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COCHAIN OPERATIONS AND SUBSPACE ARRANGEMENTS

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Abstract: We give a combinatorial description of simplicial cochain and cocycle operations mod p by certain subspace arrangements over $\mathbb{Z}/p$. Then we show that any primary or higher cohomology operation mod p has a description at the cochain level by polynomials of coface operators. We define an algebraic filtration on the space of cochain operations and give an explicit dimension formula for the space of cocyle operation of fixed algebraic degree. The filtration is also defined on Steenrod operations and leads to a new spectral sequence converging to the cohomology of Eilenberg–MacLane spaces mod p.

1 – Introduction

In [6] we gave a modification of L. Kristensen's machinery of constructing cohomology operations in algebraic topology by simplicial cochain operations which also works for higher order cohomology operations. There are some significant examples of applications of cochain operations, for example an independent proof for the Hopf invariant one theorem by the computation of Kristensen of Massey products in the Steenrod algebra [8], the examination of the β-family in stable homotopy by L. Smith using a secondary Hopf invariant ([11]) and the authors result on Brown–Kervaire invariants for Spin manifolds ([4], [5]). These results

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depend on skilful constructions of special cochain operations and heavy combinatorial calculations made by L. Kristensen. However, the algebraic structure of the full set of cochain operations remained open and Kristensen did not make computations above third order relations by combinatorial means. In this paper, we consider the combinatorial structure of cochain operations, and obtain some explicit results by relating cochain operations to subspace arrangements over finite fields. There are some well-known examples of cochain operations: The coboundary homomorphism $d = \sum(-1)^i d_i$ in the cochain complex of any space, the cup product given on the cochain level by the Alexander–Whitney formula $x \cup y = d^I x \cdot d^J y$ with d^I the front $dim(y)$-coface operator and d^J the back $dim(x)$-coface operator, and more general, the cup-i products given by Steenrod's formula [12]

$$\cup_i \colon C^m(X;\mathbb{Z}/2) \times C^n(X;\mathbb{Z}/2) \longrightarrow C^{m+n-i}(X;\mathbb{Z}/2)$$

$$x \cup_i y \;=\; \sum_{I,J} d^I x \cdot d^J y$$

where summation runs over all subsets I, J of

$$[m+n-i] := \{0, 1, \ldots, m+n-i\}$$

that are given by sequences $0 \leq k_0 < k_1 < \ldots < k_i \leq n+m-i$ such that

$$[n+m-i] - J \;=\; \{0, \ldots, k_0\} \cup \{k_1, \ldots, k_2\} \cup \ldots$$

$$[n+m-i] - I \;=\; \{k_0, \ldots, k_1\} \cup \{k_2, \ldots, k_3\} \cup \ldots$$

and $|I| = n - i$, $|J| = m - i$. Using cup-i products, the Steenrod squares are constructed explicitely by

$$sq^i(x) = x \cup_{n-i} x$$

for n-cochains mod 2. This raises the question if, in general, cohomology operations are represented by suitable polynomials in coface operators. We prove this not only for primary cohomology operations $\bmod p$, but also show that higher cohomology operations $\bmod p$ are represented by algebraic systems of equations on cochains, where the system is again given by polynomials in coface operators. This defines an algebraic filtration on the set of cochain operations and also for cohomology operations. In the last section we consider cocycle operations $\bmod p$ and compute the dimensions of filtration quotients. The algebraic filtration defines a new spectral sequence converging to the cohomology of Eilenberg–MacLane spaces $\bmod p$, and the computation of filtration quotients gives the additive structure of its E_0-term.

2 – Simplicial cochain operations

Let $C^m(-;R)$ denote the normalized cochain functor and $\tilde{C}^m(-;R)$ the full cochain functor of simplicial sets, where R is a commutative ring. Thus $\tilde{C}^m(X_\bullet;R) = map(X_m, R)$, the set of all R-valued functions on X_m, and $C^m(X_\bullet;R)$ is the subset of functions that vanish on the degenerate m-simplices. We recall from [6] the basic definitions and properties of cochain operations:

Definition 1. An **unstable cochain operation** of type (R,m,S,n) is a natural transformation from the functor $C^m(-;R)$ to the functor $C^n(-;S)$. There is no condition of linearity. Denote the set of these operations by $\mathcal{O}(R,m,S,n)$. $\square$

There is a representability result of Eilenberg–MacLane (see [9]):

$$C^m(X_\bullet;R) = mor(X_\bullet, L(R,m+1)_\bullet) \ ,$$

where mor denotes the set of simplicial maps and $L(R,m+1)_\bullet$ is the acyclic simplicial R-module which corresponds to the acyclic chain complex

$$l(R,m+1) := (\ldots \to 0 \to R \to R \to 0 \to \ldots \to 0)$$

by Dold–Kan equivalence. The representability isomorphism is given by pulling back the fundamental cochain $\iota \in C^m(L(R,m+1);R)$. This gives by the Yoneda lemma

$$\mathcal{O}(R,m,S,n) = mor(L(R,m+1)_\bullet, L(S,n+1)_\bullet) = C^n(L(R,m+1)_\bullet;S) \ .$$

As $L(R,m+1)_n = 0$ for $n < m$, it follows $\mathcal{O}(R,m,S,n) = 0$ for $n < m$. As $L(R,m)_m = R$ with $0 \in R$ being the only degenerate simplex in dimension m, it follows $\mathcal{O}(R,m,S,m) = \{f\colon R \to S \mid f(0) = 0\}$. In particular, this includes the $\mathbb{Z}$-linear maps $Hom(R,S)$ as a subset. The coboundary homomorphism $d\colon C^m(-;R) \to C^{m+1}(-;R)$ gives a further example of a cochain operation

$$d \in \mathcal{O}(R,m,R,m+1) \ .$$

As the functor $C^n(-;S)$ takes values in S-modules, we have a canonical S-module structure on $\mathcal{O}(R,m,S,n)$ which corresponds to the 'pointwise' structure in $mor(L(R,m+1)_\bullet, L(S,n+1)_\bullet)$ induced from the simplicial S-module structure on $L(S,n+1)_\bullet$. Furthermore, full cochains form an S-algebra

$\tilde{C}^n(X_\bullet; S) = map(X_n, S)$ with respect to pointwise multiplication of functions and $C^n(X_\bullet; S)$ is an ideal. Hence $\mathcal{O}(R, m, S, n)$ is an ideal in the S-algebra

$$\tilde{\mathcal{O}}(R, m, S, n) := \tilde{C}^n(L(R, m+1)_\bullet; S) ,$$

which is the set of natural transformations from the normalized cochain functor $C^m(-; R)$ to the full cochain functor $\tilde{C}^n(-; S)$. All cochain operations $a \in \mathcal{O}(R, m, S, n)$ with $n > 0$ satisfy $a(0) = 0$, where 0 denotes the zero cochain on a simplicial set $X_\bullet$. This follows easily utilizing naturality for the projection $X_\bullet \to *_\bullet$ to a point. Composition of cochain operations gives a map

$$\circ\colon \mathcal{O}(R_2, m_2, R_3, m_3) \times \mathcal{O}(R_1, m_1, R_2, m_2) \to \mathcal{O}(R_1, m_1, R_3, m_3)$$

which is R_3-linear in the left variable, *but non-linear in the right variable*, in general. In particular, $(a + a') \circ b = a \circ b + a' \circ b$ and $(a \cdot a') \circ b = (a \circ b) \cdot (a' \circ b)$, but $a \circ (b + b') \neq a \circ b + a \circ b'$ and $a \circ (b \cdot b') \neq (a \circ b) \cdot (a \circ b')$, in general.

