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GLOBAL SOLUTIONS TO SOME NONLINEAR DISSIPATIVE MILDLY DEGENERATE KIRCHHOFF EQUATIONS

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Abstract: We investigate the evolution problem

$$u_{tt} + \delta\, u_t - m\Big(\int_\Omega |\nabla u|^2\, dx\Big)\, \Delta u + f(u) \,=\, 0 \;,$$
$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x)\,, \qquad x \in \Omega, \quad t \geq 0 \;,$$

where $n \leq 3$, $\Omega \subset \mathbb{R}^n$ is a bounded open set, $\delta > 0$, and $m : [0,+\infty[\;\to [0,+\infty[$ is a locally Lipschitz continuous function, with $m(0)=0$ and $m(r)>0$ in a neighborhood of 0, and $f(u)\, u \geq 0$.

We prove that this problem has a unique global solution for positive times, provided that the initial data $(u_0, u_1) \in (H^1_0 \cap H^2)(\Omega) \times H^1_0(\Omega)$ and f satisfy suitable smallness assumptions and the non-degeneracy condition $u_0 \neq 0$. We prove also that $(u(t), u'(t), u''(t)) \to (0,0,0)$ in $(H^1_0 \cap H^2)(\Omega) \times H^1_0(\Omega) \times L^2(\Omega)$ as $t \to \infty$.

1 – Introduction

Let $\Omega \subseteq \mathbb{R}^n$ $(n \leq 3)$ be an open domain, $H := L^2(\Omega)$, with norm $\|\cdot\|$ and scalar product $\langle\cdot,\cdot\rangle$. Let us set $A := -\Delta$, with domain $D(A) := (H^1_0 \cap H^2)(\Omega)$. We consider the Cauchy problem

$$\text{(1.1)} \qquad \begin{cases} u''(t) + \delta\, u'(t) + m(\|A^{1/2} u(t)\|^2)\, Au(t) + f(u(t)) \,=\, 0\,, \qquad t \geq 0\;, \\ u(0) = u_0\,, \qquad u'(0) = u_1 \;, \end{cases}$$

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where $\delta > 0$, $m\colon [0,+\infty[\to [0,+\infty[$ is a locally Lipschitz continuous function, $f \in C^1(\mathbb{R})$ and $f(u)\,u \geq 0$.

If $\Omega = [0,L]$ is an interval of the real line, this equation is a model for the damped small transversal vibrations of an elastic string with fixed endpoints.

It is well known that the motion of a clamped string in the 3-dimensional Euclidean space is described by a system of three quasilinear hyperbolic equations, whose unknowns are the transversal displacement u and the two components of the longitudinal displacement v. Unfortunately, the three equations in the exact system cannot be uncoupled. However, in the monograph [7], Kirchhoff showed that under the *Ansatz* that $v_{tt} = o(u_{tt})$, the string tension can be assumed to be independent of x. Therefore it can be approximated by its x-average. This allows to decouple the system (see [1] for the details), leading to the following equation for the transversal motion u (that is the original form of (1.1)):

$$\rho\, h\, u_{tt} + \delta\, u_t + f \;=\; \left(m_0 + \frac{Eh}{2L}\int_0^L |u_x|^2\right) u_{xx}$$

where L is the rest-length, E is the Young modulus, ρ is the mass density, h is the cross-section area, m_0 is the initial axial tension, δ is the resistance modulus and f is a nonlinear perturbation effect.

The Kirchhoff correction, where $m(r)$ is a general stress-strain function, is less drastic that the linear approximation, which correspond to consider the tension independent of x and t.

The case $m_0 > 0$ which in mathematics gives strict hyperbolicity, physically correspond to a pre-stressed string. In this paper we are interested in strings with zero rest-tension ($m_0 = 0$), which mathematically corresponds to weak hyperbolicity. Moreover we do not limit ourselves to the case where the stress-strain function $m(r)$ has a polynomial decay at $r = 0$.

The case $\delta = 0$, $f = 0$ (free vibrations) has long been studied: the interested reader can find appropriate references in the surveys of A. Arosio [1] and S. Spagnolo [13].

In the case $\delta = 0$, $f(u) = \pm|u|^{\alpha}\, u$ with large α and $m(r) \geq \nu > 0$, P. D'Ancona and S. Spagnolo [4] proved that if $u_0, u_1 \in C_0^{\infty}(\mathbb{R}^n)$ are small, then problem (1.1) has a global solution.

The non-degenerate case (i.e. $m(r) \geq \nu > 0$) with $\delta > 0$ and $f = 0$ was considered by E. H De Brito, Y. Yamada, and K. Nishihara [2, 12, 3, 9]: they proved that for small initial data $(u_0, u_1) \in D(A) \times D(A^{1/2})$ there exists a unique global solution of (1.1) that decays exponentially as $t \to \infty$.

Degenerate equations ($m(r) \geq 0$) with $\delta > 0$, $f = 0$, were considered by K. Nishihara and Y. Yamada [10], for $m(r) = r^\gamma$ ($\gamma \geq 1$), and for a general $m(r) \geq 0$ in [5]. In [5] it was proved the existence and uniqueness of a global solution $u(t)$ of (1.1) for small initial data $(u_0, u_1) \in D(A) \times D(A^{1/2})$ with $m(\|A^{1/2}u_0\|^2) \neq 0$ and the asymptotic behaviour $(u(t), u'(t), u''(t)) \to (u_\infty, 0, 0)$ in $D(A) \times D(A^{1/2}) \times H$ as $t \to +\infty$, where either $u_\infty = 0$ or $m(\|A^{1/2}u_\infty\|^2) = 0$.

The case $m(r) \geq \nu > 0$, $\delta > 0$, $f(u) = |u|^\alpha u$ has been considered by M. Hosoya and S. Yamada [6] under the following condition:

$$0 \leq \alpha < \frac{2}{n-4} \quad \text{if } n \geq 5\,, \qquad 0 \leq \alpha < +\infty \quad \text{if } n \leq 4\ .$$

They proved that, if the initial data are small enough, problem (1.1) has a global solution which decays exponentially as $t \to +\infty$.

Degenerate equations of type (1.1) were considered by K. Ono [11] when $n \leq 3$, for $\delta > 0$, $m(r) = r^\gamma$, $f(u) \cong |u|^\alpha u$. He proved that if the initial data are small enough, $u_0 \neq 0$, and:

$$\alpha > 2\gamma - 1 \quad \text{if } n = 1, 2\,, \qquad \alpha > 4\gamma - 2 \quad \text{if } n = 3\ , \tag{1.2}$$

then problem (1.1) has a global solution, that decays with a polynomial rate as $t \to +\infty$. However the technique of [11] which (besides the result of [10]) is based on a decay Lemma by M. Nakao [8] does not seem to be extendible to more general cases.

