 and u_2 are constant functions,
- (**b**) u_1 and u_2 are monotone functions,
- (**c**) u_1 is monotone increasing (decreasing) and then u_2 has exactly one critical point which is a maximum (minimum),
- (**d**) u_1 is monotone decreasing (increasing) and then u_2 has exactly one critical point which is a minimum (maximum).

We want to rule out (c) and (d). Consider the case (c) where u_1 is monotone increasing and u_2 has a maximum at ξ_0 in (b, ∞). Since ξ_0 is maximum we have $u_2' > 0$ on (b, ξ_0) and $u_2' < 0$ on (ξ_0, ∞). Now from the equation (5.26) we have, on integration

$$u_2'(\infty) - u_2'(\xi_0) \ = \ p(u_1(\infty)) - p(\xi_0) - b\Big[u_2(\infty) - u_2(\xi_0)\Big]\ .$$

Using $u_2'(\infty) = u_2'(\xi_0) = 0$ and rearranging terms we get

$$p(u_1(\infty)) - p(u_1(\xi_0)) \ = \ b\Big[u_2(\infty) - u_2(\xi_0)\Big]\ .$$

As u_2 is decreasing on (ξ_0, ∞), $b < 0$ and $p'(u_1) < 0$ this gives us $u_1(\infty) < u_1(\xi_0)$ which contradicts the fact that u_1 is monotone increasing. Now take the case u_1 decreasing; similar reasoning give u_2 cannot have a minimun in (b, ∞) and thus (c) is ruled out. The posibility (d) is also ruled out by similar argument. Now we write (5.2a) in the following form

$$u_2 - u_{2\infty} = -u_1' - b(u_1 - u_{1\infty}) , \qquad -p(u_1) - p(u_{1\infty}) = -u_2' - b(u_2 - u_{2\infty}) \ .$$

This shows that either both u_1 and u_2 are strictly increasing or both are strictly decreasing

$$u_1' > 0 \ \text{ and } \ u_2' > 0 \quad \text{ or } \quad u_1' < 0 \ \text{ and } \ u_2' < 0 \ . \tag{5.27}$$

Using this monotonicity properties we determine the set $(u_{1\infty}, u_{2\infty})$ such that (5.22) has a solution.

Suppose $u_1^{**} < u_{1L} < u_{1\infty}$. We prove there is no solution for this case. If there is a solution by (5.27), $u_1' > 0$ for all $y > b$ and so $u_1^{**} < u_{1L} < u_{1\infty}$. Then we get $k(u_1(y)) < k(u_{1\infty})$ for all $y > b$. Integrating (5.24a) we get

$$u_1'(y) \, = \int_y^\infty e^{2b(s-y)} \Big(k(u_1(y)) - k(u_{1\infty}) \Big) \, dy$$

which says $u_1'(y) < 0$, a contradiction.

Now take the case $u_1^{**} \le u_{1\infty} < u_{1L}$. Following exactly as before it follows that there is no solution in this case.

Next take the case $u_{1L}^{*} \le u_{1\infty} < u_1^{**} < u_{1L}$. Suppose there is a solution for (2.24) then $u_1'(y) < 0$, for all $y > b$ by using (5.27), $u_2'(y) < 0$ for all $y > b$. So from (5.22a) we get

$$-u_2 - b\,u_1 + u_{2\infty} + b\,u_{1\infty} < 0 , \qquad p(u_1) - b\,u_2 - p(u_{1\infty}) + b\,u_{2\infty} < 0 \ , \tag{5.28}$$

for $y > b$. Multiplying the first inequality by $-b$ and adding to the second we get $k(u_1) < k(u_{1\infty})$ for all y which contradicts our initial assumption $k(u_{1L}) > k(u_{1\infty})$.

Now consider the case $u_{1L} \le u_1^{**} < u_{1\infty}$. Repeating a similar argument as before gives non-existence of solution.