3 – cochain operations and arrangements

In order to get a combinatorial description of cochain operations, we consider the following explicit model of $L(R, m+1)_\bullet$ (see [9]):

$$L(R, m+1)_n = C^m(\Delta^n_\bullet; R)$$

where $\Delta^n_\bullet$ denotes the simplicial standard n-simplex. The spaces $\Delta^n_\bullet$ form a cosimplicial (with respect to the index n) simplicial set, and applying the contravariant cochain functor turns n into a simplicial index. Now, every m-simplex in $\Delta^n_\bullet$ is uniquely given by a sequence

$$I = (0 \leq i_0 \leq i_1 \leq \ldots i_m \leq n)$$

where the action of face and degeneracy operations d_j, s_j is given by deleting respectively doubling the j-th number i_j. Thus the non-degenerate m-simplices are exactly the $(m+1)$-subsets of the set $[n] := \{0, 1, \ldots, n\}$, and $L(R, m+1)_n$ is the free R-module of R-valued functions on the set of $(m+1)$-subsets I of $[n]$. A basis is given by the Kronecker-functions $[I]$ defined by

$$[I](I') := \begin{cases} 1 & \text{for } I = I' \\ 0 & \text{for } I \neq I' \end{cases}$$

hence $rank(L(R,m+1)_n) = \binom{n+1}{m+1}$. The cosimplicial structure maps

$$d^j : \Delta_\bullet^{n-1} \to \Delta_\bullet^n , \quad j = 0, 1, \dots n-1 ,$$
$$s^j : \Delta_\bullet^{n+1} \to \Delta_\bullet^n , \quad j = 0, 1, \dots n+1 ,$$

are given by $d^j(I) = \{i \mid j > i \in I\} \cup \{i+1 \mid j \le i \in I\}$ (i.e., add 1 to the numbers $\ge j$), and $s^j(I) = \{i \mid j \ge i \in I\} \cup \{i-1 \mid j < i \in I\}$ (i.e., subtract 1 from the numbers $> j$). This shows that face and degeneracy operators on $L(R,m)_\bullet$ are the surjective respectively injective R-linear maps given by

$$d_j[I] = [I] \circ d^j = \begin{cases} [s^j(I)] & \text{for } j \in I \\ 0 & \text{for } j \notin I \end{cases} ,$$
$$s_j[I] = [I] \circ s^j = \begin{cases} [d^j(I)] + [d^{j+1}(I)] & \text{for } j \in I \\ [d^j(I)] & \text{for } j \notin I \end{cases} .$$

Hence the degenerate simplices in $L(R,m+1)_n$ form the union of a central **arrangement**

$$\mathcal{A}_n^m := \bigcup_{i=0}^{n-1} V_i$$
$$V_i := im\Big(s_i : L(R,m+1)_{n-1} \to L(R,m+1)_n\Big)$$

of n free submodules V_i of rank $\binom{n}{m+1}$ (for basic facts on arrangements, see [10]). We recall that the associated **intersection poset** $L(\mathcal{A})$ of an arrangement $\mathcal{A}$ is given by the set of intersections of members in $\mathcal{A}$ with respect to reverse inclusion.

Theorem 1. *The set of cochain operations $\mathcal{O}(R,m,S,n)$ can be identified with the set of S-valued functions on the free R-module $V := L(R,m+1)_n$ of rank $\binom{n+1}{m+1}$ which vanish on the arrangement $\mathcal{A}_m^n$. The intersection poset $L(\mathcal{A}_m^n)$ is the poset of subsets of $[n-1]$ truncated above the level $n-m$. The rank function is given by $\binom{n+1-r}{m+1}$ where r denotes the cardinality of the subset. If R is a finite ring of cardinality q, then $\mathcal{O}(R,m,S,n)$ is a free S-module of rank*

$$\sum_{r=0}^{n} (-1)^r \binom{n}{r} q^{\binom{n+1-r}{m+1}} .$$

Proof: The first statement directly follows from the definition of normalized cochains. By the fundamental lemma for degenerate simplices, we have for $0 \le i_1 < i_2 < \dots i_r \le n-1$ that

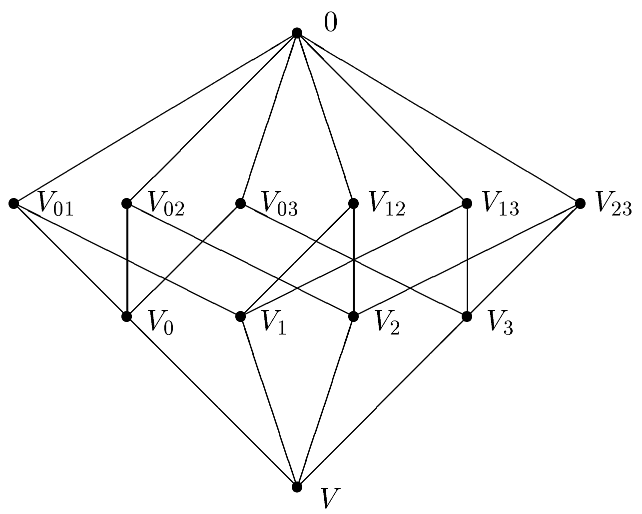
$$\begin{aligned} V_{i_1,i_2,\dots i_r} &:= im(s_{i_1}) \cap im(s_{i_2}) \cap \dots \cap im(s_{i_r}) \\ &= im\Big(s_{i_r} \dots s_{i_2} s_{i_1} : L(R,m+1)_{n-r} \to L(R,m+1)_n\Big) . \end{aligned}$$