In this paper we consider problem (1.1) where m is any non-negative locally Lipschitz continuous function, and $m(0) = 0$, $m(r) > 0$ in a neighborhood of 0. We prove that there exists a unique global solution for $(u_0, u_1) \in D(A) \times D(A^{1/2})$ provided that (u_0, u_1) and f satisfy suitable smallness assumptions (cf. Theorem 2.2) and the non-degeneracy condition $u_0 \neq 0$ holds. Moreover we prove that $u(t) \to 0$ as $t \to \infty$. (cf. Theorem 2.4).

NOTATIONS

In this paper, we denote by a_1, a_2, b_ε, a_3 some constants such that

$$\begin{aligned}
&\|u\| \leq a_1 \|A^{1/2}u\| && u \in D(A^{1/2}) && n = 1, 2, 3\ ;\\
&\|u\|_\infty \leq a_2 \|A^{1/2}u\| && u \in D(A^{1/2}) && n = 1\ ;\\
&\|u\|_\infty \leq b_\varepsilon \|Au\|^\varepsilon \|A^{1/2}u\|^{1-\varepsilon} && u \in D(A) && n = 2, \quad 0 < \varepsilon \leq 1\ ;\\
&\|u\|_\infty \leq a_3 \|Au\|^{1/2} \|A^{1/2}u\|^{1/2} && u \in D(A) && n = 3\ .
\end{aligned} \tag{1.3}$$

2 – Statement of the results

In this section we state the main results of this paper. For completeness' sake, we recall the following local existence result, which may be proved by fixed point theorems (a sketch of the proof is included in Section for the convenience of the reader).

Theorem 2.1. (Local existence) *Let* $\delta > 0$, *let* $m\colon [0,+\infty[\, \to [0,+\infty[$ *be a locally Lipschitz continuous function,* $f \in C^1(\mathbb{R})$, *and let* $(u_0, u_1) \in D(A)\times D(A^{1/2})$ *with* $m(\|A^{1/2}u_0\|^2) > 0$.

Then there exists $T > 0$ *such that problem (1.1) has a unique solution*

$$u \in C^2([0,T]; H) \cap C^1([0,T]; D(A^{1/2})) \cap C^0([0,T]; D(A)) \ .$$

Moreover, u *can be uniquely continued to a maximal solution defined in an interval* $[0, T_*[$, *and at least one of the following statements is valid:*

(i) $T_* = \infty\,;$

(ii) $\limsup_{t\to T_*^-}(\|A^{1/2}u'(t)\|^2 + \|Au(t)\|^2) = +\infty\,;$

(iii) $\liminf_{t\to T_*^-} m(\|A^{1/2}u(t)\|^2) = 0\,.$

We can state the global existence result.

Theorem 2.2. (Global existence) *Let* $\delta > 0$, *and let* $m(r)$ *be a locally Lipschitz continuous function with* $m(0) = 0$ *and* $m(r) > 0$ *on some* $]0, r_0]$. *Let us assume that* $f(y)$ *is a* C^1 *function on* $\mathbb{R}$ *satisfying one of the following conditions in some neighborhood of* $u = 0$:

either

(i) $f(y)y \geq 0$ *and:*

$$\max_{|y|\le s} |f'(y)| \;\le\; \begin{cases} C\, m(s^{2+\varepsilon}) & \text{if} \quad n = 1,\ 2 \\ C\, m(s^4) & \text{if} \quad n = 3 \end{cases} \tag{2.1}$$

or

(ii) $f(0) = 0$, $f' \geq 0$ *and:*

$$\max_{|y|\le s} |f'(y)| \;\le\; \begin{cases} C\, m(s^{2+\varepsilon})s^{-1+\varepsilon} & \text{if} \quad n = 1,\ 2 \\ C\, m(s^4)s^{-2+\varepsilon} & \text{if} \quad n = 3 \end{cases} \tag{2.2}$$

for some $\varepsilon > 0$.

Moreover let us assume that the initial data $(u_0, u_1) \in D(A)\times D(A^{1/2})$ *are small enough and satisfy the non-degeneracy condition* $u_0 \neq 0$.

Then problem (1.1) admits a unique global solution

$$u \in C^2([0,+\infty[; H) \cap C^1([0,\infty[; D(A^{1/2})) \cap C^0([0,\infty[; D(A)) \ .$$

Remark 2.3. Theorem 2.2 (i) is still true, when $n = 1$, if we replace the condition (2.1) with:

$$\sup_{|y|\leq a_2\, s} |f'(y)| \leq C\, m(s^2) \ , \tag{2.3}$$

where a_2 is a constant for which (1.3) holds. □

If $m(r) = r^\gamma$ and $|f'(u)| \leq k\,|u|^\alpha$, thanks to (2.3), by Theorem 2.2 (i), we obtain the result of [11] under the stronger assumption that:

$$\alpha \geq 2\,\gamma \ \text{ if } n=1\,, \qquad \alpha > 2\,\gamma \ \text{ if } n=2\,, \qquad \alpha \geq 4\,\gamma \ \text{ if } n=3\ .$$

On the other hand, Theorem 2.2 (ii) allows us to obtain the same conclusion of [11] with $m(r) = r^\gamma$, $f(u) = |u|^\alpha\, u$ under the assumption (1.2).

Finally we have the following result.

Theorem 2.4. (Asymptotic behaviour) *Under the assumptions of Theorem 2.2 we have that:*

(i) $m(\|A^{1/2}u(t)\|^2) > 0$ *for all* $t \geq 0$;

(ii) $(u(t), u'(t), u''(t)) \to (0,0,0)$ *in* $D(A)\times D(A^{1/2})\times H$ *as* $t \to \infty$.

The proof of Theorem 2.4 relies on a result about the asymptotic behaviour of solutions of the linearization of (1.1) (see Lemma 3.2 for the precise statement).

3 – Proofs

3.1. Local existence

Proof of Theorem 2.1 (see [11])**:** Since the argument is standard, we only sketch the main steps of the proof.