Now consider the case $u_{1L} > u_1^{**}$ and $u_{1\infty} < u_{1L}^{*}$. We show that there is solution to the problem (5.22) iff

$$-u_{2L} - b\,u_{1L} + u_{2\infty} + b\,u_{1\infty} < 0 , \quad p(u_{1L}) - b\,u_{2L} - p(u_{1\infty}) + b\,u_{2\infty} < 0 \tag{5.29a}$$

and

$$u_{2\infty} - u_{2L} \, = \int_{u_{1L}}^{u_{1\infty}} \frac{p(s) - p(u_{1\infty}) + b(u_{2\infty} - u_2(s))}{u_{2\infty} - u_2(s) + b(u_{1\infty} - u_1(s))} \, ds \ . \tag{5.29b}$$

From the equation (5.22a) we have

$$(5.30)\quad u_2 - u_{2\infty} = -u_1' - b(u_1 - u_{1\infty})\,, \quad -p(u_1) - p(u_{1\infty}) = -u_2' - b(u_2 - u_{2\infty})\ .$$

This shows that either both u_1 and u_2 is strictly increasing or both are strictly decreasing on the interval of existence and

$$\frac{du_2}{du_1} = \frac{p(s) - p(u_{1\infty}) + b(u_{2\infty} - u_2(s))}{\Big(u_{2\infty} - u_2(s) + b(u_{1\infty} - u_1(s)\Big)}\ .$$

It follows that (5.29) is necessary for existence. To prove sufficiency we consider the problem (5.22) with data satisfying (5.29). From (5.22a) we get on the interval of existence (5.28) holds. As before this leads to $k(u_1) < k(u_{1\infty})$. This shows that $u_1 < u_{1\infty}$. Using this in (5.28) we get u_2 is also bounded below. So we have u_1 and u_2 bounded decreasing solution of (5.22a) with initial conditions $u_1(0) = u_{1L}$ and $u_2(0) = u_{2L}$ with $u_{1\infty}$ and $u_{2\infty}$ satisfying (5.29). Clearly $\lim_{y\to\infty}(u_1(y), u_2(y)) = (u_{11}, u_{21})$ exists. We claim that $(u_{11}, u_{21}) = (u_{1\infty}, u_{1\infty})$. First consider u_1. There exits $y_n \in (b+n, b+n+1)$ such that $u_1(b+n+1) - u_1(b+n) = u'(y_n)$. Letting n tends to infinity we have $u_1'(y_n)$ goes to 0 as y_n go to infinity. For any sequence z_n going to infinity we have from (5.22a)

$$u_1'(z_n) - u_1'(y_n) = \Big(u_2(y_n) - u_2(z_n)\Big) + b\Big(u_1(y_n) - u_1(z_n)\Big)\ .$$

Letting n tends to infinity we get $\lim_{z_n\to\infty} u_1'(z_n) = 0$ and so $\lim_{y\to\infty} u_1'(y) = 0$. Similarly $\lim_{y\to\infty} u_2'(y) = 0$. So from (5.22a) we get

$$(5.31)\quad -u_{21} - b\,u_{11} + u_{2\infty} + b\,u_{1\infty} = 0\,, \quad p(u_{11}) - b\,u_{21} - p(u_{1\infty}) + b\,u_{2\infty} = 0\ .$$

From this we get $k(u_{11}) = k(u_{1\infty})$. As u_{11} and $u_{1\infty}$ are less than u_1^{**} it follows that $u_{11} = u_{1\infty}$. Using this in (5.31) it follows that $u_{21} = u_{2\infty}$.

By a similar reasoning we get existence for the cases $u_{1L} \leq u_{1\infty} \leq u_1^{**}$ iff

$$(5.32a)\quad -u_{2L} - b\,u_{1L} + u_{2\infty} + b\,u_{1\infty} > 0\,, \quad p(u_{1L}) - b\,u_{2L} - p(u_{1\infty}) + b\,u_{2\infty} > 0$$

and

$$(5.32b)\qquad u_{2\infty} - u_{2L} = \int_{u_{1L}}^{u_{1\infty}} \frac{p(s) - p(u_{1\infty}) + b(u_{2\infty} - u_2(s))}{u_{2\infty} - u_2(s) + b(u_{1\infty} - u_1(s))}\, ds\ .$$

and for the case $u_{1\infty} \leq u_{1L} \leq u_1^{**}$ if (5.29) is satisfied. Thus we proved the following Theorem.

Theorem 5.5.