By the injectivity of degeneracy maps, this is a free R-module of rank $\binom{n+1-r}{m+1}$. As long as this rank is not 0, i.e. for $1 \leq r \leq n-m$, the submodules $V_{i_1,i_2,\dots i_r}$ for different choices of i_j are different from each other as their intersections have lower rank. This proves the assertion on the intersection poset of $\mathcal{A}^n_m$. The complement of the arrangement is a finite set if and ony if R is finite, and then the S-valued functions form a free S-module. By the inclusion-exclusion principle, it follows that the cardinality of $\mathcal{A}^n_m$ is given by

$$\#(\mathcal{A}^n_m) \;=\; \sum_{r=1}^{n}(-1)^{r-1} \sum_{0\leq i_1<i_2<\dots i_r\leq n-1} \#(V_{i_1,i_2,\dots i_r}) \;.$$

Together with $\#(L(R,m+1)_n) = q^{\binom{n+1}{m+1}}$, this implies the formula for the rank of $\mathcal{O}(R,m,S,n)$. ∎

In order to illustrate this result we include a picture of the intersection poset $L(\mathcal{A}^4_2)$:



4 – Polynomials in coface operators

We have seen that some classical examples of cochain operations (differential, cup-i products, Steenrod squares) can be represented by polynomials in coface operators. Now we show that this is always the case over a finite field $\mathbb{F}$. First we make precise what we mean by a coface operator:

Definition 2. For $I := (0 \leq i_1 < i_2 < \ldots < i_{n-m} \leq n)$, define the **coface operator** d^I to be the natural transformation

$$d^I : \; \tilde{C}^m(-;R) \longrightarrow \tilde{C}^n(-;R)$$

defined by $d^I(x)(\sigma) := x(d_{i_1} d_{i_2} \cdots d_{i_{n-m}} \sigma)$ for any m-cochain x and n-simplex σ of some space $X_\bullet$. A **monomial in coface operators** is a finite product $d^{I_1} d^{I_2} \cdots d^{I_s}$ with respect to the pointwise multiplication in $\tilde{C}^n(-;R)$ (here, of course, the source dimensions and the target dimensions for all d^{I_j} have to coincide with m and n, respectively). A **polynomial in coface operators** is a finite R-linear combination of monomials. □

It is clear that d^I is a linear transformation, but, in general, it does *not* maps normalized cochains to normalized cochains again. For example, $(d^i x)(s_i \sigma) = x(\sigma)$ need not to vanish. Hence restriction only gives a natural transformation

$$d^I : \; C^m(-;R) \longrightarrow \tilde{C}^n(-;R) \; .$$

It seems to be a combinatorially difficult problem to determine explicitely all polynomials in coface operators that map normalized cochains to normalized cochains again (i.e., the polynomials that give elements in $\mathcal{O}(R, m, R, n)$).

Theorem 2. *For a finite field $\mathbb{F}$, any element in $a \in \tilde{\mathcal{O}}(\mathbb{F}, m, \mathbb{F}, n)$ has a representation as a polynomial in coface operators, i.e. for any $x \in C^m(X;\mathbb{F})$, it holds*

$$\begin{aligned} a(x) \; &= \; c + \sum_I c_I \, d^I(x) \; + \; \sum_{I,J} c_{I,J} \, d^I(x) \cdot d^J(x) \\ &\quad + \sum_{I,J,K} c_{I,J,K} \, d^I(x) \cdot d^J(x) \cdot d^K(x) \; + \; \ldots \; , \end{aligned}$$

where summation takes place over the $(n-m)$-subsets $I, J, K, \ldots$ of $[n]$ and c, c_I, $c_{I,J}, \ldots$ denote coefficients in $\mathbb{F}$. In particular, this also holds true for the ideal $\mathcal{O}(\mathbb{F}, m, \mathbb{F}, n)$ of cochain operations, where we have additionally $c = 0$.

For the proof, we need two lemmas:

Lemma 1. *With respect to the isomorphism*

$$\tilde{\mathcal{O}}(R, m, R, n) = \tilde{C}^n(L(R, m+1)_\bullet; R) = map(L(R, m+1)_n, R) \; ,$$

the element d^I is given by $[\bar{I}]^*$, where $[J]^* \in map(L(R, m+1)_n, R)$ denotes the dual basis of the basis $[J] \in L(R, m+1)_n$, and $\bar{I} \subset [n]$ denotes the complementary subset of $I \subset [n]$.

Proof: We have to evaluate d^I on the fundamental cochain

$$\iota \in C^m(L(R,m+1)_\bullet;R) \ .$$

Here, $L(R,m+1)_m = R\cdot M$ with $M := \{0<1<2<\ldots<m\}$ and ι is given by the obvious isomorphism $\iota\colon r\cdot M \mapsto r$. We get

$$d^I(\iota)([J]) = \iota(d_{i_1}d_{i_2}\cdots d_{i_{n-m}}[J])$$

for any basis element $[J]$ of $L(R,m+1)_n$. Application of d_j to $[J]$ gives 0 if $j\in J$ and $[s_j\,J]$ otherwise, i.e. we delete the 'hole' j in J by subtracting 1 from the numbers $>j$ of J. This shows that for any $(m+1)$-subset J of $[n]$, we have

$$d_{i_1}d_{i_2}\cdots d_{i_{n-m}}[J] = \begin{cases} M & \text{if } I \text{ is the complement of } J \text{ in } [n] \\ 0 & \text{otherwise} \end{cases} \ . \blacksquare$$

Lemma 2. *Let V be a finite dimensional vector space over a finite field $\mathbb{F}$ of order q. Then the evaluation map from the symmetric algebra of the dual space V^* to the algebra $map(V,\mathbb{F})$ is an epimorphism with kernel generated by the Frobenius map, i.e.*

$$Sym(V^*)/\langle l^q - l \mid l\in V^*\rangle = map(V,\mathbb{F}) \ .$$

If $l_1,l_2,\ldots l_d$ is a basis of V^*, then the maps

$$l_{e_1,e_2,\ldots e_d} := l_1^{e_1}\cdot l_2^{e_2}\cdots l_d^{e_d} : \ V\to\mathbb{F}$$

with all $0\le e_k<q$ form a basis of $map(V,\mathbb{F})$.