Step 1. Let us set:

$$m_0 := m(\|A^{1/2}u_0\|^2)\,, \qquad m_* := \max\left\{1, \frac{2}{m_0}\right\},$$

$$F_0 := \|A^{1/2}u_1\|^2 + m_0\,\|Au_0\|^2\,, \qquad R^2 = 3\,m_*\,F_0\ ,$$

$$c_R := \|m'\|_{L^\infty([0,a_1^2R^2])}\,, \qquad \alpha_R := 2\,c_R(a_1^2+1)\,\frac{R^2}{m_0}\ ,$$

$$f_R := \max_{|y|\le c_1 R} |f'(y)|, \qquad \text{where}\ \ c_1 := \begin{cases} a_1\,a_2 & n=1,\\ b_1 & n=2,\\ a_3\,\sqrt{a_1} & n=3\ . \end{cases}$$

Moreover let us define:

$$T\ :=\ \min\left\{\frac{\log 2}{\alpha_R},\ \frac{\delta\,\log 2}{3\,(f_R+2\,c_R\,R)^2\,a_1^2\,m_*}\right\}\ . \tag{3.1}$$

Let us set $\mathrm{C} := C^0_w([0,T];D(A)) \cap C^1_w([0,T];D(A^{1/2}))$ and let us consider the functional space

$$X_{R,T}\ :=\ \Big\{v\in \mathrm{C}:\ v(0)=u_0,\ v'(0)=u_1, \\ \|A^{1/2}u'(t)\|^2 + \|Au(t)\|^2 \le R^2,\ \ t\in[0,T]\Big\}\ .$$

This space, with the distance:

$$d(v_1,\,v_2)\ :=\ \max_{t\in[0,T]}\Big(\|(v_1-v_2)'(t)\|^2 + \|A^{1/2}(v_1-v_2)(t)\|^2\Big)^{1/2}$$

is a complete metric space.

Let us set, for $v\in X_{R,T}$:

$$c_v(t) := m(\|A^{1/2}v(t)\|^2)\ .$$

Since

$$|c_v'(t)|\ \le\ c_R\,(a_1^2+1)\,R^2\ <\ \frac{m_0}{2\,T}$$

it follows that

$$c_v(t)\ >\ \frac{m_0}{2} \qquad \forall\, t\in[0,T]\ . \tag{3.2}$$

We can therefore define

$$[\Phi(v)]\,(t) = u\ ,$$

where $u \in \mathrm{C}$ is the unique solution of the linear problem

$$\begin{cases} u''(t) + \delta\, u'(t) + c_v(t)\, Au(t) + f(v(t)) = 0\,, & t \geq 0\ , \\ u(0) = u_0\,, \quad u'(0) = u_1\ . \end{cases} \tag{3.3}$$

Step 2. We show that Φ maps $X_{R,T}$ into itself.
To this end, let us define:

$$F(t) := \|A^{1/2}u'(t)\|^2 + c_v(t)\,\|Au(t)\|^2\ .$$

By a standard computation, we obtain:

$$F'(t) \;\leq\; \alpha_R\, F + \frac{(f_R\, a_1 R)^2}{\delta}\ .$$

Hence, recalling the definition of R, since $\dfrac{e^y - 1}{y}$ is an increasing function and $\alpha_R\, T \leq \log 2$:

$$F \;\leq\; F_0\, e^{\alpha_R T} + (e^{\alpha_R T} - 1)\ \frac{(f_R\, a_1 R)^2}{\delta\, \alpha_R} \;\leq\; 3\, F_0\ .$$

Thus we have proved that

$$\|A^{1/2}u'(t)\|^2 + \|Au(t)\|^2 \;\leq\; 3\, m_*\, F_0 \;=\; R^2\ .$$

Step 3. We show that Φ is Lipschitz continuous, with a Lipschitz constant less then 1. For $v^1, v^2 \in X_{R,T}$ let us set $u^1 = \Phi(v^1)$, $u^2 = \Phi(v^2)$, $w = u^1 - u^2$. If we consider

$$F_w(t) := \|w'(t)\|^2 + c_{v_1}(t)\,\|A^{1/2}w(t)\|^2\ ,$$

then

$$F_w'(t) \;\leq\; \alpha_R\, F_w + \frac{a_1^2}{\delta}\,(2\, c_R\, R + f_R)^2\, d^2(v^1, v^2)\ .$$

Hence

$$d^2(u^1, u^2) \;\leq\; m_*(e^{\alpha_R T} - 1)\, \frac{a_1^2}{\delta\, \alpha_R}\,(2\, c_R\, R + f_R)^2\, d^2(v^1, v^2) \;\leq\; \frac{1}{2}\, d^2(v^1, v^2)\ .$$

This complete the proof of this step.

Step 4. By step 2 – step 3, the map Φ has a unique fixed point u that is a weakly solution of (1.1); moreover in a standard way (see [14]) one can prove that

$$u \in C^2([0,T];H) \cap C^1([0,T];D(A^{1/2})) \cap C^0([0,T];D(A)) \ .$$

Step 5. Let us prove the last part of the statement.

Let $[0, T_*[$ be the maximal interval where the solution exists, and let us assume by contradiction that (i), (ii), and (iii) are false. Then there exist two constants ν, M such that $m(\|A^{1/2}u(t)\|^2) \geq \nu > 0$ in a left neighborhood of T_*, and $\|A^{1/2}u'(t)\|^2 + \|Au(t)\|^2 \leq M$ for every $t \in [0, T_*[$. By (3.1) it follows that, for all S in a left neighborhood of T_*, the life span of the solution of

$$\begin{cases} w''(t) + \delta\, w'(t) + m(\|A^{1/2}w(t)\|^2)\, Aw(t) + f(w(t)) = 0\,, & t \geq S\ , \\ w(S) = u(S)\,, \quad w'(S) = u'(S)\ , \end{cases}$$

is larger then a strictly positive quantity independent of S. This contradicts the maximality of T_*. ∎

3.2. Global existence

In the sequel we need the following comparison result for ODEs (the simple proof is omitted).

Lemma 3.1. *Let $T > 0$, and let $g \in C^1([0,T[, \mathbb{R})$. Let us assume that $g(t) \geq 0$ in $[0,T[$ and that there exist two constants $c_1 > 0$, $c_2 \geq 0$ such that*

$$(g(t))' \leq -\sqrt{g(t)}\left(c_1\sqrt{g(t)} - c_2\right) \qquad \forall\, t \in [0,T[\ .$$

Then

$$\sqrt{g(t)} \leq \max\left\{\sqrt{g(0)}, \frac{c_2}{c_1}\right\}$$

for all $t \in [0,T[$. ∎

From now on we use the following notations:

$$\phi_\varepsilon(n) := \begin{cases} (a_1^\varepsilon\, a_2)^{-2/\varepsilon} & n = 1, \\ (b_\varepsilon)^{-2/\varepsilon} & n = 2, \\ (a_3)^{-4} & n = 3\ , \end{cases}$$

$$\beta := \begin{cases} 1 & \text{if } n = 1,\ 2, \\ 1/2 & \text{if } n = 3\ , \end{cases}$$

$$\mu_f(s) := \max_{|y| \le s} |f'(y)|\,, \qquad \sqrt{c} := C\ .$$

With these notations, without loss of generality, we can rewrite (2.1)–(2.2) as follows:

$$\mu_f(s^{\beta-\varepsilon}) \le \sqrt{c}\ m(s^2) \qquad s \in [0, \sqrt{r_0}] \tag{3.4}$$

for some $0 < \varepsilon < 1$ if $n = 1,\ 2$, and $\varepsilon = 0$ if $n = 3$, and:

$$\mu_f(s^{\beta-\varepsilon_0}) \le \sqrt{c}\ m(s^2)\, s^{\varepsilon_1 - 1} \quad s \in [0, \sqrt{r_0}] \tag{3.5}$$

for some $0 < \varepsilon_0 < 1$ if $n = 1, 2$, $\varepsilon_0 = 0$ if $n = 3$, and $0 < \varepsilon_1 < 1$.

