PORTUGALIAE MATHEMATICA
Vol. 60 Fasc. 1 – 2003
Nova Série

STABILITY AND CONTROLLABILITY OF THE ELECTROMAGNETO-ELASTIC SYSTEM

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Abstract: We consider the stabilization of the nonstationary electromagneto-elastic system in a bounded region with a Lipschitz boundary by means of nonlinear boundary feedbacks. This requires the validity of some stability estimate in the linear case that may be checked in some particular situations. As a consequence we get an explicit decay rate of the energy for appropriate feedbacks. By Russell's principle we further get some exact controllability results.

1 – Introduction

Let Ω be a bounded domain of $\mathbb{R}^3$ with a Lipschitz boundary Γ. In that domain we consider the non-stationary electromagneto-elastic system with non-linear boundary conditions:

$$
(1)\qquad \left\{
\begin{array}{l}
\partial_t^2 u - \nabla\sigma(u) + \xi\,\mathbf{curl}\,E = 0 \quad \text{ in } \; Q := \Omega\times\,]0,+\infty[\;, \\
\varepsilon\,\partial_t E - \mathbf{curl}\,H - \xi\,\mathbf{curl}\,\partial_t u = 0 \quad \text{ in } \; Q\;, \\
\mu\,\partial_t H + \mathbf{curl}\,E = 0 \quad \text{ in } \; Q\;, \\
\operatorname{div}(\varepsilon\,E) = \operatorname{div}(\mu\,H) = 0 \quad \text{ in } \; Q\;, \\
H\times\nu + \xi\,\partial_t u\times\nu + g_1(E\times\nu)\times\nu \;=\; 0 \quad \text{ on } \; \Sigma := \Gamma\times\,]0,+\infty[\;, \\
\sigma(u)\cdot\nu + Au + g_2(\partial_t u) = 0 \quad \text{ on } \; \Sigma\;, \\
u(0) = u_0,\;\; \partial_t u(0) = u_1 \quad \text{ in } \; \Omega\;, \\
E(0) = E_0,\;\; H(0) = H_0 \quad \text{ in } \; \Omega\;.
\end{array}
\right.
$$

Received: October 11, 2001; *Revised*: February 1, 2002.
AMS Subject Classification: 93D15, 93B05, 93C20.
Keywords: elasticity system; Maxwell's system; stability; nonlinear feedbacks.

This system models the coupling between Maxwell's system and the elastic one [8, 11], in which $E(x,t)$, $H(x,t)$ are the electric and magnetic fields at the point $x \in \Omega$ at time t; $u(x,t)$ is the displacement field at the point $x \in \Omega$ at time t. ε, μ are the electric permittivity and magnetic permeability, respectively, and are supposed to be real, positive functions in $L^\infty(\Omega)$. $\sigma(u) = (\sigma_{ij}(u))_{i,j=1}^3$ is the stress tensor given by (here and in the sequel we shall use the summation convention for repeated indices)

$$\sigma_{ij}(u) = a_{ijkl}\,\varepsilon_{kl}(u)\ ,$$

where $\varepsilon(u) = (\varepsilon_{ij}(u))_{i,j=1}^3$ is the strain tensor given by

$$\varepsilon_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\ ,$$

and the tensor $(a_{ijkl})_{i,j,k,l=1,2,3}$ is made of $C^2(\bar\Omega)$ entries such that

$$a_{ijkl} = a_{jikl} = a_{klij}\ ,$$

and satisfying the ellipticity condition

$$a_{ijkl}\,\varepsilon_{ij}\,\varepsilon_{kl} \ \geq\ \alpha\,\varepsilon_{ij}\,\varepsilon_{ij}\ , \tag{2}$$

for every symmetric tensor (ε_{ij}) and some $\alpha > 0$. Hereabove and below $\nabla\sigma(u)$ is the vector field defined by

$$\nabla\sigma(u) = (\partial_j\sigma_{ij}(u))_{i=1}^3\ .$$

As usual ν is the exterior unit normal vector to Γ. The mappings g_1, g_2 are assumed to satisfy some standard properties that will be described later on. Finally A is a positive real number and ξ is a real constant which is the coupling parameter. Indeed when $\xi = 0$ the above system decouples into two independent problems namely the elastodynamic system and Maxwell's system.

The above system is a particular case of the piezoelectric system of constitutive equations [8, 11]:

$$\begin{aligned}
\sigma_{ij} &= a_{ijkl}\,\varepsilon_{kl}(u) - e_{kij}\,E_k\ ,\\
D_i &= \varepsilon\,E_i + e_{ikl}\,\frac{\partial u_k}{\partial x_l}\ ,
\end{aligned}$$

where e_{kij} are the piezoelectric constants; the equation of motion is given by (without internal forces)

$$\partial_t^2 u_i = \partial_j\sigma_{ji}\ ,$$

and is completed by Maxwell's equations

$$\partial_t D = \mathbf{curl}\, H\,, \qquad \mu\, \partial_t H = -\,\mathbf{curl}\, E\ .$$

In the simplest case, we may take

$$e_{kji}\, \partial_j E_k = \xi(\mathbf{curl}\, E)_i\,, \qquad e_{ikl}\, \frac{\partial u_k}{\partial x_l} = -\xi(\mathbf{curl}\, u)_l\ ,$$

which yields our system (1).

The controllability of the above system with Dirichlet type boundary conditions (different from the above one) is considered in [13], while to our knowledge the stability as well as the controllability of the above system by means of Neumann type boundary conditions is not known. Our goal is then to consider the above problems adapting recent results for Maxwell's system [14, 23, 7, 25] or the elastodynamic system [1, 3, 9].

More precisely we first give a necessary and sufficient condition which guarantees the exponential decay of the energy of the solutions of (1) in the case of linear boundary conditions (i.e. $g_1(\xi) = g_2(\xi) = \xi$). This condition is in fact the validity of a stability estimate called the EE-stability estimate. We secondly check this stability estimate in some particular cases using the multiplier method. In a third step we use the so-called Russell's principle "controllability via stability" to obtain new controllability results for the above system extending results from [13]. Finally using Liu's principle [22] (based on Russell's principle) and a new integral inequality from [7] we give sufficient conditions on g_1, g_2 which lead to an explicit decay rate of the energy. Note that, contrary to the customary assumption, g_2 is not necessarily diagonal (i.e. $(g_2(\xi))_i = g_{2i}(\xi_i)$, for some functions g_{2i}), therefore an approximation scheme by globally Lipschitz functions g_2^k preserving some appropriate properties of g_2 is built. The strength of our approach lies in the fact that the controllability and stability results (with general feedbacks) are only based on the EE-stability estimate, estimate which may be obtained by different techniques, like the multiplier method [14, 23, 1, 3, 9], microlocal analysis [26], a conjunction of them [10, 7] or any method entering in a linear framework (like nonharmonic analysis for instance, see [16]). This approach was successfully initiated in [25] for Maxwell's system and is here extended to our system (1). Note that we obtain new results even for the elastodynamic system since we get stability results with different decay rates (exponential, polynomial, logarithmic, ...) and non necessarily diagonal feedbacks.

The schedule of the paper is the following one: Well-posedness of problem (1) is analyzed in Section 2 under appropriate conditions on g_1, g_2. Section 3 is first devoted to the equivalence between exponential stability of (1) by means of linear boundary feedbacks with the EE-stability estimate. We secondly check the EE-stability estimate in some particular situations. In Section 4 exact controllability results are deduced from Russell's principle. Section 5 is devoted to the stability results of (1) for general nonlinear feedbacks g_1, g_2. There the approximation of a (non globally Lipschitz) mapping by a sequence of globally Lipschitz mappings is constructed.

2 – Well-posedness of the problem

We start this section with the well-posedness of problem (1) under appropriate conditions on the mappings g_1, g_2. At the end we will check the dissipativeness of (1).

Let us introduce the Hilbert spaces (see e.g. [20, 24, 1])

$$J(\Omega,\varepsilon) = \Big\{E \in L^2(\Omega)^3 \mid \operatorname{div}(\varepsilon E) = 0 \ \text{ in } \Omega\Big\} , \tag{3}$$

$$\mathcal{H} \,=\, H^1(\Omega)^3 \times L^2(\Omega)^3 \times J(\Omega,\varepsilon) \times J(\Omega,\mu) , \tag{4}$$

equipped with the norm induced by the inner product

$$(E,E')_\varepsilon \,=\int_\Omega \varepsilon(x)\, E(x)\cdot E'(x)\,dx\,, \qquad \forall\, E,E' \in J(\Omega,\varepsilon) ,$$

$$\Big((u,v,E,H),(u',v',E',H')\Big)_{\mathcal{H}} = (u,u')_1 + (v,v')_0 + (E,E')_\varepsilon + (H,H')_\mu , \qquad \forall\,(u,v,E,H),(u',v',E',H') \in \mathcal{H} ,$$

where we have set

$$(v,v')_0 \,=\int_\Omega u(x)\cdot u'(x)\,dx ,$$

$$(v,v')_1 \,=\int_\Omega \sigma(u)(x) : \varepsilon(u')(x)\,dx \,+\, A\int_\Gamma u(x)\cdot u'(x)\,d\sigma ,$$

with the notation

$$\sigma(v) : \varepsilon(v') := \sigma_{ij}(v)\,\varepsilon_{ij}(v') .$$

Now define the (nonlinear) operator A from $\mathcal{H}$ into itself as follows:

$$\begin{aligned}
(5)\quad D(A) = \Big\{&(u,v,E,H)\in\mathcal{H} \mid \nabla\sigma(u), \mathbf{curl}\, E, \mathbf{curl}\, H \in L^2(\Omega)^3;\ v\in H^1(\Omega)^3;\\
&E\times\nu,\ H\times\nu\in L^2(\Gamma)^3 \ \text{ satisfying}\\
(6)\qquad\qquad &H\times\nu+\xi\, v\times\nu+g_1(E\times\nu)\times\nu = 0 \ \text{ on } \Gamma\ ,\\
(7)\qquad\qquad &\sigma(u)\cdot\nu+Au+g_2(v)=0 \ \text{ on } \Gamma\Big\}\ .
\end{aligned}$$

For all $(u,v,E,H)\in D(A)$ we take

$$A(u,v,E,H)=\Big(-v,-\nabla\sigma(u)+\xi\,\mathbf{curl}\,E,-\varepsilon^{-1}(\mathbf{curl}\,H+\xi\,\mathbf{curl}\,v),\mu^{-1}\,\mathbf{curl}\,E\Big)\ .$$

In order to give a meaning to the boundary conditions (6) and (7) we require that g_i, $i=1,2$, satisfies

$$(8)\qquad |g_i(E)|\le M(1+|E|)\,,\qquad \forall\, E\in\mathbb{R}^3\ ,$$

for some positive constant M. In that case for $(u,v,E,H)\in D(A)$, from Section 2 of [2] the property $\nabla\sigma(u)\in L^2(\Omega)^3$ implies that $\sigma(u)\cdot\nu$ belongs to $H^{-1/2}(\Gamma)^3$. Since the properties $u,v\in H^1(\Omega)^3$ and (8) satisfied by g_2 imply that $Au+g_2(v)$ belongs to $L^2(\Gamma)^3$, the boundary condition (7) has a meaning (in $H^{-1/2}(\Gamma)^3$) and furthermore yields $\sigma(u)\cdot\nu\in L^2(\Gamma)^3$. Similarly the properties of H, v and (8) satisfied by g_1 give a meaning to the boundary condition (6) (as an equality in $L^2(\Gamma)^3$). In summary both boundary conditions (6) and (7) have to be understood as an equality in $L^2(\Gamma)^3$.

We now see that formally problem (1) is equivalent to

$$(9)\qquad \begin{cases} \dfrac{\partial U}{\partial t}+AU=0\ ,\\ U(0)=U_0\ ,\end{cases}$$

when $U=(u,\partial_t u,E,H)$ and $U_0=(u_0,u_1,E_0,H_0)$.

We shall prove that this problem (9) has a unique solution using nonlinear semigroup theory (see e.g. [28]) by showing that A is a maximal monotone operator adapting an argument from Section 2 of [25]. But first we show the following density result.