Proof: We recall the identity $x^q - x = \prod_{a\in\mathbb{F}}(x-a)$ in the polynomial ring $\mathbb{F}[x]$. Thus, the Kronecker function $\delta_0\colon\mathbb{F}\to\mathbb{F}$ is given by the polynomial $x^{q-1}-1$. Let $v_1,v_2,\ldots v_d$ be a basis of V with dual basis $l_1,l_2,\ldots l_d$ of V^*. Then for $v\in V$, the Kronecker function $\delta_v\colon V\to\mathbb{F}$ is given by

$$\delta_v(w) = \prod_{i=1}^{d}\delta_0\Big(l_i(w-v)\Big)$$

which is a polynomial in the l_i of order $\le d(q-1)$. Moreover, the order of each l_i in δ_v is $\le q-1$. The lemma follows as the δ_v for all $v\in V$ form a basis of the vector space $map(V,\mathbb{F})$. $\blacksquare$

Now, the proof of the theorem above is a direct consequence of both lemmas:

Proof of Theorem 2: The cochain operation $a \in \tilde{\mathcal{O}}(\mathbb{F}, m, \mathbb{F}, n)$ corresponds to a function $a\colon L(\mathbb{F}, m+1)_n \to \mathbb{F}$ which by the second lemma is representable by a polynomial in the dual basis $[I]^*$ of the basis $[I]$ of $L(\mathbb{F}, m+1)_n$. By the first lemma, these linear forms are given by coface operators. If the cochain operation is normalized, it has to vanish on the zero vector $0 \in L(\mathbb{F}, m+1)_n$ because it is degenerate, which gives $c = 0$. ∎

Let $a \in \mathcal{O}(\mathbb{F}, m, \mathbb{F}, n)$ and $b \in \mathcal{O}(\mathbb{F}, k, \mathbb{F}, m)$ be cochain operations which are represented by polynomials in coface operators

$$a = \sum_{\mathcal{I}} c_{\mathcal{I}} \prod_{I \in \mathcal{I}} d^I \,, \qquad b = \sum_{\mathcal{J}} c'_{\mathcal{J}} \prod_{J \in \mathcal{J}} d^J \,,$$

where summation is over systems $\mathcal{I}, \mathcal{J}$ of $(n-m)$-subsets I, respectively $(m-k)$-subsets J. Then the composition $a \circ b \in \mathcal{O}(\mathbb{F}, k, \mathbb{F}, n)$ is a poynomial in coface operators given by

$$a \circ b \;=\; \sum_{\mathcal{I}} c_{\mathcal{I}} \prod_{I \in \mathcal{I}} \left(\sum_{\mathcal{J}} c'_{\mathcal{J}} \prod_{J \in \mathcal{J}} d^I \circ d^J \right) .$$

In contrast to the symmetric algebra $Sym(V^*)$, there exists no graduation by algebraic degree on $map(V, \mathbb{F})$ for $\mathbb{F}$ finite but only a filtration. The reason is that the kernel of the evaluation map is a non-homogeneous ideal. For example, it holds $(d^I)^q - d^I = 0$.

Definition 3. Let $\mathbb{F}$ be a finite field and s be a natural number. A cochain operation $a \in \tilde{\mathcal{O}}(\mathbb{F}, m, \mathbb{F}, n)$ is of **filtration** $\leq s$ if it can be represented by a polynomial in coface operators of degree at least s. We denote the subspace of these elements a by $\mathcal{F}^{\leq s}\tilde{\mathcal{O}}(\mathbb{F}, m, \mathbb{F}, n)$ and the intersection with normalized cochain operations by $\mathcal{F}^{\leq s}\mathcal{O}(\mathbb{F}, m, \mathbb{F}, n)$. □

From the definition, we see that the filtration is multiplicative, i.e. the pointwise product gives

$$\mathcal{F}^{\leq s}\tilde{\mathcal{O}}(\mathbb{F}, m, \mathbb{F}, n) \cdot \mathcal{F}^{\leq t}\tilde{\mathcal{O}}(\mathbb{F}, m, \mathbb{F}, n) \subset \mathcal{F}^{\leq s+t}\tilde{\mathcal{O}}(\mathbb{F}, m, \mathbb{F}, n) \;.$$

The filtration can also be given in another way. We recall the definition of cross-effects of maps $f\colon A \to B$ between abelian groups A and B from Eilenberg and MacLane [3] with a small modification concerning an additional term $f(0)$:

$$f^{(n)}\colon \prod_n A \longrightarrow B \;,$$

$$f^{(n)}(a_1, a_2, \ldots, a_n) \;:=\; \sum_{I \subset \{1,2,\ldots n\}} (-1)^{n-\#(I)}\, f\Big(\sum_{i \in I} a_i\Big) \;.$$

The sum over the empty index set is understood to be 0. Thus, $f^{(0)} = f(0)$, $f^{(1)}(a_1) = f(a_1) - f(0)$, $f^{(2)}(a_1, a_2) = f(a_1 + a_2) - f(a_1) - f(a_2) + f(0)$ and so on. It is clear that $f^{(n)}$ is symmetric in its variables and $(f+g)^{(n)} = f^{(n)} + g^{(n)}$. Moreover, the following properties hold:

Lemma 3. *If $n > 0$ and one $a_i = 0$, then $f^{(n)}(a_1, a_2, \ldots, a_n) = 0$. It holds*

$$f(a_1 + a_2 + \ldots + a_n) \; = \; \sum_{k=0}^{n} \left(\sum_{\{i_1, \ldots i_k\} \subset \{1, \ldots, n\}} f^{(k)}(a_{i_1}, a_{i_2}, \ldots, a_{i_k}) \right)$$

and

$$f^{(n)}(a_1 + a'_1, a_2, \ldots, a_n) \; =$$
$$= f^{(n)}(a_1, a_2, \ldots, a_n) + f^{(n)}(a'_1, a_2, \ldots, a_n) - f^{(n+1)}(a_1, a'_1, a_2, \ldots, a_n) \; .$$

In particular, if $f^{(n)} = 0$, then also $f^{(n+1)} = 0$, and $f^{(n+1)} = 0$ is equivalent to the multilinearity of $f^{(n)}$. ∎

As the proof is by straightforward induction we omit it. Thus we can define a **cross-effect filtration** on $map(A, B)$ by

$$\mathcal{F}^{\leq s} map(A, B) \; := \; \left\{ f \in map(A, B) \mid \; f^{(s+1)} = 0 \right\} \; .$$

The definition of cross-effects and of the filtration also make sense for topological abelian groups or simplicial abelian groups. In order to compare both filtrations for cochain operations, we need a lemma.

Lemma 4. *Let A and B be abelian groups and $g\colon A \times A \times \ldots \times A \to B$ be a multilinear map in s variables. Let $f(x) := g(x, x, \ldots, x)$, then the s-cross-effect of f is the symmetrization of g, i.e. it holds*

$$f^{(s)}(x_1, x_2, \ldots, x_s) \; = \; \sum_{\pi} g(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(s)})$$

where the sum runs over all permutations π of $\{1, 2, \ldots, s\}$.