Proof of Theorem 2.2:

Case (i)

Let us set:

$$E_0 := \|u_1\|^2 + \int_0^{\|A^{1/2}u_0\|^2} m(s)\, ds \, + \, 2 \int_\Omega \int_0^{u_0} f(s)\, ds\ ,$$

$$F_0 := \frac{\|A^{1/2}u_1\|^2}{m(\|A^{1/2}u_0\|^2)} + \|Au_0\|^2 + \frac{c}{\delta}\Big(\langle u_0, u_1\rangle + \frac{\delta}{2}\,\|u_0\|^2\Big) + \frac{c}{2\,\delta^2}\,E_0$$

$$G_0 := \max\left\{\frac{\|u_1\|}{m(\|A^{1/2}u_0\|^2)},\ \frac{2}{\delta}(\sqrt{c}\,a_1^2 + 1)\,\sqrt{F_0}\right\}$$

$$c_{F_0} := \max_{0 \le s \le a_1^2 F_0} |m'(s)|\ .$$

Let us suppose that the initial data verifies:

$$c_{F_0}\, G_0\, \sqrt{F_0}\ < \frac{\delta}{4}\,, \qquad F_0 < \min\left\{\phi_\varepsilon,\, r_0\, a_1^{-2}\right\} =: \sigma\ .$$

We prove that under these smallness assumptions the solution u of (1.1) is a global solution.

In the following let us set

$$c(t) = m(\|A^{1/2}u(t)\|^2)\ .$$

Let us assume that $m \in C^1([0, +\infty[; \mathbb{R})$, and let $[0, T_*[$ be the maximal interval where the solution exists.

Step 1. A priori estimates

Let us set

$$E(t) := \|u'(t)\|^2 + \int_0^{\|A^{1/2}u(t)\|^2} m(s)\,ds \,+\, 2\int_\Omega \int_0^u f(s)\,ds \,+\, 2\,\delta \int_0^t \|u'(s)\|^2\,ds\ .$$

Since E is a conserved energy and $f(u)u \geq 0$, we have:

$$\|u'(t)\|^2 \,+\, 2\,\delta \int_0^t \|u'(s)\|^2\,ds \,\leq\, E(0) = E_0 \qquad t \in [0, T_*[\ . \tag{3.6}$$

Furthermore, by taking the scalar product of the equation (1.1) with u, and integrating on $[0,t]$ we obtain:

$$\begin{aligned}
&\int_0^t \Big(c(s)\,\|A^{1/2}u(s)\|^2 + \langle f(u(s)), u(s)\rangle \Big)\,ds \;= \\
&\qquad = \int_0^t \|u'(s)\|^2\,ds \,+\, \langle u_0, u_1\rangle - \langle u(t), u'(t)\rangle + \frac{\delta}{2}\,\|u_0\|^2 - \frac{\delta}{2}\,\|u(t)\|^2 \\
&\qquad \leq \int_0^t \|u'(s)\|^2\,ds \,+\, \frac{\|u'(t)\|^2}{2\,\delta} + \langle u_0, u_1\rangle + \frac{\delta}{2}\,\|u_0\|^2 \\
&\qquad \leq \frac{1}{2\,\delta}\,E_0 + \langle u_0, u_1\rangle + \frac{\delta}{2}\,\|u_0\|^2\ .
\end{aligned}$$

Hence, for $t \in [0, T_*[$:

$$\int_0^t c(s)\,\|A^{1/2}u(s)\|^2\,ds \,\leq\, \frac{1}{2\,\delta}\,E_0 + \langle u_0, u_1\rangle + \frac{\delta}{2}\,\|u_0\|^2\ . \tag{3.7}$$

Step 2. Definitions – considerations

Let us set

$$T := \sup\left\{ \tau \in [0, T_*[:\ c(t) > 0,\ \left|\frac{c'(t)}{c(t)}\right| \leq \frac{\delta}{2},\ \ \|Au(t)\|^2 \leq \sigma \quad \forall\, t \in [0,\tau] \right\}. \tag{3.8}$$

We show that $T = T_*$. Let us assume by contradiction that $T < T_*$. Since $|c'(t)| \leq \dfrac{\delta}{2}\,c(t)$ in $[0, T[$ we have that

$$0 \,<\, c(0)\,e^{-\delta T/2} \,\leq\, c(T) \,\leq\, c(0)\,e^{\delta T/2}\ . \tag{3.9}$$

Moreover, by $\|Au(t)\|^2 \leq \sigma$ we obtain:

$$\|A^{1/2}u(t)\|^2 \,\leq\, a_1^2\,\|Au(t)\|^2 \,\leq\, r_0 \qquad t \in [0, T]\ .$$

Since $c(\cdot)$, $c'(\cdot)$, and $\|Au(\cdot)\|^2$ are continuous functions, by the maximality of T we have that necessarily

$$\left|\frac{c'(T)}{c(T)}\right| = \frac{\delta}{2}\ ; \tag{3.10}$$

or

$$\|Au(T)\|^2 = \sigma\ . \tag{3.11}$$

Step 3. (3.11) is false

Let us set

$$F(t) := \frac{\|A^{1/2}u'(t)\|^2}{c(t)} + \|Au(t)\|^2\ .$$

Then, a standard calculation shows that on $[0,T[$ we have:

$$\begin{aligned} F'(t) &\le -\left(2\,\delta + \frac{c'(t)}{c(t)}\right)\frac{\|A^{1/2}u'(t)\|^2}{c(t)} + \frac{2}{c(t)}\,\|A^{1/2}u'(t)\|\,\|f'(u(t))\,A^{1/2}u(t)\| \\ &\le -\frac{\delta}{2}\,\frac{\|A^{1/2}u'(t)\|^2}{c(t)} + \frac{1}{\delta c(t)}\,\|f'(u(t))\,A^{1/2}u(t)\|^2\ . \end{aligned}$$

Since $\|Au(t)\|^2 \le \phi_\varepsilon$, we have:

$$|u(t,x)| \le \|u(t)\|_\infty \le \|A^{1/2}u(t)\|^{\beta-\varepsilon}\ , \tag{3.12}$$

hence, by (3.4)

$$\begin{aligned} \|f'(u(t))\,A^{1/2}u(t)\|^2 &\le \mu_f\Big(\|A^{1/2}u(t)\|^{\beta-\varepsilon}\Big)^2\,\|A^{1/2}u(t)\|^2 \\ &\le c\,m\Big(\|A^{1/2}u(t)\|^2\Big)^2\,\|A^{1/2}u(t)\|^2\ . \end{aligned} \tag{3.13}$$

By this fact:

$$F(t) \le F(0) + \frac{c}{\delta}\int_0^t c(s)\,\|A^{1/2}u(s)\|^2\,ds\ ;$$

therefore, by (3.7)

$$F(T) \le F_0 < \sigma\ . \tag{3.14}$$

This contradicts (3.11).