(**i**) *Let (u_1, u_2) be a solution of (5.22). Then either u_1 and u_2 are constant functions or u_1 and u_2 are strictly increasing functions or both are strictly decreasing functions.*

(**ii**) *The set of $(u_{1\infty}, u_{2\infty})$ for which (5.22) has a solution is the union of (u_{1L}, u_{2L}) and a curve*

$$u_{2\infty} - u_{2L} = \int_{u_{1L}}^{u_{1\infty}} \frac{p(s) - p(u_{1\infty}) + b(u_{2\infty} - u_2(s))}{u_{2\infty} - u_2(s) + b(u_{1\infty} - u_1(s))}\, ds$$

lying in the set

$$C(u_{1L}, u_{2L}) = \Big\{(u_{1L}, u_{2L})\Big\} \cup \begin{cases} [0 < u_1 < u_{1L}^*] \cap B(u_{1L}, u_{2L}), & u_{1L} > u_1^{**}, \\ [u_{1L} < u_1 < u_1^*] \cap A(u_{1L}, u_{2L}) & \\ \qquad \cup [u_1 < u_{1L} < u_1^*] \cap B(u_{1L}, u_{2L}), & u_{1L} \le u_1^{**}. \end{cases}$$

Isentropic gas dynamics equation. Consider now the one-dimensional isentropic gas dynamics equation in Eulerian co-ordinates

$$\begin{aligned} &\partial_t \rho + \partial_x \rho u = 0\ , \\ &\partial_t(\rho u) + \partial_x(\rho u^2 + p(\rho)) = 0\ , \end{aligned} \tag{5.33}$$

with $p(\rho) = k\rho^\gamma$, $\gamma > 0$. The relaxation aproximation of (5.33)

$$\begin{aligned} &-\xi\, \rho^{\varepsilon\prime} + v_1^{\varepsilon\prime} = 0\,, \quad -\xi\, \rho^\varepsilon u^{\varepsilon\prime} + v_2^{\varepsilon\prime} = 0\ , \\ &-\xi\, v_1^{\varepsilon\prime} + a^2 \rho^{\varepsilon\prime} = \frac{1}{\varepsilon}\left(\rho^\varepsilon u^\varepsilon - v_1^\varepsilon\right), \quad -\xi\, v_2^{\varepsilon\prime} + a^2(\rho^\varepsilon u^\varepsilon)' = \frac{1}{\varepsilon}\left(\rho^\varepsilon u^{\varepsilon 2} + p(\rho) - v_2^\varepsilon\right) \end{aligned} \tag{5.34a}$$

on a bounded interval $[b, c]$ with the boundary conditions

$$\rho(b) = \rho_L, \quad \rho(c) = \rho_R\,, \qquad u(b) = u_L, \quad u(c) = u_R\ , \tag{5.34b}$$

leads to

$$\begin{aligned} &\varepsilon(a^2 - \xi^2)\, \rho'' = (2\,\varepsilon - 1)\, \xi\, \rho' + m'\ , \\ &\varepsilon(a^2 - \xi^2)\, m'' = (2\,\varepsilon - 1)\, \xi\, m' + \left(\frac{m^2}{\rho} + p(\rho)\right)' \end{aligned} \tag{5.35a}$$

on $[b, c]$ with $m = \rho\, u$ and the boundary conditions

$$\rho(b) = \rho_L, \quad \rho(c) = \rho_R\,, \qquad m(b) = \rho_L\, u_L, \quad m(c) = \rho_R\, u_R\ , \tag{5.35b}$$

with

$$v_1(\xi) = \varepsilon(a^2 - \xi^2)\,\rho' + m\,, \quad v_2(\xi) = \varepsilon(a^2 - \xi^2)\,(\rho\,u)' + \left(\frac{m^2}{\rho} + p(\rho)\right)\,. \tag{5.35c}$$

With minor modifications of the analysis of Slemrod and Tzavaras [19] we could also prove the existence of solutions with uniformly bounded variation independent of ε. The details are omitted.

REFERENCES

[1] Dafermos, C.M. – Solution of the Riemann problem for a class of hyperbolic conservation laws by the viscosity method, *Arch. Rational Mech. Anal.,* 52 (1973), 1–9.

[2] Dafermos, C.M. and DiPerna, R. – The Riemann problem for certain classes of hyperbolic systems of conservation laws, *J. Differential Equations,* 20 (1976), 90–114.