Proof: The definition of the s-cross-effect and the multilinearity of g yield

$$f^{(s)}(x_1, x_2, \ldots, x_s) \; = \sum_{I \subset \{1,2,\ldots,s\}} (-1)^{s - \#I} \sum_{i_1, \ldots, i_s \in I} f(x_{i_1}, x_{i_2}, \ldots, x_{i_s}) \; .$$

Thus every s-tupel $(i_1, i_2, \ldots, i_s)$ occurs in the whole sum as often as there are subsets I with $\{i_1, i_2, \ldots, i_s\} \subset I \subset \{1, 2, \ldots, s\}$. Denote the cardinality of

$\{i_1, i_2, \ldots, i_s\}$ by k and the cardinality of I by i, then the number of these I is given by $\binom{s-k}{i-k}$. Thus we obtain

$$f^{(s)}(x_1, x_2, \ldots, x_s) \;=\; \sum_{i_1, i_2, \ldots, i_s \in \{1,2,\ldots,s\}} c_{i_1, i_2, \ldots, i_s}\; f(x_{i_1}, x_{i_2}, \ldots, x_{i_s}) \;,$$

where

$$c_{i_1, i_2, \ldots, i_s} \;=\; \sum_{k \le i \le s} (-1)^{s-i} \binom{s-k}{i-k} \;=\; (1-1)^{s-k} \;=\; \begin{cases} 0 & \text{if } k < s \\ 1 & \text{if } k = s \end{cases} \;.$$

Hence, only the s-tuples $(i_1, i_2, \ldots, i_s)$ with pairwise different entries contribute to the sum (with coefficient 1). ∎

Proposition 1. *Let $\mathbb{F}$ be a finite field. Then, with respect to the isomorphism $\mathcal{O}(\mathbb{F}, m, \mathbb{F}, n) = mor(L(\mathbb{F}, m+1)_\bullet, L(\mathbb{F}, n+1)_\bullet)$, the filtration of a cochain operation coincides with the cross-effect filtration.*

Proof: For a simplicial abelian group $A_\bullet$, we have $mor(A_\bullet, L(R, n+1)_\bullet) = C^n(A_\bullet; R) \subset map(A_n, R)$. Hence the cross-effects of a map in $mor(L(R,M)_\bullet, L(R,n)_\bullet)$ correspond exactly to the cross-effects of the corresponding map in $map(L(R,m)_n, R)$ using suitable sums of copies of $L(R,m)_\bullet$ for $A_\bullet$. Thus we have to show that for a finite dimensional vector space V over a finite field $\mathbb{F}$, the polynomial filtration in $map(V, \mathbb{F})$ coincides with the cross-effect filtration. For a monomial $f(v) = l_1(v)\, l_2(v) \cdots l_s(v)$ in linear forms $l_i \in Hom(V, \mathbb{F})$, the above lemma shows that

$$f^{(s)}(v_1, v_2, \ldots, v_s) \;=\; \sum_\pi l_1(v_{\pi(1)})\, l_2(v_{\pi(2)}) \cdots\, l_s(v_{\pi(s)}) \;.$$

In particular, $f^{(s)}$ is multilinear and thus $f^{(s+1)} = 0$, showing that a map of polynomial filtration $\le s$ also has cross-effect filtration $\le s$. Furthermore, by Lemma 2, a basis $l_1, l_2, \ldots l_d$ of V^* yields a basis $l_{e_1, e_2, \ldots e_d} := l_1^{e_1} \cdot l_2^{e_2} \cdots l_d^{e_d}$ of $map(V, \mathbb{F})$, where all $0 \le e_k < q$. By definition, $l_{e_1, e_2, \ldots, e_d}$ has polynomial filtration precisely given by $s := \sum_{k=1}^d e_k$. We have to show that also the cross-effect filtration is precisely s, i.e. $l^{(s)}_{e_1, e_2, \ldots, e_d} \ne 0$. Here we have to be careful as the symmetrization does not run over the symmetric group in d letters but in s letters. In particular, for the power l^e of a single linear form l, we get $(l^e)^{(e)}(v_1, v_2, \ldots, v_e) = e!\, l(v_1)\, l(v_2) \cdots\, l(v_e)$. Hence in the symmetrization $l^{(s)}_{e_1, e_2, \ldots, e_d}$, the summand

$$l_1(v_1)\, l_1(v_2) \cdots\, l_1(v_{e_1}) \cdot l_2(v_{e_1+1})\, l_2(v_{e_1+2}) \cdots\, l_2(v_{e_1+e_2}) \cdots$$
$$\cdots\, l_d(v_{s-e_d+1})\, l_d(v_{s-e_d+2}) \cdots\, l_d(v_s)$$

appears exactly $e_1!\,e_2!\cdots e_d!$ times. As $0 \le e_k < q$, this factor is non-zero in $\mathbb{F}$, thus $l^{(s)}_{e_1,e_2,\ldots,e_d} \neq 0$. ∎

In general, this result is not true for other coefficients. As an example consider the map $f\colon \mathbb{Z} \to \mathbb{Z}$ given by $f(n) := \binom{n}{2}$. It has cross-effect filtration 2, but it cannot be represented as a polynomial in linear forms with *integer* coefficients. If $a \in \mathcal{O}(\mathbb{F},m,\mathbb{F},n)$ and $b \in \mathcal{O}(\mathbb{F},k,\mathbb{F},m)$ are cochain operations which are represented by polynomials in coface operators of order s and t, respectively, then the composition $a \circ b \in \mathcal{O}(\mathbb{F},k,\mathbb{F},n)$ is a cochain operation which is represented by a polynomial in coface operators of order $\le st$. Thus, composition behaves multiplicative with repect to the filtration:

$$\circ\colon\; \mathcal{F}^{\le s}\tilde{\mathcal{O}}(\mathbb{F},m,\mathbb{F},n) \times \mathcal{F}^{\le t}\tilde{\mathcal{O}}(\mathbb{F},k,\mathbb{F},m) \longrightarrow \mathcal{F}^{\le st}\tilde{\mathcal{O}}(\mathbb{F},k,\mathbb{F},n)\ .$$

For cochain operations of cross-effect filtration ≤ 1, we have the following result:

Lemma 5. *The cochain operations in $\mathcal{O}(R,m,S,n)$, $n > 0$, of cross-effect filtration ≤ 1 are given by the space of linear cochain operations. It holds*

$$Hom(L(R,m+1)_\bullet, L(S,n+1)_\bullet) \;=\; \begin{cases} Hom(R,S) & \text{iff } n=m \\ d\circ Hom(R,S) & \text{iff } n=m+1 \\ 0 & \text{otherwise} \end{cases}\ .$$

Proof: For $n > 0$, we have $a(0) = 0$ which proves the first statement. The assertion on linear operations follows straightforward using Dold–Kan equivalence [9]

$$\begin{aligned} &Hom(L(R,m+1)_\bullet, L(S,n+1)_\bullet) \;= \\ &\qquad\qquad = \; Hom_{\text{chain}}(l(R,m+1)_*, l(S,n+1)_*)\ . \end{aligned}$$ ∎

Unfortunately, Dold–Kan equivalence cannot be applied to obtain the non-linear maps in $\mathcal{O}(R,m,S,n)$. We will determine the subspaces of higher filtration explicitely in the modified case of cocycle operations (see Section 6).