Step 4. (3.10) is false

Let us define $G(t) := \dfrac{\|u'(t)\|}{c(t)}$. By a simple computation, on $[0,T[$ we obtain:

$$(G^2(t))' \le -\delta\,G^2(t) + 2\,G(t)\,\|Au(t)\| + 2\,G(t)\,\frac{\|f(u(t))\|}{c(t)}\ .$$

Since $f(0) = 0$, by (3.4) (see (3.12)–(3.13)) we have:

$$\begin{aligned} \int_\Omega f(u(t,x))^2 &= \int_\Omega f'(\xi_u(t,x))^2\, u^2(t,x) \\ &\leq \mu_f\Big(\|A^{1/2}u(t)\|^{\beta-\varepsilon}\Big)^2\, \|u(t)\|^2 \\ &\leq c\, a_1^4\, m\Big(\|A^{1/2}u(t)\|^2\Big)^2\, \|Au(t)\|^2 \,. \end{aligned} \tag{3.15}$$

By this fact, using the analogous of (3.14) for $t \in [0, T[$:

$$(G^2(t))' \leq -G(t)\left(\delta\, G(t) - 2\,(1 + \sqrt{c}\; a_1^2)\,\sqrt{F_0}\right) .$$

Hence, applying Lemma 3.1 with $g = G^2$ we have:

$$G(T) \leq \max\left\{G(0),\ \frac{2\,(1+\sqrt{c}\; a_1^2)}{\delta}\,\sqrt{F_0}\right\} = G_0 \,. \tag{3.16}$$

By (3.14)–(3.16), we have then

$$\begin{aligned} \left|\frac{c'(T)}{c(T)}\right| &= \left|\frac{2\,m'(|A^{1/2}u(T)|^2)\,\langle u'(T), Au(T)\rangle}{c(T)}\right| \\ &\leq 2 \max_{0\leq r\leq a_1^2 F_0} |m'(r)|\, \frac{|u'(T)|}{c(T)}\, |Au(T)| \\ &\leq 2\, c_{F_0}\, G_0\, \sqrt{F_0} < \frac{\delta}{2} \,. \end{aligned}$$

This contradicts (3.10).

Step 5. Conclusion

Let us assume by contradiction that $T_* < +\infty$. By (3.9) and (3.14) it follows that

$$\liminf_{t\to T_*^-} m(\|A^{1/2}u(t)\|^2) \geq m(\|A^{1/2}u_0\|^2)\, e^{-\delta T_*/2} > 0 \,,$$

$$\limsup_{t\to T_*^-} \|A^{1/2}u'(t)\|^2 + \|Au(t)\|^2 \leq \max\Big\{1,\, c(0)\, e^{\delta T_*/2}\Big\}\, F_0 < +\infty \,.$$

By the last statement of Theorem 2.1 this is a contradiction. This completes the proof if m' is continuous. If m is only locally Lipschitz continuous, thesis follows from a standard approximation argument. ∎

Case (ii)

Let us set:

$$E_0 := \frac{\|u_1\|^2}{m(\|A^{1/2}u_0\|^2)} + \|A^{1/2}u_0\|^2\,, \qquad F(0) := \frac{\|A^{1/2}u_1\|^2}{m(\|A^{1/2}u_0\|^2)} + \|Au_0\|^2$$

$$H_0 := \frac{\delta}{\varepsilon_1}\,\|A^{1/2}u_0\|^{\varepsilon_1} + \frac{\langle Au_0, u_1\rangle}{\|A^{1/2}u_0\|^{2-\varepsilon_1}}\,, \qquad c_1 := \sup_{0\le s\le 1} |m'(s)|\ .$$

Moreover let us define:

$$\sigma := \min\Big\{\phi_{\varepsilon_0},\, r_0\, a_1^{-2}\Big\}\,, \qquad \sigma_1 := \delta\,\frac{1 - E(0)^{1-\varepsilon_1/2}}{c\, a_1^2}\,, \qquad \sigma_2 := \frac{\delta(\sigma - F(0))}{c}\ .$$

Let us suppose that, for a suitable λ:

$$0 < \lambda < \lambda_0 := \min\{\sigma_1, \sigma_2\}\,, \qquad a_1^2\Big(H_0 + \frac{2\,c_1}{\delta}\,F_0\Big) < \lambda\,, \qquad c_1\, G_0 \sqrt{F_0} < \frac{\delta}{4}\,,$$

where

$$G_0 \;:=\; \min\Bigg\{\frac{\|u_1\|}{m(\|A^{1/2}u_0\|^2)},\, \frac{2}{\delta}(\sqrt{F_0} + \sqrt{c}\; a_1^{1+\varepsilon_1} F_0^{\varepsilon_1/2})\Bigg\}$$

$$F_0 \;:=\; F(0) + \frac{c\,\lambda}{\delta}\Big(E(0)^{1-2/\varepsilon_1} + \frac{c\,a_1^2}{\delta}\,\lambda\Big)^{\frac{\varepsilon_1}{2-\varepsilon_1}}\ .$$

Then we prove that under these smallness conditions the solution u of (1.1) is a global solution.

In the following let us set

$$c(t) = m(\|A^{1/2}u(t)\|^2)\ .$$

Let us assume that $m \in C^1([0,+\infty[,\mathbb{R})$ and let $[0, T_*[$ be the maximal interval where the solution exists.

Step 1. Definitions – considerations

Let us set

$$T := \sup\Bigg\{\tau \in [0, T_*[:\ c(t) > 0,\ \left|\frac{c'(t)}{c(t)}\right| \le \frac{\delta}{2},\ \|Au(t)\|^2 \le \sigma \quad \int_0^t c(s)\,\|A^{1/2}u(s)\|^{\varepsilon_1}\,ds \;\le\; \lambda \;\; \forall\, t \in [0,\tau]\Bigg\}\ . \tag{3.17}$$

We show that $T = T_*$. Let us assume by contradiction that $T < T_*$.