[3] Dubois, F. and LeFloch, P.G. – Boundary conditions for nonlinear hyperbolic systems of conservation laws, *J. Differential Equations,* 71 (1988), 93–122.

[4] Fan, H.-T. – A vanishing viscosity approach on the dynamics of phase transitions in van der Waals fluids, *J. Differential Equations,* 103 (1993), 179–204.

[5] Fan, H.-T. and Slemrod, M. – *The Riemann problem for systems of conservation laws of mixed type*, in "Shock Induces Transitions and Phase Structures in General Media" (R. Fosdick, E. Dunn and H. Slemrod, Eds.), IMA Vol. Math. Appl. Vol. 52, Springer-Verlag, New York, pp. 61–91.

[6] Joseph, K.T. and LeFloch, P.G. – *Boundary layers in weak solutions of hyperbolic conservation laws*, Preprint, Institute of Mathematics and its Applications, Minneapolis, Avril 1996.

[7] Joseph, K.T. and LeFloch, P.G. – Boundary layers in weak solutions of hyperbolic conservation laws, *Arch. Rational Mech. Anal.,* 147 (1999), 47–88.

[8] Joseph, K.T. and LeFloch, P.G. – Boundary layers in weak solutions of hyperbolic conservation laws. II, *Comm. Pure Applied Anal,* 1 (2002), 51–76.

[9] Jin, S. and Xin, Z.-P. – The relaxation schemes for systems of conservation laws in arbitrary space dimensions, *Comm. Pure Appl. Math.,* 48 (1995), 235–276.

[10] Lax, P.D. – *Hyperbolic Systems of Conservation Laws and the Mathematical Theory of Shock Waves*, SIAM, Philadelphia, 1973.

[11] LeFloch, P.G. – *Hyperbolic Systems of Conservation Laws: The Theory of Classical and Nonclassical Shock Waves*, Lectures in Mathematics, ETH Zurich, Birkhauser (2002).

[12] LeFloch, P.G. and Rohde, C. – The zero diffusion-dispersion limit for the Riemann problem, *Indiana Univ. Math. J.,* 50 (2002).

[13] LeFloch, P.G. and Tzavaras, A.E. – In preparation.

[14] Liu, T.P. – Hyperbolic conservation laws with relaxation, *Comm. Math. Phys.,* 108 (1987), 153–175.

[15] Mawhin, J. – *Topological Degree Methods in Nonlinear Boundary Value Problems*, CBMS 40, AMS, Providence, Rhode Island, 1979.

[16] Natalini, R. – *Recent results on hyperbolic relaxation problems*, in "Analysis of Systems of Conservation Laws" (Aachen, 1997), Chapman & Hall/CRC Monogr. Surv. Pure Appl. Math., 99, pp. 128–198.

[17] Slemrod, M. – *Vanishing viscosity-capillarity approach to the Riemann problem for a van der Waals fluid*, in "Non-classical continuum mechanics" (Durham, 1986), London Math. Soc. Lecture Note Ser., Vol. 122, 1987, Cambridge Univ. Press, Cambridge, pp. 325–335.

[18] Slemrod, M. – A limiting viscosity approach to the Riemann problem for materials exhibiting change of phase, *Arch. Rational Mech. Anal.,* 105 (1989), 327–365.

[19] Slemrod, M. and Tzavaras, A.E. – A limiting viscosity approach for the Riemann problem in isentropic gas dynamics, *Indiana Univ. Math. J.,* 38 (1989), 1047–1074.

[20] Smoller, J. – *Shock Waves and Reaction–Diffusion Equations*, Springer-Verlag, 1983.

[21] Tzavaras, A.E. – Wave interactions and variation estimates for self-similar viscous limits in systems of conservation laws, *Arch. Rational Mech. Anal.,* 135 (1996), 1–60.

K.T. Joseph,
School of Mathematics, Tata Institute of Fundamental Research,
Homi Bhabha Road, Bombay 400005 – INDIA
E-mail: `ktj@math.tifr.res.in`

and

P.G. LeFloch,
Centre de Mathématiques Appliquées and Centre National de la Recherche Scientifique,
U.M.R. 7641, Ecole Polytechnique, 91128 Palaiseau – FRANCE
E-mail: `lefloch@cmapx.polytechnique.fr`