5 – Representation of primary and higher cohomology operations

We recall the well-known result that unstable primary cohomology operations of type (R,m,S,n) can be identified with the cohomology group

$$A(R,m,S,n) \;:=\; [K(R,m)_\bullet, K(S,n)_\bullet] \;=\; H^n(K(R,m)_\bullet; S)$$

of the Eilenberg–MacLane space $K(R,m)_\bullet$. Here, the connection between $L(R,m)_\bullet$ and $K(R,m)_\bullet$ (standard simplicial model) is given as follows [9]: The differential $d\colon L(R,m)_\bullet \to L(R,m+1)_\bullet$ is a homomorphism of simplicial R-modules with $ker(d) = K(R,m-1)_\bullet$ and also $im(d) = K(R,m)_\bullet$. In other words, we have

$$K(R,m)_n = Z^m(\Delta^n_\bullet;R) = B^m(\Delta^n_\bullet;R) \ .$$

The space $L(R,m)_\bullet$ can also be identified with the simplicial path space $PK(R,m)_\bullet$ and d with the projection map p followed by inclusion. Kristensen [7] gave a representation of stable primary cohomology operations by cochain operations which we generalized in [6] to the unstable case and also for higher order cohomology operations. We recall from [6] the representation result for primary unstable cohomology operations:

Theorem 3. *Any unstable primary cohomology operation α of type (R,m,S,n) is represented by a cochain operation $a \in \mathcal{O}(R,m,S,n)$ such that*

$$d \circ a \circ d = 0 \ .$$

For a cohomology class $\xi \in H^m(X_\bullet;R)$, we have $\alpha(\xi) = a(x) \mod B^n(X_\bullet;S)$, where $x \in Z^m(X_\bullet;R)$ represents ξ. There is an isomorphism

$$A(R,m,S,n) = \frac{\Big\{a \in \mathcal{O}(R,m,S,n) \mid \ d\circ a\circ d = 0\Big\}}{\Big\{a \in \mathcal{O}(R,m,S,n) \mid d\circ a = 0\Big\} + \Big\{a \in \mathcal{O}(R,m,S,n) \mid a \circ d = 0\Big\}} \ . \blacksquare$$

We remark that the condition $d \circ a \circ d = 0$ is equivalent to the existence of a cochain operation $a' \in \mathcal{O}(R,m-1,S,n-1)$ such that

$$a \circ d = d \circ a' \ ,$$

see [6]. Up to sign, the operation a' represents the cohomology suspension $\Omega\alpha$ of α, i.e. the action of α on a suspension of a space. We are mainly interested in the case where the coefficients R and S are a finite field $\mathbb{F}$. From our representation result of cochain operations by polynomials in coface operators, we get

Corollary 1. *Any unstable primary cohomology α operation of type $(\mathbb{F},m,\mathbb{F},n)$ has a representation on the cochain level by a polynomial a in coface operators, i.e. for any cohomology class $\xi \in H^m(X;\mathbb{F})$, we have*

$$\alpha(\xi) = \ a(x) \ \mathrm{mod}\, B^n(X;\mathbb{F}) \ ,$$

where $x \in Z^m(X;\mathbb{F})$ is any cocycle representing ξ. $\blacksquare$

This allows us to define an algebraic filtration also on the set of primary cohomology operations:

Definition 4. Define $\mathcal{F}^{\leq s}A(\mathbb{F},m,\mathbb{F},n)$ to be set of cohomology operations which have a representation by some polynomial in coface operators of filtration $\leq s$. This filtration behaves multiplicative with respect to composition of cohomology operations. □

By the results of Serre and Cartan [13], the structure of

$$A(\mathbb{Z}/p,m,\mathbb{Z}/p,n) = H^n(K(\mathbb{Z}/p,m);\mathbb{Z}/p)$$

is well-known (given by certain polynomials in Steenrod operations). It would be interesting to compute the filtration explicitely for these operations, at least in the stable range $n < 2m$ where it defines a filtration on the Steenrod algebra. Now we come to the representation of higher cohomology operations. We use the construction of these operations by cochain operations which we have proved in [6]. Here we need to consider also the **multi-variable case** of cochain operations. For example, consider natural transformations $\mathcal{O}$ from the product $C^m(-;R)\times C^{m'}(-;R')$ to the product $C^n(-;S)\times C^{n'}(-;S')$. From the representability of cochains, we get

$$\begin{aligned}\mathcal{O} &= mor\Big(L(R,m+1)_\bullet \times L(R',m'+1)_\bullet,\ L(S,n+1)_\bullet \times L(S',n'+1)_\bullet\Big)\\ &= C^n\Big(L(R,m{+}1)_\bullet{\times}\, L(R',m'{+}1)_\bullet;S\Big)\times C^{n'}\Big(L(R,m{+}1)_\bullet{\times}\, L(R',m'{+}1)_\bullet;S'\Big).\end{aligned}$$

Thus the case of several output variables can be reduced to the one variable case. For the input variables, we have to be more careful. We have

$$\begin{aligned}&C^n\Big(L(R,m+1)_\bullet \times L(R',m'+1)_\bullet;S\Big)\ \subset\\ &\qquad\subset\ map\Big(L(R,m+1)_n \times L(R',m'+1)_n,S\Big)\end{aligned}$$

but the normalization condition (cochains have to vanish on degenerate simplices) does not decompose as the degenerate simplices of a product of simplicial sets are not given by pairs of degenerate simplices of the factors. The arrangements which arise in this way are combinatorially more involved. Nevertheless, if all coefficients R, R' and S are a finite field $\mathbb{F}$, it holds

$$\begin{aligned}&map\Big(L(\mathbb{F},m+1)_n \times L(\mathbb{F},m'+1)_n,\mathbb{F}\Big)\ =\\ &\qquad=\ map(L(\mathbb{F},m+1)_n,\mathbb{F})\otimes map(L(\mathbb{F},m'+1)_n,\mathbb{F})\ .\end{aligned}$$