Firstly let us remark that in $[0, T]$ we have:

$$\|A^{1/2}u(t)\|^2 \,\leq\, a_1^2\,\|Au(t)\|^2 \,\leq\, r_0 \ .$$

Furthermore, since $|c'(t)| \leq \dfrac{\delta}{2}\,c(t)$ in $[0, T[$, we have that

$$0 \,<\, c(0)\,e^{-\delta T/2} \,\leq\, c(T) \,\leq\, c(0)\,e^{\delta T/2} \ . \tag{3.18}$$

Since $c(t)$, $c'(t)$, $\displaystyle\int_0^t c(s)\,\|A^{1/2}u(s)\|^{\varepsilon_1}\,ds$ and $\|Au(t)\|^2$ are continuous functions, by the maximality of T we have that necessarily

$$\left|\frac{c'(T)}{c(T)}\right| = \frac{\delta}{2} \ ; \tag{3.19}$$

or

$$\|Au(T)\|^2 = \sigma \ , \tag{3.20}$$

or

$$\int_0^T c(s)\,\|A^{1/2}u(s)\|^{\varepsilon_1}\,ds \,=\, \lambda \ . \tag{3.21}$$

Step 2. (3.20) is false

Let us set:

$$E(t) := \frac{\|u'(t)\|^2}{c(t)} + \|A^{1/2}u(t)\|^2 \ .$$

Hence a simple calculation show that in $[0, T[$ we have:

$$E'(t) \leq -\frac{\delta}{2}\,\frac{\|u'(t)\|^2}{c(t)} + \frac{\|f(u(t))\|^2}{\delta\,c(t)} \ ;$$

therefore, as in (3.15), using (3.5), and $\|Au(t)\|^2 \leq \phi_{\varepsilon_0}$ (see (3.12)):

$$E'(t) \,\leq\, \frac{c\,a_1^2}{\delta}\,c(t)\,\|A^{1/2}u(t)\|^{2\,\varepsilon_1} \,\leq\, \frac{c\,a_1^2}{\delta}\,c(t)\,\|A^{1/2}u(t)\|^{\varepsilon_1}E^{\varepsilon_1/2} \ .$$

By this fact, since $\lambda \leq \lambda_0$ we have:

$$E(t)^{1-\varepsilon_1/2} \,\leq\, E(0)^{1-\varepsilon_1/2} + \frac{c\,a_1^2}{\delta}\,\lambda =: \gamma \,\leq\, 1 \ . \tag{3.22}$$

We can now estimate

$$F(t) := \frac{\|A^{1/2}u'(t)\|^2}{c(t)} + \|Au(t)\|^2 \ .$$

In fact, by using an estimate likes (3.13), we have:

$$F'(t) \;\leq\; -\frac{\delta}{2}\,\frac{\|A^{1/2}u'(t)\|^2}{c(t)} + \frac{c\,c(t)}{\delta}\,\|A^{1/2}u(t)\|^{\varepsilon_1}E^{\varepsilon_1/2}(t)\ ,$$

hence:

$$F(T)+\frac{\delta}{2}\int_0^T\frac{\|A^{1/2}u'(s)\|^2}{c(s)}\,ds \leq F(0)+\frac{c\,\lambda}{\delta}\,\gamma^{\frac{\varepsilon_1}{2-\varepsilon_1}} = F_0 \leq F(0)+\frac{c\,\lambda}{\delta} < \sigma\ . \tag{3.23}$$

Step 3. (3.21) is false

By taking the scalar product of the equation (1.1) with $\dfrac{Au}{\|A^{1/2}u\|^{2-\varepsilon_1}}$ we obtain:

$$\left(\frac{\langle u'(t),Au(t)\rangle}{\|A^{1/2}u(t)\|^{2-\varepsilon_1}} + \frac{\delta}{\varepsilon_1}\,\|A^{1/2}u(t)\|^{\varepsilon_1}\right)' - \frac{\|A^{1/2}u'(t)\|^2}{\|A^{1/2}u(t)\|^{2-\varepsilon_1}} + \frac{c(t)\,\|Au(t)\|^2}{\|A^{1/2}u(t)\|^{2-\varepsilon_1}} +$$
$$+\ \frac{(2-\varepsilon_1)\,\langle u'(t),Au(t)\rangle^2}{\|A^{1/2}u(t)\|^{4-\varepsilon_1}} + \frac{\Big\langle f'(u(t))\,A^{1/2}u(t),\ A^{1/2}u(t)\Big\rangle}{\|A^{1/2}u(t)\|^{2-\varepsilon_1}} \;=\; 0\ .$$

Hence integrating on $[0,T[$, since $f'\geq 0$, using (3.22)–(3.23):

$$\begin{aligned}
\int_0^T c(t)\,\frac{\|Au(t)\|^2}{\|A^{1/2}u(t)\|^{2-\varepsilon_1}}\,dt \;&\leq\; -\left(\frac{\langle u'(T),Au(T)\rangle}{\|A^{1/2}u(T)\|^{2-\varepsilon_1}} + \frac{\delta}{\varepsilon_1}\,\|A^{1/2}u(T)\|^{\varepsilon_1}\right)\\
&\qquad +\ H_0\ +\int_0^T\frac{\|A^{1/2}u'(t)\|^2}{c(t)}\,\frac{c(t)}{\|A^{1/2}u(t)\|^{2-\varepsilon_1}}\,dt\\
&\leq\; H_0+\frac{\varepsilon_1}{2\,\delta}\,\frac{\|A^{1/2}u'(T)\|^2}{\|A^{1/2}u(t)\|^{2-\varepsilon_1}}\ +\ c_1\int_0^T\frac{\|A^{1/2}u'(t)\|^2}{c(t)}\,dt\\
&\leq\; H_0+c_1\left(\frac{\|A^{1/2}u'(T)\|^2}{\delta\,c(T)}\ +\int_0^T\frac{\|A^{1/2}u'(t)\|^2}{c(t)}\,dt\right)\\
&\leq\; H_0+\frac{2\,c_1}{\delta}\,F_0\ .
\end{aligned}$$

Therefore, by the smallness assumptions on the initial data:

$$\int_0^T c(t)\,\|A^{1/2}u(t)\|^{\varepsilon_1}\,dt \;\leq\; a_1^2\left(H_0+\frac{2\,c_1}{\delta}\,F_0\right)\;<\;\lambda\ .$$

Step 4. (3.19) is false

Proceeding as in the proof of case (i), step 4, we can now estimate $G(t) := \dfrac{\|u'(t)\|}{c(t)}$ as follows:

$$\begin{aligned}(G^2(t))' &\leq -\delta\, G^2(t) + 2\, G\Big(\|Au(t)\| + \frac{\|f(u(t))\|}{c(t)}\Big)\\ &\leq -G(t)\left(\delta\, G(t) - 2\sqrt{F_0} + \sqrt{c}\, a_1^{1+\varepsilon_1} F_0^{\varepsilon_1/2}\right),\end{aligned}$$

hence, applying Lemma 3.1, with $g := G^2$ we obtain $G(t) \leq G_0$. Then as in the proof of case (i), step 4:

$$\left|\frac{c'(T)}{c(T)}\right| \leq 2\, c_1\, G_0 \sqrt{F_0} < \frac{\delta}{2}\,.$$

Step 5. Conclusion

We can conclude as in step 5 of the proof of case (i). ∎

Proof of Remark 2.3: It is enough to replace σ with $r_0\, a_1^{-2}$ (taking $\varepsilon = 0$) and to proceed as in the proof of Theorem (2.2) (i). ∎

3.3. Asymptotic behaviour

In order to study the asymptotic behaviour of the solutions of (1.1), we consider the linearized problem

$$\left\{\begin{array}{l} v''(t) + \delta\, v'(t) + c(t)\, Av(t) + f(t,x) = 0\,, \qquad t \geq 0\,,\\ v(0) = v_0\,, \qquad v'(0) = v_1\,. \end{array}\right. \tag{3.24}$$

In the following lemma we examine the asymptotic behaviour of the solutions of (3.24).