Lemma 2.1. *If $g_1(0) = g_2(0) = 0$, the domain of the operator A is dense in $\mathcal{H}$.*

Proof: By Lemma 2.3 of [25] $P_\varepsilon \mathcal{D}(\Omega)^3 \times P_\mu \mathcal{D}(\Omega)^3$ is dense in $J(\Omega,\varepsilon) \times J(\Omega,\mu)$ when P_ε is the projection on $J(\Omega,\varepsilon)$ in $L^2(\Omega)^3$ endowed with the inner product $(\cdot,\cdot)_\varepsilon$. Consequently the space $\mathcal{D}(\Omega)^3 \times \mathcal{D}(\Omega)^3 \times P_\varepsilon \mathcal{D}(\Omega)^3 \times P_\mu \mathcal{D}(\Omega)^3$ is dense in $\mathcal{H}$.

The conclusion follows from the inclusion

$$\mathcal{D}(\Omega)^3 \times \mathcal{D}(\Omega)^3 \times P_\varepsilon \mathcal{D}(\Omega)^3 \times P_\mu \mathcal{D}(\Omega)^3 \subset D(A) ,$$

consequence of the assumption $g_1(0) = g_2(0) = 0$. ∎

Lemma 2.2. *For $i = 1$ or 2 assume that the mapping $g_i \colon \mathbb{R}^3 \to \mathbb{R}^3$ is continuous and satisfies (8) as well as*

$$(g_i(E) - g_i(F)) \cdot (E - F) \ge 0 , \quad \forall E, F \in \mathbb{R}^3 \ \textit{(monotonicity)} , \tag{10}$$

$$g_i(0) = 0 , \tag{11}$$

$$g_1(E) \cdot E \ge m\,|E|^2 , \quad \forall E \in \mathbb{R}^3 : \ |E| \ge 1 , \tag{12}$$

for some positive constant m. Then A is a maximal monotone operator.

Proof: We start with the monotonicity: A is monotone if and only if

$$(AU - AV, U - V)_{\mathcal{H}} \ge 0 , \quad \forall U, V \in D(A) .$$

From the definition of A and the inner product in $\mathcal{H}$, this is equivalent to

$$\begin{aligned}(v - v', u - u')_1 \,+\, &\Big(\nabla\sigma(u - u') - \xi\,\mathbf{curl}(E - E'),\, v - v'\Big)_0 + \\ &+ \int_\Omega \Big\{(E - E') \cdot \Big(\mathbf{curl}(H - H') + \xi\,\mathbf{curl}(v - v')\Big)\Big\}\,dx \,- \\ &\qquad - \int_\Omega \mathbf{curl}(E - E') \cdot (H - H')\,dx \ \le\ 0 ,\end{aligned}$$

for any $(u, v, E, H), (u', v', E', H') \in D(A)$. Lemma 2.2 of [25] and Green's formula yield equivalently

$$\begin{aligned}(v - v', u - u')_1 \,-\, &\int_\Omega \sigma(u - u') : \varepsilon(v - v')\,dx \,+ \\ &+ \xi \int_\Omega \Big\{\mathbf{curl}(v - v') \cdot (E - E') - \mathbf{curl}(E - E') \cdot (v - v')\Big\}\,dx \,+ \\ &+ \int_\Gamma \Big\{(\sigma(u - u') \cdot \nu) \cdot (v - v') + ((E - E') \times \nu) \cdot (H - H')\Big\}\,d\sigma \ \le\ 0 ,\end{aligned}$$

for any $(u,v,E,H),(u',v',E',H') \in D(A)$. Applying again Green's formula in the third term of the above left-hand side and using the definition of the inner product $(\cdot,\cdot)_1$, we get equivalently

$$A\int_\Gamma (v-v')\cdot(u-u')\,d\sigma \;-\; \xi\int_\Gamma (v-v')\times\nu\cdot(E-E')\,d\sigma \;+\; \\ +\int_\Gamma \Big\{(\sigma(u-u')\cdot\nu)\cdot(v-v') + ((E-E')\times\nu)\cdot(H-H')\Big\}\,d\sigma \;\le\; 0\;,$$

for any $(u,v,E,H),(u',v',E',H') \in D(A)$. Using the boundary conditions (6) and (7), we arrive at

$$\int_\Gamma \Big\{\Big(g_1(E\times\nu)-g_1(E'\times\nu)\Big)\cdot(E\times\nu-E'\times\nu)+\Big(g_2(v)-g_2(v)\Big)\cdot(v-v')\Big\}\,d\sigma \;\ge\; 0\;,$$

for any $(u,v,E,H),(u',v',E',H') \in D(A)$. We then conclude using the monotonicity assumption (10) on g_1 and g_2.

Let us now pass to the maximality. This means that for all (f,g,F,G) in $\mathcal{H}$, we are looking for (u,v,E,H) in $D(A)$ such that

$$(I+A)\,(u,v,E,H) = (f,g,F,G)\,. \tag{13}$$

From the definition of A, this equivalently means

$$\begin{cases} u - v \;=\; f\;, \\ v - \nabla\sigma(u) + \xi\,\mathbf{curl}\,E \;=\; g\;, \\ E - \varepsilon^{-1}(\mathbf{curl}\,H + \xi\,\mathbf{curl}\,v) \;=\; F\;, \\ H + \mu^{-1}\,\mathbf{curl}\,E \;=\; G\;. \end{cases} \tag{14}$$

The first and fourth equations allow to eliminate H and u, since they are respectively equivalent to

$$u = v + f\;, \tag{15}$$

$$H = G - \mu^{-1}\,\mathbf{curl}\,E\;. \tag{16}$$

Substituting these expressions in the second and third equations yields formally

$$v - \nabla\sigma(v) + \xi\,\mathbf{curl}\,E \;=\; \nabla\sigma(f) + g\;, \tag{17}$$

$$\varepsilon\,E + \mathbf{curl}(\mu^{-1}\,\mathbf{curl}\,E) - \xi\,\mathbf{curl}\,v \;=\; \varepsilon\,F + \mathbf{curl}\,G\;. \tag{18}$$

This system in (v,E) will be uniquely defined by adding boundary conditions on u and E. Indeed using the identities (15) and (16), we see that (6) and (7) are formally equivalent to

(19) $-\mu^{-1}\,\mathbf{curl}\,E\times\nu+\xi\,v\times\nu+g_1(E\times\nu)\times\nu=-G\times\nu-\mu^{-1}\,\mathbf{curl}\,f\times\nu\quad$ on Γ,

(20) $\sigma(v)\cdot\nu+Av+g_2(v)=-\sigma(f)\cdot\nu-Af\quad$ on Γ .

By formal integration by parts we remark that the variational formulation of the system (17)–(18) with the boundary conditions (19)–(20) is the following one: Find $(v,E)\in V$ such that

$$a\Big((v,E),(v',E')\Big)=F(v',E')\,,\qquad\forall\,(v',E')\in V\ , \tag{21}$$

where the Hilbert space V is given by $V=H^1(\Omega)^3\times W_\varepsilon$ when W_ε is defined by

$$W_\varepsilon=\Big\{E\in L^2(\Omega)^3\mid\ \mathbf{curl}\,E\in L^2(\Omega)^3,\ \operatorname{div}(\varepsilon E)\in L^2(\Omega)\ \text{ and }\ E\times\nu\in L^2(\Gamma)^3\Big\}\ ,$$

with the norm

$$\|E\|^2_{W_\varepsilon}=\int_\Omega\Big(|E|^2+|\,\mathbf{curl}\,E|^2+|\operatorname{div}(\varepsilon E)|^2\Big)\,dx\,+\int_\Gamma|E\times\nu|^2\,d\sigma\ ,$$

the form a is defined by

$$\begin{aligned}a\Big((v,E),(v',E')\Big)=&\int_\Omega\Big\{\sigma(v):\varepsilon(v')+v\cdot v'\Big\}\,dx\\&+\int_\Omega\Big\{\mu^{-1}\,\mathbf{curl}\,E\cdot\mathbf{curl}\,E'+\varepsilon\,E\cdot E'+s\operatorname{div}(\varepsilon E)\ \operatorname{div}(\varepsilon E')\Big\}\,dx\\&+\xi\int_\Omega\Big\{E\cdot\mathbf{curl}\,v'-E'\cdot\mathbf{curl}\,v\Big\}\,dx\\&+\int_\Gamma\Big\{g_1(E\times\nu)\cdot E'\times\nu+g_2(v)\cdot v'+Av\cdot v'\Big\}\,d\sigma\\&-\xi\int_\Gamma\Big\{(v\times\nu)\cdot E'+(E\times\nu)\cdot v\Big\}\,d\sigma\ ,\end{aligned}$$

$s>0$ being a parameter appropriately chosen later on. Finally the form F is defined by

$$F(v',E')\,=\int_\Omega\Big\{g\cdot v'-\sigma(f):\varepsilon(v')+\varepsilon\,F\cdot E'+G\cdot\mathbf{curl}\,E'\Big\}\,dx\,-\,A\int_\Gamma f\cdot v'\,d\sigma\ .$$

Let us now introduce the mapping

$$\mathcal{A}:V\to V':(v,E)\to\mathcal{A}(v,E)\ ,$$

where $\mathcal{A}(v,E)((v',E'))=a((v,E),(v',E'))$. As F belongs to V', the solvability of (21) is equivalent to the surjectivity of $\mathcal{A}$. For that purpose we make use of

Corollary 2.2 of [28] which proves that $\mathcal{A}$ is surjective if $\mathcal{A}$ is monotone, hemicontinuous, bounded and coercive, properties that we now check.

From the monotonicity of g_1 and g_2, we directly see that $\mathcal{A}$ is monotone. Moreover the continuity of g_1 and g_2 leads to the hemicontinuity of $\mathcal{A}$ while the property (8) of g_1 and g_2 implies the boundedness of $\mathcal{A}$. It then remains to show that $\mathcal{A}$ is coercive, i.e.,

$$\frac{\mathcal{A}(v,E)((v,E))}{\|(v,E)\|_V} = \frac{a((v,E),(v,E))}{\|(v,E)\|_V} \to +\infty \quad \text{as} \quad \|(v,E)\|_V \to +\infty \,. \tag{22}$$

For a fixed $E \in L^2(\Omega)^3$ we set

$$\Gamma_E^+ = \Big\{x \in \Gamma\colon\ |(E\times\nu)(x)| > 1\Big\}\,, \quad \Gamma_E^- = \Big\{x \in \Gamma\colon\ |(E\times\nu)(x)| \le 1\Big\}\,.$$

The properties of g_1 and g_2 imply that

$$\begin{aligned} a\Big((v,E),(v,E)\Big) \;\ge\; &\int_\Omega \Big\{\sigma(v):\varepsilon(v) + |v|^2\Big\}\,dx \\ &+ \int_\Omega \Big\{\mu^{-1}|\,\mathbf{curl}\,E|^2 + \varepsilon\,|E|^2 + s|\operatorname{div}(\varepsilon E)|^2\Big\}\,dx \\ &+ m\int_{\Gamma_E^+} |E\times\nu|^2\,d\sigma\,. \end{aligned}$$

Moreover from the definition of Γ_E^-, the ellipticity assumption (2) and Korn's inequality, there exists $c>0$ such that

$$\begin{aligned} \|(v,E)\|_V^2 \;\le\; c&\int_\Omega \Big\{\sigma(v):\varepsilon(v)+|v|^2\Big\}\,dx \\ &+ \int_\Omega \Big(|E|^2 + |\,\mathbf{curl}\,E|^2 + |\operatorname{div}(\varepsilon E)|^2\Big)\,dx + \int_{\Gamma_E^+} |E\times\nu|^2 d\sigma + |\Gamma|\,. \end{aligned}$$

These two inequalities show that there exists a positive contant β (independent on v and E) such that

$$a\Big((v,E),(v,E)\Big) \;\ge\; \beta\Big(\|(v,E)\|_V^2 - |\Gamma|\Big)\,.$$

This leads to (22).