It is straightforward to see that our proof concerning the representability of cochain operations by polynomials of coface operators also works here, giving that any multi-variable cochain operation with coefficients in $\mathbb{F}$ also has a representation as a polynomial in coface operators. The difference from the one variable case is that we have to label the coface operators in order to know to which cochain variable they have to be applied, and that mixed products may occur. Indeed, the cochain formula for the cup-i product in the introduction is a typical example. Clearly, all we have said also holds for more than two variables (as long as the number of variables is finite). We need some notation for multi-variable operations [6]. We consider series $M = (m_k)_{k\geq 0}$ of natural numbers $m_k \in \mathbb{N}$ where $m_k \neq 0$ for only finitely many k. The number m_k counts the number of variables we need in dimension k. For such series M, N, let

$$L_\bullet^M := \prod_{k\geq 0} L(\mathbb{F}, k+1)_\bullet^{m_k}\ , \qquad K_\bullet^M := \prod_{k\geq 0} K(\mathbb{F}, k)_\bullet^{m_k}\ ,$$

hence

$$\begin{aligned} C^M(X_\bullet;\mathbb{F}) &:= \prod_{k\geq 0} C^k(X_\bullet;\mathbb{F})^{m_k} = mor(X_\bullet, L_\bullet^M)\ , \\ Z^M(X_\bullet;\mathbb{F}) &:= \prod_{k\geq 0} Z^k(X_\bullet;\mathbb{F})^{m_k} = mor(X_\bullet, K_\bullet^M)\ , \\ H^M(X_\bullet;\mathbb{F}) &:= \prod_{k\geq 0} H^k(X_\bullet;\mathbb{F})^{m_k} = [X_\bullet, K_\bullet^M]\ , \end{aligned}$$

and

$$\mathcal{O}_M^N := mor(L_\bullet^M, L_\bullet^N) = C^N(L_\bullet^M;\mathbb{F})\ .$$

Be careful with the index shift in $L_\bullet^M$ which we introduced in order to make notation for the representation of multi-variable cochains more convenient. We also need some notation for manipulation of series: for a series $M = (m_k)_{k\geq 0}$ let $M' := (m_{k+1})_{k\geq 0}$, i.e. shift M down by one step. For example, the suspension isomorphism in reduced cohomology reads as

$$\bar{H}^M(\Sigma X_\bullet;\mathbb{F}) = \bar{H}^{M'}(X_\bullet;\mathbb{F})\ .$$

Furthermore, the sum of two series M and N is defined as $M+N := (m_k+n_k)_{k\geq 0}$. In [6] we have showed

Theorem 4. *Let ϕ be an unstable higher cohomology operation of order s which is defined on a subset of $H^M(X_\bullet;\mathbb{F})$ and takes values in $H^N(X_\bullet;\mathbb{F})$ (modulo* *indeteminacy, i.e. values are subsets of $H^N(X_\bullet;\mathbb{F})$). Then ϕ can be represented by a system of cochain operations $a_1, a_2, \dots a_s$ and $a'_1, a'_2, \dots, a'_s$ living in $a_i \in \mathcal{O}^{N_i}_{M_i}$ and $a'_i \in \mathcal{O}^{N'_i}_{M'_i}$ with $M_i := M + N'_1 + N'_2 + \dots + N'_{i-1}$ for some series N_1, $N_2, \dots$, $N_s := N$. In particular, $M_1 = M$. The cochain operations have to satisfy the following system $(*)$ of equations for $1 \le i \le s-1$:*

$$(*):\quad a_i\Big(dx_1,\ dx_2 + a'_1(x_1),\ dx_3 + a'_2(x_1,x_2),\ dx_4 + a'_3(x_1,x_2,x_3),\ \dots \\ \dots,\ dx_i + a'_{i-1}(x_1,x_2,\dots,x_{i-1})\Big) \;=\; da'_i(x_1,x_2,...,x_i)\ .$$

Here, $x_1 \in H^{M'}(-;\mathbb{F})$ and $x_i \in H^{N''_i}(-;\mathbb{F})$ for $i > 1$ denote variables (arbitrary cochains on any space). Equivalently, the system $(*)$ has to be satisfied for the universal cochains $x_1 = \iota_{M'} \in H^{M'}(L^{M'}_\bullet;\mathbb{F})$ and $x_i = \iota_{N''_i} \in H^{N''_i}(L^{N''_i}_\bullet;\mathbb{F})$ for $i > 1$. Conversely, any system of cochain operations $a_1, a_2, \dots a_s$ and $a'_1, a'_2, \dots, a'_s$ satisfying these equations defines an unstable higher cohomology operation. Evaluation of ϕ on a cohomology class $\xi \in H^M(X_\bullet;\mathbb{F})$ with representing cocycle $x_1 \in Z^M(X_\bullet;\mathbb{F})$ is given by the set of cocycles $z \in Z^N(X_\bullet;\mathbb{F})$ modulo coboundaries with

$$z \;=\; a_s(x_1,x_2,\dots,x_s)$$

where the cochains $x_i \in C^{N'_i}(X_\bullet;\mathbb{F})$ for $1 \le i \le s-1$ have to satisfy the system $(**)$ of equations

$$(**):\quad a_i(x_1,x_2,\dots,x_i) = dx_{i+1}\ .\ \blacksquare$$

With some care concerning the non-linear character of these systems of equations, condition $(*)$ can also be formulated in matrix notation

$$a_i \circ \begin{pmatrix} d & 0 & 0 & \dots & 0 \\ a'_1 & d & 0 & \dots & 0 \\ a'_2 & & d & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ a'_{i-1} & & & & d \end{pmatrix} = d \circ a'_i$$

for $1 \le i \le s$. Evaluation $(**)$ is then given by the following set of cocycles:

$$\left\{ a_s(x_1,x_2,\dots,x_s) \;\middle|\; \begin{pmatrix} d & 0 & 0 & \dots & 0 \\ -a_1 & d & 0 & \dots & 0 \\ -a_2 & & d & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ -a_{s-1} & & & & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_s \end{pmatrix} = 0 \right\}.$$

Thus we get:

Corollary 2. *For an unstable higher cohomology operation ϕ of order s with $\mathbb{F}$-coefficients, the representing cochain operations $a_1, a_2, \ldots a_s$ and $a'_1, a'_2, \ldots, a'_s$ in the theorem above are given by polynomials in coface operators. Evaluation of ϕ on a cohomology class ξ is given by application of a_s to the set of solutions of the sytem of algebraic equations $(**)$ for the cocycles $x_1, x_2, \ldots, x_s$.* ∎

6 – Cocycle operations and coordinate arrangements

In [6], we have showed that instead of cochain operations, we can also use cocycle operations in order to construct higher cohomology operations. As we will see here, cocycle operations have some combinatorial advantages compared with cochain operations.