Lemma 3.2. *Let $\delta > 0$. Let $c\colon [0,+\infty[\, \to\,]0,+\infty[$ be a Lipschitz continuous bounded function such that*

$$\left|\frac{c'(t)}{c(t)}\right| \leq \frac{\delta}{2} \qquad \textit{for a.e. } t \geq 0\,.$$

Let $f\colon [0,+\infty[\, \times \Omega \to \mathbb{R}$ be a continuous function such that $f(t,\cdot) \in D(A^{1/2})$ for all $t \geq 0$ and

$$\int_0^{+\infty} \frac{1}{c(s)}\, \|A^{1/2} f(s)\|^2\, ds < +\infty\,, \qquad \sup_{t\geq 0} \frac{\|f(t)\|}{c(t)} < +\infty\,.$$

Let v be the unique global solution of (3.24) with $(v_0, v_1) \in D(A) \times D(A^{1/2})$. Then there exists $v_\infty \in D(A)$ such that

$$v(t) \longrightarrow v_\infty \quad \text{in } D(A)\ , \tag{3.25}$$

$$v'(t) \longrightarrow 0 \quad \text{in } D(A^{1/2})\ , \tag{3.26}$$

as $t \to \infty$. Furthermore, if $v_\infty \neq 0$, then necessarily $c(t) \to 0$ as $t \to \infty$.

Proof of Lemma 3.2:

Step 1. Let us consider the function

$$H(t) := \frac{\|A^{1/2}v'(t)\|^2}{c(t)} + \|Av(t)\|^2 - \frac{1}{\delta}\int_0^t \frac{1}{c(s)}\|A^{1/2}f(s)\|^2\,ds\ .$$

A simple computation shows that

$$H'(t) \leq -\frac{\delta}{2}\frac{\|A^{1/2}v'(t)\|^2}{c(t)}\ . \tag{3.27}$$

By this fact we obtain:

1. for all $t \geq 0$:

$$\frac{\|A^{1/2}v'(t)\|^2}{c(t)} + \|Av(t)\|^2 + \frac{\delta}{2}\int_0^t \frac{\|A^{1/2}v'(s)\|^2}{c(s)}\,ds \leq$$
$$\leq \frac{\|A^{1/2}v_1\|^2}{c(0)} + \|Av_0\|^2 + \int_0^{+\infty}\frac{1}{\delta\, c(s)}\|A^{1/2}f(s,\cdot)\|^2\,ds =: \gamma_0\ .$$

2. Since the function $c(\cdot)$ is bounded then:

$$\int_0^{+\infty}\|A^{1/2}v'(t)\|^2\,dt < +\infty \tag{3.28}$$

3. The function H is non-increasing, hence there exists:

$$F_\infty := \lim_{t\to\infty}\frac{\|A^{1/2}v'(t)\|^2}{c(t)} + \|Av(t)\|^2\ .$$

If $F_\infty = 0$, then (3.25) holds true with $v_\infty = 0$. Since the function c is bounded, then also (3.26) follows from $F_\infty = 0$.

Therefore from now on we assume that $F_\infty > 0$.

Step 2. We show that

$$\int_0^\infty c(t)\,\|Av(t)\|^2\,dt\ <\ +\infty\ . \tag{3.29}$$

Indeed, taking the scalar product of the equation with Av and integrating on $[0,T]$, it follows that

$$\begin{aligned}\int_0^T c(t)\,\|Av(t)\|^2\,dt\ &=\ \langle v_1, Av_0\rangle + \frac{\delta}{2}\,\|A^{1/2}v_0\|^2\ -\int_0^T\langle A^{1/2}f(t), A^{1/2}u(t)\rangle\,dt\\ &\qquad -\langle v'(T), Av(T)\rangle - \frac{\delta}{2}\,\|A^{1/2}v(T)\|^2 + \int_0^T\|A^{1/2}v'(t)\|^2\,dt\\ &\le\ \langle v_1, Av_0\rangle + \frac{\delta}{2}\,\|A^{1/2}v_0\|^2 + \frac{1}{2\,\delta}\,\|c\|_\infty\,\frac{\|A^{1/2}v'(T)\|^2}{c(T)}\\ &\qquad + \|c\|_\infty\int_0^T\frac{\|A^{1/2}v'(t)\|^2}{c(t)}\,dt\\ &\qquad + \frac{1}{2\,a_1^2}\int_0^T c(t)\,\|A^{1/2}u(t)\|^2\,dt\ + \frac{a_1^2}{2}\int_0^T\frac{\|A^{1/2}f(t)\|^2}{c(t)}\,dt\\ &\le\ \langle v_1, Av_0\rangle + \frac{\delta}{2}\,\|A^{1/2}v_0\|^2 + \left(\frac{2}{\delta}\,\|c\|_\infty + \frac{\delta\,a_1^2}{2}\right)\gamma_0\\ &\qquad + \frac{1}{2}\int_0^T c(t)\,\|Au(t)\|^2\,dt\ .\end{aligned}$$

Hence

$$\int_0^T c(t)\,\|Av(t)\|^2\,dt\ \le\ 2\left(\langle v_1, Av_0\rangle + \frac{\delta}{2}\,\|A^{1/2}v_0\|^2 + \left(\frac{2}{\delta}\,\|c\|_\infty + \frac{\delta\,a_1^2}{2}\right)\gamma_0\right)\ .$$

Passing to the limit as $T\to\infty$, we obtain (3.29).

Step 3. From (3.28) and (3.29) it follows that

$$\int_0^\infty c(t)\left(\frac{\|A^{1/2}v'(t)\|^2}{c(t)} + \|Av(t)\|^2\right)dt\ <\ +\infty\ .$$

Since, for $t\ge\overline{T}$:

$$\frac{\|A^{1/2}v'(t)\|^2}{c(t)} + \|Av(t)\|^2\ \ge\ \frac{F_\infty}{2}\ >\ 0\ ,$$

then also

$$\int_0^\infty c(t)\,dt\ <\ +\infty\ . \tag{3.30}$$

Since $c(\cdot)$ is Lipschitz continuous, it follows that $c(t) \to 0$ as $t \to \infty$. Since $|A^{1/2}v'(t)|^2 \leq c(t)\,\gamma_0$, then (3.26) is proved.