At this stage we need to show that the solution $(v,E)\in V$ of (21) and u, H given respectively by (15), (16) are such that (u,v,E,H) belongs to $D(A)$ and satisfies (13) (or equivalently (14)). We first show that εE is divergence free by taking test functions $v'=0$ and $E'=\nabla\phi$ with $\phi\in D(\Delta_\varepsilon^{\mathrm{Dir}})$, where $D(\Delta_\varepsilon^{\mathrm{Dir}})$ is the domain of the operator $\Delta_\varepsilon^{\mathrm{Dir}}$ with Dirichlet boundary conditions defined by

$$\begin{aligned} &D(\Delta_\varepsilon^{\mathrm{Dir}}) = \Big\{\phi \in \overset{\circ}{H}{}^1(\Omega)\mid \Delta_\varepsilon\phi := \operatorname{div}(\varepsilon\nabla\phi)\in L^2(\Omega)\Big\}\,, \\ &\Delta_\varepsilon^{\mathrm{Dir}}\phi = \Delta_\varepsilon\phi\,, \quad \forall\,\phi\in D(\Delta_\varepsilon^{\mathrm{Dir}})\,. \end{aligned}$$

In that case (21) becomes (the boundary terms disappear since ϕ and v' are equal to zero on Γ)

$$\int_\Omega \Big\{\varepsilon\, E\cdot\nabla\phi + s\,\mathrm{div}(\varepsilon E)\,\Delta_\varepsilon\phi\Big\}\,dx \;=\; \int_\Omega \varepsilon\, F\cdot\nabla\phi\,dx\,, \qquad \forall\,\phi\in D(\Delta_\varepsilon^{\mathrm{Dir}})\ .$$

Since $\varepsilon\, E$ and $\varepsilon\, F$ have a divergence in $L^2(\Omega)$, by Green's formula in the above left-hand side and right-hand side (allowed since ϕ is in $H^1(\Omega)$, see for instance [2]), we obtain

$$\int_\Omega \mathrm{div}(\varepsilon\, E)\,\{\phi + s\,\Delta_\varepsilon\phi\}\,dx \;=\; 0\,, \qquad \forall\,\phi\in D(\Delta_\varepsilon^{\mathrm{Dir}})\ ,$$

since $\varepsilon\, F$ is divergence free. Taking $s>0$ such that $-s^{-1}$ is not an eigenvalue of $\Delta_\varepsilon^{\mathrm{Dir}}$ (always possible since $\Delta_\varepsilon^{\mathrm{Dir}}$ is a negative selfadjoint operator with a discrete spectrum), we conclude that

$$\mathrm{div}(\varepsilon\, E) = 0 \quad \text{ in } \ \Omega\ .$$

Using this fact and the identities (15) and (16), we see that (21) is equivalent to

$$\begin{aligned}
&\int_\Omega \Big\{\sigma(u):\varepsilon(v') + v\cdot v'\Big\}\,dx + \int_\Omega \Big\{-H\cdot\mathbf{curl}\,E' + \varepsilon\, E\cdot E'\Big\}\,dx \;+\\
&\quad + \xi\int_\Omega \Big\{E\cdot\mathbf{curl}\,v' - E'\cdot\mathbf{curl}\,v\Big\}\,dx + \int_\Gamma \Big\{g_1(E\times\nu)\cdot E'\times\nu + g_2(v)\cdot v' + Au\cdot v'\Big\}\,d\sigma\\
&\quad - \xi\int_\Gamma \Big\{(v\times\nu)\cdot E' + (E\times\nu)\cdot v'\Big\}\,d\sigma \;=\; \int_\Omega \Big\{g\cdot v' + \varepsilon\, F\cdot E'\Big\}\,dx\,, \quad \forall\,(v',E')\in V.
\end{aligned}$$

First taking test functions v' in $\mathcal{D}(\Omega)^3$ and $E'=0$, we get

$$\nabla\sigma(u) + v + \xi\,\mathbf{curl}\,E = g \quad \text{ in } \ \mathcal{D}'(\Omega)\ .$$

This implies the second identity in (14) as well as the regularity $\nabla\sigma(u)\in L^2(\Omega)^3$ (from the fact that v, $\mathbf{curl}\,E$ as well as g belongs to that space).

Second we take test functions $v'=0$ and $E'=P_\varepsilon\chi$ with $\chi\in\mathcal{D}(\Omega)^3$ by Lemma 2.3 of [25] we get

$$\varepsilon\, E - \mathbf{curl}\,H - \xi\,\mathbf{curl}\,v \;=\; \varepsilon\, F \quad \text{ in } \ \mathcal{D}'(\Omega)\ ,$$

since one readily checks that

$$\int_\Omega P_\varepsilon\chi\cdot\mathbf{curl}\,v\;dx \;=\; \int_\Omega \chi\cdot\mathbf{curl}\,v\;dx\ .$$

This means that the third identity in (14) holds as well as the regularity $\mathbf{curl}\, H \in L^2(\Omega)$.

Thirdly taking test functions $v' \in H^1(\Omega)^3$ and $E' = P_\varepsilon \chi$ with $\chi \in C^\infty(\bar{\Omega})^3$ by the property $\mathbf{curl}\, P_\varepsilon\chi = \mathbf{curl}\,\chi$ (see Remark 2.4 of [25]) and Green's formula (see Section 2 of [2] and Lemma 2.2 of [25]), we get

$$\begin{aligned}\langle \sigma(u)\cdot\nu,\, v'\rangle \;&-\; \int_\Gamma (H\times\nu)\cdot E'\,d\sigma \;+\; \xi \int_\Gamma (E\times\nu)\cdot v'\,d\sigma \\ &+\; \int_\Gamma \Big\{ g_1(E\times\nu)\cdot E'\times\nu + g_2(v)\cdot v' + Au\cdot v' \Big\}\,d\sigma \\ &-\; \xi \int_\Gamma \Big\{ (v\times\nu)\cdot E' + (E\times\nu)\cdot v' \Big\}\,d\sigma \;=\; 0\ ,\end{aligned}$$

where $\langle\cdot,\cdot\rangle$ means here the duality bracket between $H^{-1/2}(\Gamma)^3$ and $H^{-1/2}(\Gamma)^3$.

Since $E'\times\nu = P_\varepsilon\chi\times\nu = \chi\times\nu$ on Γ (cf. Remark 2.4 of [25]) the above identity is equivalent to

$$\begin{aligned}\langle \sigma(u)\cdot\nu,\, v'\rangle \;&-\; \int_\Gamma (H\times\nu)\cdot\chi\;d\sigma \;+\; \xi \int_\Gamma (E\times\nu)\cdot v'\;d\sigma \\ &+\; \int_\Gamma \Big\{ -g_1(E\times\nu)\times\nu\cdot\chi + g_2(v)\cdot v' + Au\cdot v' \Big\}\,d\sigma \\ &-\; \xi \int_\Gamma \Big\{ (v\times\nu)\cdot\chi + (E\times\nu)\cdot v' \Big\}\,d\sigma \;=\; 0\ .\end{aligned}$$

This leads to the boundary conditions (6) and (7) since v' (resp. χ) was arbitrary in $H^1(\Omega)^3$ (resp. in $C^\infty(\bar{\Omega})^3$) whose trace belongs to a dense subspace of $L^2(\Gamma)^3$.

Finally from (16) and the fact that μG is divergence free, μH is also divergence free. ∎

Nonlinear semigroup theory [28] allows to conclude the following existence results:

Corollary 2.3. *If g_1 and g_2 satisfy the assumptions of Lemma 2.2, for all $(u_0, u_1, E_0, H_0) \in \mathcal{H}$, the problem (1) admits a unique (weak) solution (u, E, H) satisfying $(u, \partial_t u, E, H) \in C(\mathbb{R}_+, \mathcal{H})$, or equivalently $u \in C^1(\mathbb{R}_+, L^2(\Omega)^3) \cap C(\mathbb{R}_+, H^1(\Omega)^3)$, $E \in C(\mathbb{R}_+, J(\Omega,\varepsilon))$ and $H \in C(\mathbb{R}_+, J(\Omega,\mu))$. If moreover (u_0, u_1, E_0, H_0) belongs to $D(A)$, the problem (1) admits a unique (strong) solution (u, E, H) satisfying $(u, \partial_t u, E, H) \in W^{1,\infty}(\mathbb{R}_+, \mathcal{H}) \cap L^\infty(\mathbb{R}_+, D(A))$, or equivalently satisfying $u \in W^{2,\infty}(\mathbb{R}_+, L^2(\Omega)^3) \cap W^{1,\infty}(\mathbb{R}_+, H^1(\Omega)^3) \cap L^\infty(\mathbb{R}_+, D_E(\Omega))$, $E \in W^{1,\infty}(\mathbb{R}_+, J(\Omega,\varepsilon)) \cap L^\infty(\mathbb{R}_+, W_\varepsilon)$, $H \in W^{1,\infty}(\mathbb{R}_+, J(\Omega,\mu)) \cap L^\infty(\mathbb{R}_+, W_\mu)$, and satisfying (6)–(7) for a.e. t, where the space $D_E(\Omega)$ is defined by*

$$D_E(\Omega) := \Big\{ u \in H^1(\Omega) \mid \nabla\sigma(u) \in L^2(\Omega)^3 \Big\}\ . \;\blacksquare$$

Remark 2.4. If we assume that g_2 is globally Lipschitz, then for (u, v, E, H) in $D(A)$, the regularities $u \in H^1(\Omega)^3$ and $v \in H^1(\Omega)^3$ imply that $Au + g_2(v)$ belongs to $H^{1/2}(\Gamma)^3$. Consequently from the boundary condition (7) one gets

$$\sigma(u) \cdot \nu \in H^{1/2}(\Gamma)^3 .$$

For a smooth boundary Γ (C^2 is sufficient), standard regularity results (see for instance [6]) imply that

$$u \in H^2(\Omega)^3 .$$

If Ω is a Lipschitz polyhedron, in the sense that Ω is a bounded, simply connected Lipschitz domain with piecewise plane boundary, and if the tensor (a_{ijkl}) corresponds to the Lamé system, then the results of [17] yields

$$u \in H^{3/2+\delta}(\Omega)^3 ,$$

for some $\delta > 0$ small enough. In both cases we actually have the continuous embedding

$$\Big\{u \in H^1(\Omega) \mid \nabla\sigma(u) \in L^2(\Omega)^3 \text{ and } \sigma(u)\cdot\nu \in H^{1/2}(\Gamma)^3\Big\} \hookrightarrow H^{3/2+\delta}(\Omega)^3 , \tag{23}$$

for some $\delta > 0$. Under this assumption the strong solution (u, E, H) satisfies

$$u \in L^\infty(\mathbb{R}_+, H^{3/2+\delta}(\Omega)^3) .$$

This regularity result will be used in Section 5 as a sufficient condition for the application of Liu's principle. □

We finish this section by showing the dissipativeness of our system.

Lemma 2.5. *If g_1 and g_2 satisfy the assumptions of Lemma 2.2, then the energy*

$$\begin{aligned} \mathcal{E}(t) = &\frac{1}{2} \int_\Omega \Big(|\partial_t u(x,t)|^2 + \sigma(u)(x,t) : \varepsilon(u)(x,t)\Big)\, dx \\ &+ \frac{A}{2} \int_\Gamma |u(x,t)|^2 \, d\sigma(x) \\ &+ \frac{1}{2} \int_\Omega \Big(\varepsilon(x)\, |E(x,t)|^2 + \mu(x)\, |H(x,t)|^2\Big)\, dx \end{aligned} \tag{24}$$

is non-increasing. Moreover for $(u_0, u_1, E_0, H_0) \in D(A)$, *we have for all* $0 \le S < T < \infty$

$$\mathcal{E}(S) - \mathcal{E}(T) = \int_S^T \int_\Gamma \Big\{ g_1(E(t) \times \nu) \cdot E(t) \times \nu + \, g_2(u'(t)) \cdot u'(t) \Big\} \, d\sigma \, dt \ , \tag{25}$$

and for all $t \ge 0$

$$\mathcal{E}'(t) = -\int_\Gamma \Big\{ g_1(E(t) \times \nu) \cdot E(t) \times \nu + \, g_2(u'(t)) \cdot u'(t) \Big\} \, d\sigma \ . \tag{26}$$

Proof: Since $D(A)$ is dense in $\mathcal{H}$ it suffices to show (26). For $(u_0, u_1, E_0, H_0) \in D(A)$, from the regularity of u, E, H, we have

$$\begin{aligned} \mathcal{E}'(t) \;=\; & \int_\Omega \Big\{ \partial_t^2 u \cdot \partial_t u + \sigma(u) : \varepsilon(\partial_t u) \Big\} \, dx \; + \; A \int_\Gamma \partial_t u \cdot u \; d\sigma \\ & + \int_\Omega \Big\{ \varepsilon \, E \cdot \partial_t E + \mu \, H \cdot \partial_t H \Big\} \, dx \ . \end{aligned}$$

By (1), we get

$$\begin{aligned} \mathcal{E}'(t) \;=\; & \int_\Omega \Big\{ \partial_t u \cdot \Big(\nabla \sigma(u) - \xi \, \mathbf{curl}\, E \Big) + \sigma(u) : \varepsilon(\partial_t u) \Big\} \, dx \; + \; A \int_\Gamma \partial_t u \cdot u \; d\sigma \\ & + \int_\Omega \Big\{ E \cdot (\mathbf{curl}\, H + \xi \, \mathbf{curl}\, \partial_t u) - H \cdot \mathbf{curl}\, E \Big\} \, dx \\ =\; & -\Big(A\Big(u(t), \partial_t u(t), E(t), H(t)\Big), \; \Big(u(t), \partial_t u(t), E(t), H(t)\Big) \Big)_{\mathcal{H}} \ . \end{aligned}$$

We conclude by Lemma 2.2. ∎

3 – Exponential stability in the linear case

Let us start with the following definition.

Definition 3.1. *We say that* Ω *satisfies the EE-stability estimate if there exist* $T > 0$ *and two non negative constants* C_1, C_2 *(which may depend on* T*) with* $C_1 < T$ *such that*

$$\int_0^T \mathcal{E}(t) \, dt \;\le\; C_1 \, \mathcal{E}(0) + C_2 \int_0^T \int_\Gamma \Big(|u'(t)|^2 + |E(t) \times \nu|^2 \Big) \, d\sigma \, dt \ , \tag{27}$$

for all solution (u, E, H) *of (1) with* $g_1(\xi) = g_2(\xi) = \xi$. □

We now show that the EE-stability estimate is a necessary and sufficient condition for the exponential stability with linear feedbacks.

Theorem 3.2. *Ω satisfies the EE-stability estimate if and only if there exist two positive constants M and ω such that*

$$\mathcal{E}(t) \le M e^{-\omega t}\, \mathcal{E}(0)\ , \tag{28}$$

for all solution (u, E, H) of (1) with $g_1(\xi) = g_2(\xi) = \xi$.

Proof: As in Lemma 3.2 of [25] we prove that the EE-stability estimate is equivalent to the so-called observability estimate

$$\mathcal{E}(T)\ \le\ C \int_0^T \int_\Gamma \Big(|u'(t)|^2 + |E(t) \times \nu|^2 \Big)\, d\sigma\, dt\ ,$$

for some positive constant C (which may depend on T). This estimate, the identity (25) of Lemma 2.5 and standard arguments about uniform stabilization of hyperbolic system (see for instance [27, 25]) yield the conclusion. ∎

Using the so-called multiplier method we give examples of domains satisfying the EE-stability estimate for some particular coefficients a_{ijkl} and ε, μ.

Theorem 3.3. *Assume that Ω is a bounded, simply connected Lipschitz domain with a C^2-boundary and that Ω is strictly star-shaped with respect to the origin. Suppose further that the coefficients ε, μ and a_{ijkl} are constants in the whole domain Ω. Then Ω satisfies the EE-stability estimate.*

Proof: It suffices to show that the estimate (27) holds for any strong solution (u, E, H) of (1) with $g_1(\xi) = g_2(\xi) = \xi$ and appropriate constants T, C_1, C_2.

We take the standard multiplier $m(x) = x$ and prove that the following identity holds for all $T \ge 0$:

$$\begin{aligned} \int_0^T \mathcal{E}(t)\, dt\ &=\ I_1 + I_2 + \xi \Big[\int_\Omega (m\cdot\nabla)\, u \cdot H\, dx \Big]_0^T \\ &\quad - \Big[\int_\Omega \Big((m\cdot\nabla)\, u + u \Big) \cdot \partial_t u\, dx \Big]_0^T\ , \end{aligned} \tag{29}$$

where we have set

$$I_1 = \frac{1}{2}\int_{\Sigma_T}\Big\{m\cdot\nu\Big(\varepsilon\,|E|^2+\mu\,|H|^2\Big)-2\,\varepsilon(m\cdot E)\,(E\cdot\nu) \\ -\,2\,\mu(m\cdot H)\,(H\cdot\nu)-2\,\xi\,\mu(H\cdot\nu)\,m\cdot\partial_t u\Big\}\,d\sigma\,dt\ ,$$

$$I_2 = \frac{1}{2}\int_{\Sigma_T}\Big\{2\Big((m\cdot\nabla)\,u+u\Big)\cdot(\sigma(u)\cdot\nu) \\ +\,m\cdot\nu\Big(|\partial_t u|^2-\sigma(u):\varepsilon(u)\Big)+A|u|^2\Big\}\,d\sigma\,dt\ .$$

Indeed starting from the identity (consequence of (1))

$$0=\int_{Q_T}\Big\{\Big((m\cdot\nabla)\,u+u\Big)\cdot\Big(\partial_t^2 u-\nabla\sigma(u)+\xi\,\mathbf{curl}\,E\Big) \\ +\,\mu(m\times H)\cdot(\varepsilon\,\partial_t E-\mathbf{curl}\,H-\xi\,\mathbf{curl}\,\partial_t u) \\ +\,\Big(\varepsilon(E\times m)-\xi\Big((m\cdot\nabla)\,u+u\Big)\Big)\cdot(\mu\,\partial_t H+\mathbf{curl}\,E)\Big\}\,dx\,dt$$

and using some Green's formulas we arrive at (29) as in [1, 25].

We have now to estimate each term of the right-hand side of (29). For the estimation of I_1 we remark that (see Lemma 8.21 of [15])

$$m\cdot\nu\,(\varepsilon|E|^2+\mu|H|^2)-2\,\varepsilon(m\cdot E)\,(E\cdot\nu)-2\,\mu(m\cdot H)\,(H\cdot\nu)-2\,\xi\,\mu(H\cdot\nu)\,m\cdot\partial_t u\ \le$$

$$\le\ (m\cdot\nu)\Big\{\varepsilon\,|E_\tau|^2+\mu\,|H_\tau|^2\Big\}+\varepsilon\,\frac{(m_\tau\cdot E_\tau)^2}{(m\cdot\nu)^2}+\mu\,\frac{(m_\tau\cdot H_\tau+\xi\,m\cdot\partial_t u)^2}{(m\cdot\nu)^2}\quad \text{on } \Sigma_T.$$

where $m_\tau = m-(m\cdot\nu)\,\nu$ denotes the tangent component of m. Consequently using the boundary condition in (1) there exists a positive constant C_3 such that

$$I_1\ \le\ C_3\int_{\Sigma_T}\Big\{|E\times\nu|^2+\xi^2\,|\partial_t u|^2\Big\}\,d\sigma\,dt\ . \tag{30}$$

Following the arguments of [3] (since the second boundary condition is the same than for the elasticity system) we can show that there exists a positive constant C_4 (depending on Ω and the constant α in (2)) such that

$$I_2\ \le\ C_4\Big(\mathcal{E}(0)+\int_{\Sigma_T}\Big\{|u|^2+|\partial_t u|^2\Big\}\,d\sigma\,dt\Big)\ .$$

By Lemma 3.4 below we conclude that for all $\theta>0$ there exists a constant $C(\theta)>0$ (which does not depend on T but depends on θ, the domain, the coefficients a_{ijkl},μ and ξ) such that

$$I_2\ \le\ \Big(C_4+C(\theta)\Big)\,\mathcal{E}(0)\ +\ C_4\int_{\Sigma_T}|\partial_t u|^2\,d\sigma\,dt\ +\ \theta\int_0^T\mathcal{E}(t)\,dt\ . \tag{31}$$

Finally Cauchy–Schwarz's inequality, the definition of the energy and since the energy is non-increasing we get

$$\left|\xi\left[\int_\Omega (m\cdot\nabla)\, u\cdot H\, dx\right]_0^T - \left[\int_\Omega \Big((m\cdot\nabla)\, u + u\Big)\cdot \partial_t u\, dx\right]_0^T\right| \;\le\; C_5\,\mathcal{E}(0)\;, \tag{32}$$

for some positive constant C_5 independent of T.

The estimates (30), (31) and (32) in (29) yields for all $\theta > 0$ a constant $C(\theta) > 0$ (which does not depend on T but depends on θ, the domain, the coefficients a_{ijkl}, μ and ξ) such that

$$\int_0^T \mathcal{E}(t)\, dt \;\le\; C(\theta)\left(\mathcal{E}(0) + \int_{\Sigma_T}\Big\{|E\times\nu|^2 + \xi\,|\partial_t u|^2\Big\}\, d\sigma\, dt\right) + \theta\int_0^T \mathcal{E}(t)\, dt\;. \tag{33}$$

Taking $\theta < 1$ we arrive at the estimate (27). ∎

Lemma 3.4. *Let (u, E, H) be a strong solution of (1) with $g_1(\xi) = g_2(\xi) = \xi$. Then for all $\theta > 0$ there exists a constant $C(\theta) > 0$ (which does not depend on T but depends on θ, the domain, the coefficients a_{ijkl}, μ and the parameter ξ) such that*

$$\int_{\Sigma_T} |u|^2\, d\sigma\, dt \;\le\; C(\theta)\,\mathcal{E}(0) + \theta\int_0^T \mathcal{E}(t)\, dt\;. \tag{34}$$

Proof: As in [5] for each $t \ge 0$ we consider the solution z (depending on t) of

$$\begin{cases} \nabla\sigma(z) = 0 \quad \text{in } \Omega\;, \\ z = u \quad \text{on } \Gamma\;. \end{cases} \tag{35}$$

This solution is characterized by $z = w + u$ where $w \in H_0^1(\Omega)^3$ is the unique solution of

$$\int_\Omega \sigma(w) : \varepsilon(v)\; dx \;=\; -\int_\Omega \sigma(u) : \varepsilon(v)\; dx\,, \qquad \forall\, v \in H_0^1(\Omega)^3\;.$$

This identity implies that

$$\int_\Omega \sigma(z) : \varepsilon(u)\; dx \;=\; \int_\Omega \sigma(z) : \varepsilon(z)\; dx \;\ge\; 0\;. \tag{36}$$

Moreover the ellipticity assumption (2) and Korn's inequality yield a positive constant C_6 such that

$$\int_\Omega |z|^2\, dx \;\le\; C_6\int_\Gamma |u|^2\, d\sigma \;\le\; \frac{2\,C_6}{A}\,\mathcal{E}(t)\;, \tag{37}$$

$$\int_\Omega |\partial_t z|^2\, dx \;\le\; C_6\int_\Gamma |\partial_t u|^2\, d\sigma \;\le\; -C_6\,\mathcal{E}'(t)\;, \tag{38}$$

this last estimate coming from the identity (26).