Step 4. We show that (3.25) holds true with the additional assumptions that $(v_0, v_1) \in D(A^2)\times D(A^{3/2})$, $f(t,\cdot) \in D(A^{3/2})$ for every t and

$$\int_0^{+\infty} \frac{\|A^{3/2}f(t)\|}{c(t)}\,dt < +\infty\,, \qquad \sup_{t\geq 0} \frac{\|Af(t)\|}{c(t)} < +\infty\ . \tag{3.31}$$

To this end, let us introduce the function

$$\hat{H}(t) := \frac{\|A^{3/2}v'(t)\|^2}{c(t)} + \|A^2 v(t)\|^2 - \frac{1}{\delta}\int_0^t \frac{1}{c(s)}\,\|A^{3/2}f(s)\|^2\,ds\ .$$

As in Step 1, it is possible to prove that $\hat{H}$ is non-increasing, and that for every $t \geq 0$:

$$\|A^2 v(t)\|^2 \leq \hat{\gamma}_0\ .$$

Now let us consider the function $\hat{G}(t) := \dfrac{\|Av'(t)\|}{c(t)}$. We have that:

$$(\hat{G}(t)^2)' \leq -\hat{G}(t)\left\{\delta\,\hat{G}(t) - 2\Big(\sqrt{\hat{\gamma}_0} + \sup_{t\geq 0}\frac{\|Af(t)\|}{c(t)}\Big)\right\},$$

hence, by Lemma 3.1 with $g = \hat{G}^2$, it follows that

$$\hat{G}(t) \leq \max\left\{\hat{G}(0),\ \frac{2}{\delta}\Big(\sqrt{\hat{\gamma}_0} + \sup_{t\geq 0}\frac{\|Af(t)\|}{c(t)}\Big)\right\}.$$

By (3.30), this implies that

$$\int_0^{\infty} \|Av'(t)\|\,dt < +\infty$$

and therefore $Av(t)$ has a limit as $t \to \infty$.

Step 5. We show that (3.25) holds true for every initial data $(v_0, v_1) \in D(A)\times D(A^{1/2})$.

To this end, let us consider a sequence $\{(v_{0n}, v_{1n})\} \subseteq D(A^2)\times D(A^{3/2})$ converging to (v_0, v_1) in $D(A)\times D(A^{1/2})$ and f_n as in step 4, with:

$$\int_0^{+\infty} \frac{1}{c(t)}\,\|A^{1/2}(f(t) - f_n(t))\|^2\,dt \to 0 \quad \text{as } n \to +\infty\ .$$

Let $\{v_n\}$ be the corresponding solutions of (3.24), and let us set $w_n := v - v_n$. Since w_n is a solution of (3.24), with $f - f_n$ in place of f, we have that

$$\frac{\|A^{1/2}w_n'(t)\|^2}{c(t)} + \|Aw_n(t)\|^2 \;\leq$$

$$\leq\; \frac{\|A^{1/2}(v_{1,n} - v_1)\|^2}{c(0)} + \|A(v_{0,n} - v_0)\|^2 + \frac{1}{\delta}\int_0^{+\infty}\frac{1}{c(t)}\,\|A^{1/2}(f(t) - f_n(t))\|^2\,dt\;.$$

This proves that $\{Av_n\} \to Av$ uniformly in $[0,+\infty[$. Since $Av_n(t)$ has a limit as $t \to \infty$ for every $n \in \mathbb{N}$ (see Step 4), then necessarily $Av(t)$ has a limit as $t \to \infty$.

This completes the proof of (3.25). ∎

Proof of Theorem 2.4: We use the same notations as in the proof of Theorem 2.2 case (i) (resp. case (ii)). Let us first remark that u is the solution of (3.24) with

$$c(t) = m(\|A^{1/2}u(t)\|^2)\,, \qquad (v_0, v_1) = (u_0, u_1)\,, \qquad f(t,x) = f(u(t,x))\;.$$

In Step 2 of the proof of Theorem 2.2 case (i) (resp. Step 1 of case (ii)), we showed that $c(t) > 0$ for every $t \geq 0$ (this proves statement (i)), and

$$\left|\frac{c'(t)}{c(t)}\right| \leq \frac{\delta}{2} \qquad \forall\, t \geq 0\;.$$

Moreover in this step we proved also that $\|A^{1/2}u\| \leq r_0$, hence $c(\cdot)$ is bounded. Since m is locally Lipschitz continuous, and $\|A^{1/2}u'(t)\|^2 \leq F(t)\,c(t) \leq F_0\,c(t)$ (see (3.14) (resp. (3.23))), then it turns out that $c(\cdot)$ is globally Lipschitz continuous. Finally, as in (3.13), (3.15), by step 1 (resp. step 1):

$$\int_0^{+\infty}\frac{\|A^{1/2}f(u(t))\|^2}{c(t)}\,dt \;\leq\; c\int_0^{+\infty}c(t)\,\|A^{1/2}u(t)\|^2 \;<\; +\infty\,,$$

$$\frac{\|f(u(t))\|^2}{c^2(t)} \;\leq\; \frac{c\,a_1^2\,c^2(t)\,\|A^{1/2}u(t)\|^2}{c^2(t)} \;<\; c_0$$

$\Big($ resp.

$$\int_0^{+\infty}\frac{\|A^{1/2}f(u(t))\|^2}{c(t)}\,dt \;\leq\; c\int_0^{+\infty}c(t)\,\|A^{1/2}u(t)\|^{2\varepsilon_1} \;<\; +\infty\,,$$

$$\frac{\|f(u(t))\|^2}{c^2(t)} \;\leq\; c\,a_1^2\,\|A^{1/2}u(t)\|^{2\varepsilon_1} \;<\; c_0 \;\;\Big)$$

for some c_0 independent on t.

By Lemma 3.2, there exists $u_\infty \in D(A)$ such that $u \to u_\infty$ in $D(A)$ and $u' \to 0$ in $D(A^{1/2})$. Let us assume that $u_\infty \neq 0$, then by the last statement of Lemma 3.2 we have that $c(t) \to 0$ as $t \to \infty$, hence

$$0 = \lim_{t\to\infty} m(\|A^{1/2}u(t)\|^2) \; = \; m(\|A^{1/2}u_\infty\|^2) \; .$$

Since $\|A^{1/2}u_\infty\|^2 \le r_0$, it follows that $u_\infty = 0$. Furthermore, using the equation (1.1), $u'' \to 0$ in H. ∎

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