Now multiplying the first identity of (1) by z and integrating on Q_T we get

$$\int_{Q_T} z \cdot \left(\partial_t^2 u - \nabla\sigma(u) + \xi\,\mathbf{curl}\,E\right) dx\,dt \;=\; 0\;.$$

By Green's formula we obtain

$$\int_{Q_T}\left(z \cdot \partial_t^2 u + \sigma(u) : \varepsilon(z) + \xi\, z \cdot \mathbf{curl}\,E\right) dx\,dt \;-\; \int_{\Sigma_T} z \cdot (\sigma(u) \cdot \nu)\, d\sigma\,dt \;=\; 0\;.$$

Using the second boundary condition in (1) and the boundary condition in (35), we obtain

$$A\int_{\Sigma_T}|u|^2\,d\sigma\,dt \;=\; -\int_{\Sigma_T} u\cdot\partial_t u\;d\sigma\,dt - \int_{Q_T}\left(z\cdot\partial_t^2 u + \sigma(u):\varepsilon(z) + \xi z\cdot\mathbf{curl}\,E\right) dx\,dt\,.$$

Owing to (36) we arrive at

$$A\int_{\Sigma_T}|u|^2\,d\sigma\,dt \;\le\; -\int_{\Sigma_T} u\cdot\partial_t u\;d\sigma\,dt \;-\; \int_{Q_T} z\cdot(\partial_t^2 u + \xi\,\mathbf{curl}\,E)\,dx\,dt\;.$$

The third equation of (1) then leads to

$$A\int_{\Sigma_T}|u|^2\,d\sigma\,dt \;\le\; -\int_{\Sigma_T} u\cdot\partial_t u\;d\sigma\,dt \;-\; \int_{Q_T} z\cdot(\partial_t^2 u - \xi\,\mu\,\partial_t H)\;dx\,dt\;.$$

Integrating by parts in t we finally have

$$\begin{aligned} A\int_{\Sigma_T}|u|^2\,d\sigma\,dt \;\le\; & -\int_{\Sigma_T} u\cdot\partial_t u\;d\sigma\,dt \\ & + \int_{Q_T}\partial_t z\cdot(\partial_t u - \xi\,\mu\,H)\,dx\,dt \\ & - \left[\int_\Omega z\cdot(\partial_t u - \xi\,\mu\,H)\,dx\right]_0^T\;. \end{aligned} \tag{39}$$

It remains to estimate each term of this right-hand side. For the first term applying successively Cauchy–Schwarz's inequality, Young's inequality ($ab \le \theta a^2 + \frac{b^2}{4\,\theta}$ for all $\theta > 0$ and all real numbers a, b) and the identity (26) we may write

$$\begin{aligned} \left|\int_{\Sigma_T} u\cdot\partial_t u\;d\sigma\,dt\right| \;&\le\; \frac{A}{2}\int_{\Sigma_T}|u|^2\,d\sigma\,dt \;+\; \frac{1}{2A}\int_{\Sigma_T}|\partial_t u|^2\,d\sigma\,dt \\ &\le\; \frac{A}{2}\int_{\Sigma_T}|u|^2\,d\sigma\,dt \;-\; \frac{1}{2A}\int_0^T \mathcal{E}'(t)\,dt\;. \end{aligned}$$

Since the energy is non-negative, we arrive at

$$\left|\int_{\Sigma_T} u\cdot\partial_t u\,d\sigma\,dt\right| \;\le\; \frac{A}{2}\int_{\Sigma_T}|u|^2\,d\sigma\,dt \;+\; \frac{1}{2A}\,\mathcal{E}(0)\;. \tag{40}$$

For the second term we use successively Cauchy–Schwarz's inequality, Young's inequality, the estimate (38) and the definition of the energy to get for all $\theta>0$

$$\begin{aligned}\left|\int_{Q_T}\partial_t z\cdot(\partial_t u-\xi\,\mu\,H)\,dx\,dt\right| &\le \frac{1}{4\,\theta}\int_{Q_T}|\partial_t z|^2\,dx\,dt \;+\;\theta\int_{Q_T}|\partial_t u-\xi\,\mu\,H|^2\,dx\,dt\\ &\le -\frac{C_6}{4\,\theta}\int_0^T\mathcal{E}'(t)\,dt \;+\;(1+\xi^2\mu)\,\theta\int_0^T\mathcal{E}(t)\,dt\;.\end{aligned}$$

Changing θ into $(1+\xi^2\mu)\,\theta$ we conclude for all $\theta>0$ that

$$\left|\int_{Q_T}\partial_t z\cdot(\partial_t u-\xi\,\mu\,H)\,dx\,dt\right| \;\le\; \frac{C_7}{\theta}\,\mathcal{E}(0)\;+\;\theta\int_0^T\mathcal{E}(t)\,dt\;, \tag{41}$$

for some positive constant C_7 (depending on the domain, the coefficients a_{ijkl}, μ and the parameter ξ).

For the third term the application of Cauchy–Schwarz's inequality, the estimate (37) and the definition of the energy directly get $C_8>0$ (with the same dependance as above) such that

$$\left|\left[\int_{\Omega} z\cdot(\partial_t u-\xi\,\mu\,H)\,dx\right]_0^T\right| \;\le\; C_8\Big(\mathcal{E}(0)+\mathcal{E}(T)\Big)\;\le\;2\,C_8\,\mathcal{E}(0) \tag{42}$$

since the energy is non-decreasing.

The estimates (40) to (42) into the estimate (39) yields the conclusion. ∎

Remark 3.5. Extension of these results to the case of a nonsmooth boundary or piecewise constant coefficients can be obtained using the ideas from [13, 25]. □

4 – Exact controllability results

Using the exponential decay in the linear case and Russell's principle we deduce the exact controllability of our electromagneto-elastic system extending previous results from [13]. More precisely for all $(u_0,u_1,E_0,H_0)\in\mathcal{H}$, we are looking for a time $T>0$ and controls $J_1,J_2\in L^2(\Gamma\times\,]0,T[)^3$ such that the solution

(u, E, H) of

$$
(43) \qquad \begin{cases} \partial_t^2 u - \nabla\sigma(u) + \xi\, \mathbf{curl}\, E = 0 \quad \text{in } \; Q_T := \Omega \times]0, T[\;, \\ \varepsilon\, \partial_t E - \mathbf{curl}\, H - \xi\, \mathbf{curl}\, \partial_t u = 0 \quad \text{in } \; Q_T \;, \\ \mu\, \partial_t H + \mathbf{curl}\, E = 0 \quad \text{in } \; Q_T \;, \\ \operatorname{div}(\varepsilon\, E) = \operatorname{div}(\mu\, H) = 0 \quad \text{in } \; Q_T \;, \\ H \times \nu - \xi\, \partial_t u \times \nu = J_1 \quad \text{on } \; \Sigma_T := \Gamma \times]0, T[\;, \\ \sigma(u) \cdot \nu + A u = J_2 \quad \text{on } \; \Sigma_T \;, \\ u(0) = u_0, \;\; \partial_t u(0) = u_1, \;\; E(0) = E_0, \;\; H(0) = H_0 \quad \text{in } \; \Omega \;, \end{cases}
$$

satisfies

$$
(44) \qquad u(T) = \partial_t u(T) = E(T) = H(T) = 0 \;.
$$

Theorem 4.1. *If Ω satisfies the EE-stability estimate, then for $T > 0$ sufficiently large, for all $(E_0, H_0) \in \mathcal{H}$ there exist controls $J_1, J_2 \in L^2(\Sigma_T)^3$ satisfying*

$$
(45) \qquad J_1 \cdot \nu = 0 \quad on \;\; \Sigma_T \;,
$$

such that the solution $(u, \partial_t u, E, H) \in C([0, T], \mathcal{H})$ of (43) is at rest a time T, i.e., satisfies (44).

Proof: The proof is quite standard [4, 12, 18, 19, 21], we give it for the sake of completeness. Moreover for further purposes we prefer to solve the inverse problem: Given (y_0, y_1, P_0, Q_0) in $\mathcal{H}$, we are looking for $K_1, K_2 \in L^2(\Sigma_T)^3$ where K_1 satisfies (45) and such that the solution $(y, \partial_t y, P, Q) \in C([0, T], \mathcal{H})$ of

$$
(46) \qquad \begin{cases} \partial_t^2 y - \nabla\sigma(y) + \xi\, \mathbf{curl}\, P = 0 \quad \text{in } \; Q_T \;, \\ \varepsilon\, \partial_t P - \mathbf{curl}\, Q - \xi\, \mathbf{curl}\, \partial_t y = 0 \quad \text{in } \; Q_T \;, \\ \mu\, \partial_t Q + \mathbf{curl}\, P = 0 \quad \text{in } \; Q_T \;, \\ \operatorname{div}(\varepsilon\, P) = \operatorname{div}(\mu\, Q) = 0 \quad \text{in } \; Q_T \;, \\ Q \times \nu + \xi\, \partial_t y \times \nu = K_1 \quad \text{on } \; \Sigma_T \;, \\ \sigma(y) \cdot \nu + A y = K_2 \quad \text{on } \; \Sigma_T \;, \\ y(T) = y_0, \;\; \partial_t y(T) = y_0, \;\; P(T) = P_0, \;\; Q(T) = Q_0 \quad \text{in } \; \Omega \;, \end{cases}
$$

satisfies

$$
(47) \qquad y(0) = \partial_t y(0) = P(0) = H(0) = 0 \;.
$$

If the above problem has a solution the conclusion follows by setting

$$u(t) = -y(T-t)\,, \qquad E(t) = -P(T-t)\,, \qquad H(t) = Q(T-t)\ .$$

We solve problem (46) and (47), using a backward and an inward electro-magneto-elastic system with linear boundary conditions: First given (v_0, v_1, F_0, I_0) in $\mathcal{H}$, we consider $(v, \partial_t v, F, I) \in C([0,T], \mathcal{H})$ the unique solution of

$$(48)\qquad \begin{cases} \partial_t^2 v - \nabla\sigma(v) + \xi\,\mathbf{curl}\,F = 0 \quad \text{in } Q_T\ ,\\ \varepsilon\,\partial_t F - \mathbf{curl}\,I - \xi\,\mathbf{curl}\,\partial_t v = 0 \quad \text{in } Q_T\ ,\\ \mu\,\partial_t I + \mathbf{curl}\,F = 0 \quad \text{in } Q_T\ ,\\ \operatorname{div}(\varepsilon F) = \operatorname{div}(\mu I) = 0 \quad \text{in } Q_T\ ,\\ I\times\nu + \xi\,\partial_t v\times\nu - (F\times\nu)\times\nu = 0 \quad \text{on } \Sigma_T\ ,\\ \sigma(v)\cdot\nu + Av - \partial_t v = 0 \ \text{ on } \Sigma_T\ ,\\ v(T) = v_0,\ \ \partial_t v(0) = v_1,\ \ F(T) = F_0,\ \ I(T) = I_0 \ \text{ in } \Omega\ . \end{cases}$$

Its existence following from Corollary 2.3 by setting $\tilde u(t) = -v(T-t)$, $\tilde E(t) = -F(T-t)$ and $\tilde H(t) = I(T-t)$. Moreover applying Theorem 3.2 to $(\tilde u(t), \tilde E(t), \tilde H(t))$ its energy is exponentially stable, i.e.,

$$(49)\qquad \mathcal{E}\Big(v(t), \partial_t v(t), F(t), I(t)\Big) \le M\, e^{-\omega(T-t)}\, \mathcal{E}(v_0, v_1, F_0, I_0)\ ,$$

where $\mathcal{E}(v(t), \partial_t v(t), F(t), I(t))$ is the expression (24) when u, $\partial_t u$, E, H are respectivelly replaced by $v, \partial_t v, F, I$.

Second we consider $(w, \partial_t w, G, J) \in C([0,T], \mathcal{H})$ the unique solution of (whose existence and uniqueness still follow from Corollary 2.3)

$$(50)\qquad \begin{cases} \partial_t^2 w - \nabla\sigma(w) + \xi\,\mathbf{curl}\,G = 0 \quad \text{in } Q_T\ ,\\ \varepsilon\,\partial_t G - \mathbf{curl}\,J - \xi\,\mathbf{curl}\,\partial_t w = 0 \quad \text{in } Q_T\ ,\\ \mu\,\partial_t J + \mathbf{curl}\,G = 0 \quad \text{in } Q_T\ ,\\ \operatorname{div}(\varepsilon G) = \operatorname{div}(\mu J) = 0 \quad \text{in } Q_T\ ,\\ J\times\nu + \xi\,\partial_t w\times\nu + (G\times\nu)\times\nu = 0 \quad \text{on } \Sigma_T\ ,\\ \sigma(w)\cdot\nu + Aw + \partial_t w = 0 \quad \text{on } \Sigma_T\ ,\\ w(0) = v(0),\ \ \partial_t w(0) = \partial_t v(0),\ \ G(0) = F(0),\ \ J(0) = I(0) \quad \text{in } \Omega\ . \end{cases}$$

We now take $y = w - v$, $P = G - F$ and $Q = J - I$. From (48) and (50), (y, P, Q) satisfies (46) with

$$K_1 = -(G\times\nu) \times \nu - (F\times\nu) \times \nu \ , \tag{51}$$

$$K_2 = -\partial_t w - \partial_t v \ . \tag{52}$$

Let us further consider the mapping Λ from $\mathcal{H}$ to $\mathcal{H}$ defined by

$$\Lambda((v_0, v_1, F_0, I_0)) = (w(T), \partial_t w(T), G(T), J(T)) \ .$$

For $T > 0$ such that $d := Me^{-\omega T} < 1$, the mapping $\Lambda - I$ is invertible since $\|\Lambda\|_{\mathcal{L}(\mathcal{H},\mathcal{H})} = \sqrt{d}$. Indeed using successively the definition of Λ, Lemma 2.5, the initial conditions of problem (50) and the estimate (49) we get

$$\begin{aligned}
\|\Lambda((v_0, v_1, F_0, I_0))\|^2_{\mathcal{H}} &= 2\,\mathcal{E}(w(t), \partial_t w, G(T), J(T)) \\
&\leq 2\,\mathcal{E}(w(0), , \partial_t w(0), G(0), J(0)) \\
&\leq 2\,\mathcal{E}(v(0), \partial_t v(0), F(0), I(0)) \\
&\leq 2\,Me^{-\omega T}\mathcal{E}(v_0, v_1, F_0, I_0) \\
&= d\,\|(v_0, v_1, F_0, I_0)\|^2_{\mathcal{H}} \ .
\end{aligned}$$

Since $\Lambda - I$ is invertible for any $(y_0, y_1, P_0, Q_0) \in \mathcal{H}$, there exists a unique $(v_0, v_1, F_0, I_0) \in \mathcal{H}$ such that

$$(y_0, y_1, P_0, Q_0) = (\Lambda - I)\,(v_0, v_1, F_0, I_0) \ . \tag{53}$$

This means that given an initial datum $(y_0, y_1, P_0, Q_0) \in \mathcal{H}$ for problem (46), there exists a unique initial datum $(v_0, v_1, F_0, I_0) \in \mathcal{H}$ for problem (48), which allows to build (v, F, I) solution of this last problem, (w, G, J) solution of problem (50) and finally $(y, P, Q) = (v, F, I) - (w, G, J)$ solution of (46)–(47) (with final datum in accordance with (53)).

We complete the proof by showing that K_1 and K_2 belong to $L^2(\Sigma_T)^3$. For that purpose, we remark that Lemma 2.5 (identity (25) applied to $(\tilde{u}, \tilde{E}, \tilde{H})$ and (w, G, J)) yields

$$\begin{aligned}
\mathcal{E}(v(T), \partial_t v(T), F(T), I(T)) \ - \ \mathcal{E}(v(0), \partial_t v(0), F(0), I(0)) \ = \\
= \int_{\Sigma_T} \Big\{ |F(t) \times \nu|^2 + |\partial_t v(t)|^2 \Big\} \ d\sigma\, dt \ ,
\end{aligned}$$

$$\begin{aligned}
\mathcal{E}(w(0), \partial_t w(0), G(0), J(0)) \ - \ \mathcal{E}(w(T), \partial_t w(T), G(T), J(T)) \ = \\
= \int_{\Sigma_T} \Big\{ |G(t) \times \nu|^2 + |\partial_t w(t)|^2 \Big\} \ d\sigma\, dt \ .
\end{aligned}$$

Summing these two identities and using the initial conditions of problem (50), the final conditions of (48) and the definition of Λ, we obtain

$$\begin{aligned}\int_{\Sigma_T}\Big\{|F\times\nu|^2 \;+\; |G\times\nu|^2 + |\partial_t v(t)|^2 + |\partial_t w(t)|^2\Big\}\, d\sigma\, dt \;=\\ = \mathcal{E}(v(T),\partial_t v(T),F(T),I(T)) - \mathcal{E}(w(T),\partial_t w(T),G(T),J(T))\\ \le \frac{1}{2}\,\|(v_0,v_1,F_0,I_0)\|^2_{\mathcal{H}}\;.\end{aligned}$$

Using the identity (53) and the boundedness of $(I-\Lambda)^{-1}$ we finally arrive at the estimate

$$\begin{aligned}\int_{\Sigma_T}\Big\{|F\times\nu|^2 + |G\times\nu|^2 \;+\; |\partial_t v(t)|^2 + |\partial_t w(t)|^2\Big\}\, d\sigma\, dt \;\le\\ \le \frac{1}{2}\,\|(I-\Lambda)^{-1}(y_0,y_1,P_0,Q_0)\|^2_{\mathcal{H}}\\ \le \frac{1}{2(1-\sqrt{d})^2}\,\|(y_0,y_1,P_0,Q_0)\|^2_{\mathcal{H}}\;.\end{aligned} \tag{54}$$

This proves that K_1 (resp. K_2) given by (51) (resp. (52)) belongs to $L^2(\Sigma_T)^3$. ∎

5 – Stability in the nonlinear case

Here we use Liu's principle, a new integral inequality and an adequate approximated scheme of g_2 by globally Lipschitz functions g_2^k preserving the properties of g_2 to deduce decay rates of the energy for nonlinear feedbacks g_1, g_2.

We first recall the integral inequality obtained similarly to Theorem 9.1 of [15] and proved in detail in [7].

Theorem 5.1. *Let $\mathcal{E}\colon [0,+\infty)\to[0,+\infty)$ be a non-increasing mapping satisfying*

$$\int_S^\infty \phi(\mathcal{E}(t))\, dt \;\le\; T\,\mathcal{E}(S)\,, \qquad \forall\, S\ge 0\;, \tag{55}$$

for some $T>0$ and some strictly increasing convex mapping ϕ from $[0,+\infty)$ to $[0,+\infty)$ such that $\phi(0)=0$. Then there exist $t_1>0$ and c_1 depending on T and $\mathcal{E}(0)$ such that

$$\mathcal{E}(t)\le \phi^{-1}\Big(\frac{\psi^{-1}(c_1\,t)}{c_1\,T\,t}\Big)\,, \qquad \forall\, t\ge t_1\;, \tag{56}$$

where ψ is defined by

$$\psi(t)=\int_t^1 \frac{1}{\phi(t)}\, dt\,, \qquad \forall\, t>0\;. \;∎ \tag{57}$$

We now give the consequence of this result to our electromagneto-elastic system.

Theorem 5.2. *Assume that g_1 and g_2 satisfies the assumptions of Lemma 2.2, as well as*

$$|E|^2 + |g_i(E)|^2 \le G(g_i(E)\cdot E)\,, \qquad \forall\, |E| \le 1, \quad i = 1, 2\,, \tag{58}$$

for some concave strictly increasing function $G : [0,\infty) \to [0,\infty)$ such that $G(0) = 0$. Suppose further that g_2 satisfies (12). If Ω satisfies the EE-stability estimate and if the embedding (23) holds for some $\delta > 0$ (see Remark 2.4 for sufficient conditions which guarantee this embedding), then there exist $c_2, c_3 > 0$ and $T_1 > 0$ (depending on T, $\mathcal{E}(0)$ and $|\Gamma|$) such that

$$\mathcal{E}(t) \le c_3\, G\left(\frac{\psi^{-1}(c_2\, t)}{c_2\, T^2\, |\Gamma|\, t}\right), \qquad \forall\, t \ge T_1\,, \tag{59}$$

for all solution $(u(t), E(t), H(t))$ of (1), where ψ is given by (57) for ϕ defined by

$$\phi(s) \;=\; T\,|\Gamma|\, G^{-1}\left(\frac{s}{c_3}\right)\,, \tag{60}$$

Proof: First we prove the theorem under the additional hypothesis that g_2 is globally Lipschitz continuous. This assumption will be removed at the end using an approximation scheme (cf. Lemma 5.5).

Thanks to Lemma 2.1 it suffices to prove (59) for data in $D(A)$. In that case let (u, E, H) be the solution of (1) and consider (y, P, Q) the solution of problem (46) and (47) with $y(T) = u(T)$, $\partial_t y(T) = \partial_t u(T)$, $P(T) = E(T)$ and $Q(T) = H(T)$ with $T > 0$ sufficiently large (whose existence was est ablished in Theorem 4.1). By (1) and (46) we may write (this identity is meaningful thanks to Remark 2.4)

$$\begin{aligned}
0 \;=\; \int_{Q_T} \Big\{ &\partial_t y \cdot \Big(\partial_t^2 u - \nabla\sigma(u) + \xi\,\mathbf{curl}\, E\Big) \\
&+\; \partial_t u \cdot \Big(\partial_t^2 y - \nabla\sigma(y) + \xi\,\mathbf{curl}\, P\Big) \\
&+\; \varepsilon\, P \cdot \Big(\partial_t E - \varepsilon^{-1}(\mathbf{curl}\, H + \xi\,\mathbf{curl}\,\partial_t u)\Big) + \mu\, Q \cdot (\partial_t H + \mu^{-1}\,\mathbf{curl}\, E) \\
&+ \varepsilon\, E \cdot \Big(\partial_t P - \varepsilon^{-1}(\mathbf{curl}\, Q + \xi\,\mathbf{curl}\,\partial_t y)\Big) + \mu\, H \cdot (\partial_t Q + \mu^{-1}\,\mathbf{curl}\, P)\Big\}\, dx\, dt\,.
\end{aligned}$$

By Green's formula (integration by parts in x, cf. Lemma 2.2 of [25]) this identity becomes

$$\begin{aligned} 0 = \int_{Q_T} & \Big\{\partial_t y \cdot \partial_t^2 u + \partial_t u \cdot \partial_t^2 y + \varepsilon(P\cdot\partial_t E + E\cdot\partial_t P) \\ & + \mu(Q\cdot\partial_t H + H\cdot\partial_t Q) + \sigma(u):\varepsilon(\partial_t y) + \sigma(y):\varepsilon(\partial_t u)\Big\}\, dx\, dt \\ & - \int_{\Sigma_T} \Big\{(\sigma(u)\cdot\nu)\cdot(\partial_t y) + (\sigma(y)\cdot\nu)\cdot(\partial_t u)\Big\}\, d\sigma\, dt \\ & + \int_{\Sigma_T} \Big\{(H\times\nu)\cdot P + (Q\times\nu)\cdot E + \xi\Big(((\partial_t y\times\nu)\cdot E) + (\partial_t u\times\nu)\cdot P\Big)\Big\}\, d\sigma\, dt\,. \end{aligned}$$

Using the boundary conditions in (1) and (46), we get

$$\begin{aligned} 0 = & \int_{Q_T} \partial_t\Big\{y\cdot\partial_t u + \varepsilon\, P\cdot E + \mu\, Q\cdot H + \sigma(u):\varepsilon(y)\Big\}\, dx\, dt \\ & + \int_{\Sigma_T} A\, \partial_t\{y\cdot u\}\, d\sigma\, dt \\ & + \int_{\Sigma_T} \Big\{\partial_t y\cdot g_2(\partial_t u) - \partial_t u\cdot K_2 - P\cdot\Big(g_1(E\times\nu)\times\nu\Big) + E\cdot K_1\Big\}\, d\sigma\, dt\,. \end{aligned}$$

Now integration by parts in t and taking into account the initial/final conditions in (1) and (46), we obtain

$$\mathcal{E}(T) = -\frac{1}{2}\int_{\Sigma_T} \Big\{\partial_t u\cdot K_2 - \partial_t y\cdot g_2(\partial_t u) - (P\times\nu)\cdot g_1(E\times\nu) - E\cdot K_1\Big\}\, d\sigma\, dt\,.$$

Cauchy–Schwarz's inequality in $\mathbb{R}^3$ and the property (45) satisfied by K_1 yield

$$\mathcal{E}(T) \le \frac{1}{2}\int_{\Sigma_T} \Big\{|\partial_t u|\,|K_2| + |\partial_t y|\,|g_2(\partial_t u)| + |P\times\nu|\,|g_1(E\times\nu)| + |E\times\nu|\,|K_1|\Big\}\, d\sigma\, dt\,.$$

Now Cauchy–Schwarz's inequality in $L^2(\Sigma_T)$ yields

$$\begin{aligned} \mathcal{E}(T) \le & \Big(\int_{\Sigma_T} |\partial_t u|^2\, d\sigma\, dt\Big)^{1/2} \Big(\int_{\Sigma_T} |K_2|^2\, d\sigma\, dt\Big)^{1/2} \\ & + \Big(\int_{\Sigma_T} |\partial_t y|^2\, d\sigma\, dt\Big)^{1/2} \Big(\int_{\Sigma_T} |g_2(\partial_t u)|^2\, d\sigma\, dt\Big)^{1/2} \\ & + \Big(\int_{\Sigma_T} |P\times\nu|^2\, d\sigma\, dt\Big)^{1/2} \Big(\int_{\Sigma_T} |g_1(E\times\nu)|^2\, d\sigma\, dt\Big)^{1/2} \\ & + \Big(\int_{\Sigma_T} |E\times\nu|^2\, d\sigma\, dt\Big)^{1/2} \Big(\int_{\Sigma_T} |K_1|^2\, d\sigma\, dt\Big)^{1/2}\,. \end{aligned} \tag{61}$$

Let us remark that the estimate (54) and the final conditions on $(y, \partial_t y, P, Q)$ yield

$$\int_{\Sigma_T} \Big\{ |F \times \nu|^2 + |G \times \nu|^2 + |\partial_t v|^2 + |\partial_t w|^2 \Big\} \, d\sigma \, dt \;\le\; \frac{1}{(1-\sqrt{d})^2} \, \mathcal{E}(T) \; .$$

This estimate, the definition of y $(y = w - v)$, of P $(P = G - F)$, the definition (51) (resp. (52)) of K_1 (resp. of K_2) and Cauchy–Schwarz's inequality lead to

$$\int_{\Sigma_T} |K_i|^2 \, d\sigma \, dt \;\le\; \frac{2}{(1-\sqrt{d})^2} \, \mathcal{E}(T) \, , \qquad i = 1, 2 \; ,$$

$$\int_{\Sigma_T} |P \times \nu|^2 \, d\sigma \, dt \;\le\; \frac{2}{(1-\sqrt{d})^2} \, \mathcal{E}(T) \; ,$$

$$\int_{\Sigma_T} |\partial_t y|^2 \, d\sigma \, dt \;\le\; \frac{2}{(1-\sqrt{d})^2} \, \mathcal{E}(T) \; .$$

These estimates in (61) lead to

$$\mathcal{E}(T) \;\le\; c_4 \int_{\Sigma_T} \Big(|g_2(\partial_t u)|^2 + |\partial_t u|^2 + |g_1(E \times \nu)|^2 + |E \times \nu|^2 \Big) \, d\sigma \, dt \; , \tag{62}$$

for some positive constant c_4 depending on d (so depending on T).

We now estimate the right-hand side of (62) as follows: Introduce

$$\Sigma_T^+ = \Big\{ (x,t) \in \Sigma_T \mid |E(x,t) \times \nu(x)| > 1 \Big\} \; ,$$

$$\Sigma_T^- = \Big\{ (x,t) \in \Sigma_T \mid |E(x,t) \times \nu(x)| \le 1 \Big\} \; .$$

By the assumptions (8) and (12) satisfied by g_1 we may write

$$\int_{\Sigma_T^+} \Big(|g_1(E \times \nu)|^2 + |E \times \nu|^2 \Big) \, d\sigma \, dt \;\le\; c_5 \int_{\Sigma_T^+} (E \times \nu) \cdot g_1(E \times \nu) \, d\sigma \, dt \; ,$$

for some positive constant c_5. By (25) and the property $g_i(\xi) \cdot \xi \ge 0$ (consequence of (10) and (11)) satisfied by g_i, $i = 1, 2$, we arrive at

$$\int_{\Sigma_T^+} \Big(|g_1(E \times \nu)|^2 + |E \times \nu|^2 \Big) \, d\sigma \, dt \;\le\; c_5 (\mathcal{E}(0) - \mathcal{E}(T)) \; . \tag{63}$$

Similarly by the assumption (58) satisfied by g_1 we have

$$\int_{\Sigma_T^-} \Big(|g_1(E \times \nu)|^2 + |E \times \nu|^2 \Big) \, d\sigma \, dt \;\le\; \int_{\Sigma_T^-} G\Big((E \times \nu) \cdot g_1(E \times \nu) \Big) \, d\sigma \, dt \; .$$

Jensen's inequality then yields

$$\int_{\Sigma_T^-}\Big(|g_1(E\times\nu)|^2+|E\times\nu|^2\Big)\,d\sigma\,dt \;\le\; |\Sigma_T|\,G\Big(\frac{1}{|\Sigma_T|}\int_{\Sigma_T^-}(E\times\nu)\cdot g_1(E\times\nu)\,d\sigma\,dt\Big)\,.$$

By (25), the property $g_i(\xi)\cdot\xi\ge 0$, $i=1,2$, and the monotonicity of G, we arrive at

$$\int_{\Sigma_T^-}\Big(|g_1(E\times\nu)|^2+|E\times\nu|^2\Big)\,d\sigma\,dt \;\le\; |\Sigma_T|\,G\Big(\frac{\mathcal{E}(0)-\mathcal{E}(T)}{|\Sigma_T|}\Big)\,. \tag{64}$$

In the same way using the properties of g_2, the property $g_i(\xi)\cdot\xi\ge 0$, $i=1,2$ and the identity (25) we can prove the estimate

$$\int_{\Sigma_T}\Big\{|g_2(\partial_t u)|^2+|\partial_t u|^2\Big\}\,d\sigma\,dt \;\le\; c_6\Big(\mathcal{E}(0)-\mathcal{E}(T)+|\Sigma_T|\,G\Big(\frac{\mathcal{E}(0)-\mathcal{E}(T)}{|\Sigma_T|}\Big)\Big)\,, \tag{65}$$

for some positive constant c_6.

The estimates (63), (64) and (65) into the estimate (62) give

$$\mathcal{E}(T)\;\le\; c_7\Big(\mathcal{E}(0)-\mathcal{E}(T)+G\Big(\frac{\mathcal{E}(0)-\mathcal{E}(T)}{|\Sigma_T|}\Big)\Big)\,,$$

for some positive constant c_7 (depending on T and $|\Gamma|$). This finally leads to

$$\mathcal{E}(0)\;=\;\mathcal{E}(0)-\mathcal{E}(T)+\mathcal{E}(T)\;\le\;\max\{1,c_7\}\,\Big(\mathcal{E}(0)-\mathcal{E}(T)+G\Big(\frac{\mathcal{E}(0)-\mathcal{E}(T)}{|\Sigma_T|}\Big)\Big)\,.$$

Using this argument in $[t,t+T]$ instead of $[0,T]$ we have shown that

$$\mathcal{E}(t)\;\le\;\max\{1,c_7\}\,\Big(\mathcal{E}(t)-\mathcal{E}(t+T)+G\Big(\frac{\mathcal{E}(t)-\mathcal{E}(t+T)}{|\Sigma_T|}\Big)\Big)\,.$$

As $\frac{\mathcal{E}(t)-\mathcal{E}(t+T)}{|\Sigma_T|}\le\frac{\mathcal{E}(t)}{|\Sigma_T|}\le\frac{\mathcal{E}(0)}{|\Sigma_T|}$, the concavity of G yields a constant c_8 (depending continuously on T, $\mathcal{E}(0)$ and $|\Gamma|$) such that

$$\frac{\mathcal{E}(t)-\mathcal{E}(t+T)}{|\Sigma_T|}\;\le\; c_8\,G\Big(\frac{\mathcal{E}(t)-\mathcal{E}(t+T)}{|\Sigma_T|}\Big)\,.$$

These two estimates lead to

$$\mathcal{E}(t)\;\le\; c_3\,G\Big(\frac{\mathcal{E}(t)-\mathcal{E}(t+T)}{|\Sigma_T|}\Big)\;=\;\phi^{-1}\Big(\mathcal{E}(t)-\mathcal{E}(t+T)\Big)\,,\qquad\forall\, t\ge 0\,, \tag{66}$$

when we recall that ϕ was defined by (60) and $c_3>0$ depends only on T, $\mathcal{E}(0)$ and $|\Gamma|$.

We conclude by Theorem 5.1 since Lemma 5.1 of [7] shows that the estimate (66) guarantees that $\mathcal{E}$ actually satisfies (55).

If g_2 is no more globally Lipschitz continuous, by Lemma 5.5 below there exists a sequence of globally Lipschitz continuous mappings g_2^k, $k \in \mathbb{N}^\star$ satisfying (8), (10), (11), (12) for $|x| \geq 2$ as well as (58) for $|x| \leq 2$ with $\hat{G}$ instead of G which is simply a multiple of G (independant of k). For each k let $(u_k(t), E_k(t), H_k(t))$ be a solution of (1) with g_2 replaced by g_2^k. Applying the above arguments for each k we get the estimate

$$\mathcal{E}_k(t) \leq c_3\, G\left(\frac{\psi^{-1}(c_2\, t)}{c_2\, T^2\, |\Gamma|\, t}\right), \quad \forall\, t \geq T_1\ , \tag{67}$$

where $\mathcal{E}_k(t)$ denotes the energy of $(u_k(t), E_k(t), H_k(t))$, the constants and the functions G and ψ hereabove being independent of k. We conclude thanks to Lemma 5.6 below which shows that

$$(u_k(t), \partial_t u_k(t), E_k(t), H_k(t)) \to (u_k(t), \partial_t u_k(t), E_k(t), H_k(t)) \quad \text{in } \mathcal{H} \text{ as } k \to \infty\,,$$

for all $t \in \mathbb{R}^+$. ∎

Remark 5.3. Examples of functions g_1 and g_2 leading to an explicit decay rate (59) are given in [7, 25]. Let us notice that exponential, polynomial or logarithmic decays are available for appropriate feedbacks. □

Remark 5.4. As already mentioned, for $\xi = 0$ the system (1) is split into the elastodynamic system and Maxwell's system. In that case the conjunction of Theorems 3.3 and 5.2 gives stability results for any strictly star-shaped domain with a smooth boundary for the elastodynamic system with general nonlinear feedback g_2 satisfying the assumptions of Theorem 5.2. This result improves some earlier results from [1, 3, 9] where g_2 is chosen diagonal and exponential or polynomial decays only are available. □

We now prove the approximation scheme of a non globally Lipschitz continuous mapping g by a sequence of globally Lipschitz continuous mappings, adapting Lemma 9.9 of [15] to the vectorial case and other growth properties on g.

Lemma 5.5. *Let $g : \mathbb{R}^3 \to \mathbb{R}^3$ be a continuous mapping satisfying (8), (10), (11), (12), as well as (58). Then there exists a sequence of globally Lipschitz* *continuous mappings $g_k : \mathbb{R}^3 \to \mathbb{R}^3$, $k \in \mathbb{N}^\star$, satisfying (8) (with the same constant than g), (10), (11), as well as*

$$g_k(x) \cdot x \geq m' \, |x|^2 \,, \quad \forall \, x \in \mathbb{R}^3 : \ |x| \geq 2 \,, \tag{68}$$

$$|x|^2 + |g_k(x)|^2 \leq \gamma \, G(g_k(x) \cdot x) \,, \quad \forall \, |x| \leq 2 \,, \tag{69}$$

for some positive constant m' and γ independent of k and satisfying furthermore

$$|g_k(x)| \leq |g(x)| \,, \quad \forall \, x \in \mathbb{R}^3, \ \ k \in \mathbb{N}^\star \,, \tag{70}$$

$$g_k(x) \to g(x) \quad \text{as} \ \ k \to \infty, \ \ \forall \, x \in \mathbb{R}^3 \,. \tag{71}$$

Proof: Similarly to Lemma 9.9 of [15] we take

$$g_k(x) = g\Big((I + k^{-1} g)^{-1}(x) \Big) \,, \quad \forall \, x \in \mathbb{R}^3, \ \ k \in \mathbb{N}^\star \,.$$

Note that this definition is meaningful since $I + k^{-1} g$ is invertible due to the monotonicity of g. The definition of g_k directly leads to (71).

Let us now show the monotonicity of g_k. For $x_i \in \mathbb{R}^3$, $i = 1, 2$, we set

$$y_i = (I + k^{-1} g)^{-1}(x_i) \,, \quad i = 1, 2 \,. \tag{72}$$

This directly yields

$$g(y_i) = g_k(x_i) \,, \quad i = 1, 2 \,, \tag{73}$$

$$y_i + k^{-1} g(y_i) = x_i \,, \quad i = 1, 2 \,. \tag{74}$$

By difference we obtain

$$k^{-1}\Big(g(y_1) - g(y_2) \Big) \;=\; x_1 - x_2 - (y_1 - y_2) \,. \tag{75}$$

Taking the inner product with $x_1 - x_2$ and using Cauchy–Schwarz's inequality we obtain

$$k^{-1}\Big(g(y_1) - g(y_2) \Big) \cdot (x_1 - x_2) \;\geq\; |x_1 - x_2| \, \Big(|x_1 - x_2| - |y_1 - y_2| \Big) \,. \tag{76}$$

In (75) taking the inner product with $y_1 - y_2$ and using the monotonicity of g, we get

$$|y_1 - y_2| \leq |x_1 - x_2| \,. \tag{77}$$

Using this estimate in (76) and using the identities (73) we conclude the monotonicity of g_k.

For the globally Lipschitz continuity of g_k, in (75) we take the inner product with $g(y_1) - g(y_2)$ to obtain

$$k^{-1}|g(y_1) - g(y_2)|^2 = (x_1 - x_2) \cdot \Big(g(y_1) - g(y_2)\Big) - (y_1 - y_2) \cdot \Big(g(y_1) - g(y_2)\Big) .$$

The monotonicity of g and Cauchy-Schwarz's inequality allow to conclude that

$$k^{-1}|g(y_1) - g(y_2)| \leq |x_1 - x_2| ,$$

which shows the globally Lipschitz continuity of g_k owing to (73).

Let us now show that (70) holds: As before for a fixed $x \in \mathbb{R}^3$, we set

$$y = (I + k^{-1}g)^{-1}(x) , \tag{78}$$

which yields

$$g(y) = g_k(x) , \tag{79}$$

$$y + k^{-1}g(y) = x . \tag{80}$$

In this last identity taking the inner product with $g(x) - g(y)$ we get

$$k^{-1}g(y) \cdot \Big(g(x) - g(y)\Big) = (x - y) \cdot \Big(g(x) - g(y)\Big) .$$

From the monotonicity of g and Cauchy-Schwarz's inequality we arrive at

$$|g(y)| \leq |g(x)| ,$$

and we conclude by (79).

The estimate (8) with the same constant than g (resp. the property (11)) for g_k follow from (70) and the estimate (8) (resp. (11)) satisfied by g.

Before going on let us establish the estimate (with the above notation)

$$g_k(x) \cdot x \geq g(y) \cdot y . \tag{81}$$

Indeed thanks to (79) this is equivalent to

$$g(y) \cdot (x - y) \geq 0 ,$$

which holds owing to (80).

We are ready to prove the estimate (68). For a fixed $|x| \geq 2$, let y be given by (78).

(**a**) If $|y| \leq |x|/2$, then by (79) and (80) we may write

$$k^{-1} g_k(x) \cdot x \;=\; k^{-1} g(y) \cdot x \;=\; (x-y) \cdot x \;.$$

By the above assumption on $|y|$ we obtain

$$k^{-1} g_k(x) \cdot x \;\geq\; |x|^2/2 \;,$$

which proves that

(82) $$g_k(x) \cdot x \;\geq\; |x|^2/2 \;.$$

(**b**) If $|y| \geq |x|/2$, then $|y| \geq 1$ and by (12) satisfied by g we get

$$g(y) \cdot y \;\geq\; m\,|y|^2 \;\geq\; \frac{m}{4}\,|x|^2 \;.$$

Owing to the estimate (81), we arrive at

(83) $$g_k(x) \cdot x \;\geq\; \frac{m}{4}\,|x|^2 \;.$$

In conclusion the estimates (82) and (83) show that g_k satisfies (68) with $m' = \min\{1/2, m/4\}$.

It remains to show (69). First we remark that the properties (8), (12) and (58) satisfied by g imply

(84) $$|x|^2 + |g(x)|^2 \;\leq\; c\,G(g(x) \cdot x)\,, \qquad \forall\, |x| \leq 2\;,$$

for some positive constant c depending only on G. Indeed for $1 \leq |x| \leq 2$ by (8) and (12) we have

$$\begin{aligned} &|x|^2 + |g(x)|^2 \;\leq\; \frac{1+4M^2}{m}\, g(x) \cdot x\;, \\ &|g(x) \cdot x| \;\leq\; 6\,M\;. \end{aligned}$$

Moreover the concavity of G yields a constant C (depending continuously on $6M$) such that

(85) $$z \;\leq\; C\,G(z)\,, \qquad \forall\, 0 \leq z \leq 6M\;.$$

These last three estimates lead to (84) with $c = \max\{1, C\,\frac{1+4M^2}{m}\}$.

For a fixed $|x| \leq 2$, let y be given by (78). By (77) we have $|y| \leq |x|$ and therefore $|y| \leq 2$.

(**a**) If $|y| \leq |x|/2$, then (82) still holds and moreover

$$|g_k(x) \cdot x| \;=\; |g(y) \cdot x| \;\leq\; M(1+|y|)\,|x| \;\leq\; 4\,M \;.$$

Therefore by (85) and (82) we conclude

$$|x|^2 \;\leq\; 2\,C\,G(g_k(x) \cdot x) \;. \tag{86}$$

(**b**) If $|y| \geq |x|/2$, then by (81) and the monotonicity of G, we have

$$G(g_k(x) \cdot x) \;\geq\; G(g(y) \cdot y) \;.$$

By (84) (valid since $|y| \leq 2$) we obtain

$$|x|^2 \;\leq\; 4\,|y|^2 \;\leq\; 4\,c\,G(g(y) \cdot y) \;\leq\; 4\,c\,G(g_k(x) \cdot x) \;. \tag{87}$$

The estimates (86) and (87) show that

$$|x|^2 \leq \tilde{\gamma}\,G(g_k(x) \cdot x)\,, \qquad \forall\,|x| \leq 2\;, \tag{88}$$

with $\tilde{\gamma} = 2\max\{C, 2c\}$.

For the estimation of $|g_k(x)|^2$, we simply remark that (79) and (84) yield

$$|g_k(x)|^2 \;=\; |g(y)|^2 \;\leq\; c\,G(g(y) \cdot y) \;.$$

From (81) and the monotonicity of G, we conclude

$$|g_k(x)|^2 \leq c\,G(g_k(x) \cdot x)\,, \qquad \forall\,|x| \leq 2\;. \tag{89}$$

The sum of (88) and (89) yields (69) with $\gamma = \tilde{\gamma} + c$.

The proof is complete since all the requested properties on g_k were established. ∎

Let us finish this section by the convergence result needed in the proof of Theorem 5.2.

Lemma 5.6. *Let g_1 and g_2 satisfy the assumptions of Lemma 2.2, and let g_2^k, $k \in \mathbb{N}^\star$ be a sequence of mappings satisfying the assumptions of Lemma 2.2 (with constants independent of k) and such that*

$$g_2^k(x) \to g_2(x) \quad \text{as} \;\; k \to \infty, \;\; \forall\,x \in \mathbb{R}^3 \;.$$

For $(u_0, u_1, E_0, H_0) \in \mathcal{H}$, let (u, E, H) be the unique weak solution of problem (1) and for all $k \in \mathbb{N}^\star$ let (u_k, E_k, H_k) be the unique weak solution of problem (1) with g_2^k instead of g_2. Then for all $t \in \mathbb{R}^+$, it holds

$$(u_k(t), \partial_t u_k(t), E_k(t), H_k(t)) \to (u_k(t), \partial_t u_k(t), E_k(t), H_k(t)) \quad \textit{in } \mathcal{H} \ \textit{ as } k \to \infty \ .$$

Proof: For all $k \in \mathbb{N}^\star$ let us denote by A_k the operator introduced in Lemma 2.2 associated with our system (1) with g_1 and g_2^k. By Theorem 7.3 of [15], we only need to show that

$$(90) \qquad (I + A_k)^{-1}(f, g, F, G) \to (I + A)^{-1}(f, g, F, G) \quad \text{in } \mathcal{H} \ \text{ as } k \to \infty \ ,$$

for all $(f, g, F, G) \in \mathcal{H}$. If we denote by $(u, v, E, H) = (I + A)^{-1}(f, g, F, G)$ and $(u_k, v_k, E_k, H_k) = (I + A)^{-1}(f, g, F, G)$, then Lemma 2.2 showed that (15) and (16) hold, as well as

$$(91) \qquad u_k = v_k + f \ ,$$

$$(92) \qquad H_k = G - \mu^{-1} \,\mathbf{curl}\, E_k \ .$$

and that

$$(93) \qquad a\Big((v, E), (v', E')\Big) = a_k\Big((v_k, E_k), (v', E')\Big) \ , \qquad \forall\, (v', E') \in V \ .$$

By taking the difference between (15) and (91) as well as the difference between (16) and (92), and using the fact that $\varepsilon(E - E_k)$ is divergence free, we see that the convergence (90) holds if

$$(94) \qquad \|v - v_k\|_1 + \|E - E_k\|_{H(\mathbf{curl})} \to 0, \qquad \text{as } \ k \to \infty \ ,$$

where $\|E\|_{H(\mathbf{curl})} = \|E\|_0 + \|\,\mathbf{curl}\, E\|_0$ and $\|\cdot\|_l$ means the $H^l(\Omega)^3$-norm.

For that purpose we remark that (93) is equivalent to

$$a_k\Big((v_k, E_k), (v', E')\Big) - a_k\Big((v, E), (v', E')\Big) = \\ = a\Big((v, E), (v', E')\Big) - a_k\Big((v, E), (v', E')\Big) \ , \qquad \forall\, (v', E') \in V \ .$$

Taking $v' = v_k - v$ and $E' = E_k - E$, and using the monotonicity of g_1 and g_2^k, we get

$$\int_\Omega \Big\{ \sigma(v_k - v) : \varepsilon(v_k - v) + |v_k - v|^2 \Big\}\, dx \ + \\ + \int_\Omega \Big\{ \mu^{-1} \,|\,\mathbf{curl}(E_k - E)|^2 + \varepsilon\, |E_k - E)|^2 \Big\}\, dx \ \le \\ \le \int_\Gamma \Big(g_2(v) - g_2^k(v) \Big) \cdot (v_k - v) \ d\sigma \ .$$

By Korn's inequality, Cauchy–Schwarz's inequality and a standard trace theorem, there exists a positive constant K (independent of k) such that

$$\|v-v_k\|_1 + \|E-E_k\|_{H(\mathbf{curl})} \;\leq\; K\,\|g_2(v)-g_2^k(v)\|_{L^2(\Gamma)} \; . \tag{95}$$

As the property (8) satisfied by g_2 as well as g_2^k implies

$$|g_2(v(x)) - g_2^k(v(x))| \;\leq\; 2\,M\,(1+|v(x)|) \;,$$

we conclude that the right-hand side of (95) tends to zero as k goes to infinity by Lebesgue's bounded convergence Theorem. ∎

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