Definition 5. A **cocycle operation** is a natural transformation from $Z^m(-;R)$ to $Z^n(-;S)$. Denote the set of these cocycle operations by $\mathcal{Z}(R, m, S, n)$. □

Using the representability theorem for cocycles [9]

$$Z^n(X_\bullet; R) \;=\; mor(X_\bullet, K(R,n)_\bullet)$$

we get

$$\mathcal{Z}(R,m,S,n) \;=\; mor(K(R,m)_\bullet, K(S,n)_\bullet) \;=\; Z^n(K(R,m)_\bullet; S) \;.$$

For the Eilenberg–MacLane space, we will not use the explicit model $K(R,m)_n = Z^m(\Delta^n_\bullet; R)$ but an equivalent one (see also [2]).

Definition 6. Let $S^m_\bullet := \Delta^m_\bullet / \partial\Delta^m_\bullet$ be the **simplicial sphere** which has exactly two non-degenerate simplices: the base point $*$ in dimension 0 and the top cell σ in dimension m. Hence, for $n < m$, S^m_n only consists of the (degenerate) base point $*$, but for $n \geq m$, we have

$$S^m_n \;:=\; \Big\{ *, \sigma_I \;|\; I := (0 \leq i_1 < i_2 < \ldots < i_{n-m} < n) \Big\} \;,$$

where we have set $\sigma_I := s_{i_{n-m}} \ldots s_{i_2} s_{i_1} \sigma$. In particular, it holds $\# S^m_n = 1 + \binom{n}{m}$. □

Now we construct the Eilenberg–Maclane space by application of the Bousfield–Kan reduced free R-module functor to the simplicial sphere [1]. Form the free simplicial R-modul $R[S^m_\bullet]$ on $S^m_\bullet$ and reduce it by dividing out the simplicial submodule $R[*]$ generated by the base point. Then it holds

$$K(R,m)_\bullet \approx K'(R,m)_\bullet := R[S^m_\bullet]/R[*] \ ,$$

where the isomorphism is given as follows. The set of m-simplices of $R[S^m_\bullet]/R[*]$ is the free R-module on σ, which gives a canonical m-cocycle $z\colon R\cdot\sigma \to R$ by the identity of R. The corresponding map

$$\zeta\colon\ R[S^m_\bullet]/R[*] \to K(R,m)_\bullet$$

is our isomorphism. Hence, ζ is the R-linear extension of the map $\sigma_I \mapsto s_{i_{n-m}} \cdots$ $\ldots\ s_{i_2}s_{i_1}[\{0,1,\ldots m\}]$, with $[\{0,1,\ldots,m\}] \in K(R,m)_m = L(R,m+1)_m = R\cdot[\{0,1,\ldots,m\}]$ denoting the canonical generator. In the following, we denote $K'(\mathbb{F},m)_\bullet$ by $K^m_\bullet$. A basis of K^m_n is given by the simplices σ_I, hence K^m_n has rank $\binom{n}{m}$. The degenerate simplices form an arrangement

$$W_i := im\left(s_i\colon K^m_{n-1} \to K^m_n\right)\ , \qquad 0 \le i \le n-1\ ,$$

of n subspaces $W_0, W_1, \ldots, W_{n-1}$ of rank $\binom{n-1}{m}$. By the fundamental lemma for degenerate simplices, we have

$$W_i\ = \bigoplus_{I:i\in I} \mathbb{F}\cdot\sigma_I\ ,$$

hence the W_i are spanned by basis vectors and the arrangement

$$\overline{\mathcal{A}}^n_m\ :=\ W_0 \cup W_1 \cup \ldots \cup W_{n-1}\ \subset\ K^m_n$$

is a **coordinate arrangement**, i.e. the subspaces forming the arrangement are spanned by subsets of basis vectors of a fixed basis. In the following, we prove for $K^m_\bullet$ and its cochains similar results as for $L(\mathbb{F},m+1)_\bullet$. Note that using the canonical isomorphism ζ, the set of n-cochains on $K^m_\bullet$ can be identified with the natural transformations from $Z^m(-;\mathbb{F})$ to $C^n(-;\mathbb{F})$, and with $mor(K(\mathbb{F},m)_\bullet, L(\mathbb{F},n+1)_\bullet)$.

Theorem 5. *The intersection poset of the coordinate arrangement $\overline{\mathcal{A}}^n_m$ is the poset of subsets of $[n-1]$ truncated above the level $n-m$. The rank function is given by $\binom{n-r}{m}$ where r denotes the cardinality of the subset. The set $C^n(K^m_\bullet;\mathbb{F})$ is an $\mathbb{F}$-vector space of rank*

$$\sum_{r=0}^{n}(-1)^r\binom{n}{r}\,q^{\binom{n-r}{m}}\ .$$

Proof: We have

$$W_{i_1,1_2,\ldots,i_r} := W_{i_1} \cap W_{i_2} \cap \ldots \cap W_{i_r} = \bigoplus_{I:\{i_1,i_2,\ldots,i_r\} \subset I} \mathbb{F} \cdot \sigma_I$$

where I runs over the $(n-m)$-subsets of $\{0,1,\ldots,n-1\}$. Thus, for $1 \le r \le n-m$, the r-fold intersections $W_{i_1,1_2,\ldots,i_r}$ correspond to the r-subsets of $\{0,1,\ldots,n-1\}$, and for $r > n-m$, the intersections are 0. The rank of $W_{i_1,1_2,\ldots,i_r}$ is equal to the number of $(n-m)$-subsets I of $\{0,1,\ldots,n-1\}$ which contain $i_1, i_2, \ldots, i_r$. This number is clearly given by $\binom{n-r}{n-m-r} = \binom{n-r}{m}$. Then the inclusion-exclusion principle gives the rank of $C^n(K^m_\bullet;\mathbb{F})$. ∎

As every cochain on a simplicial subset $Y_\bullet$ of a simplicial set $X_\bullet$ can be extended to a cochain on $X_\bullet$, a cocycle operation in $Z^n(K^m_\bullet;\mathbb{F}) \subset C^n(K^m_\bullet;\mathbb{F})$ has an extension to a cochain operation in $C^n(L(\mathbb{F}, m+1)_\bullet;\mathbb{F})$. Because the isomorphism ζ is linear, this implies that a cocycle operation also has a representation as a polynomial in coface operators. The definition of algebraic filtration $\mathcal{F}^{\le s}$ and cross-effect filtration in $\mathcal{Z}^n_m$ again make sense and coincide as in the case of $\mathcal{O}(\mathbb{F}, m, \mathbb{F}, n)$. Here, in the case of cocycle operations, we are able to determine explicitely a basis and the filtration quotients of $C^n(K^m_\bullet;\mathbb{F})$. We will need a lemma concerning the vanishing ideal of coordinate arrangements.

Lemma 6. *Let V be a finite dimensional vector space over some finite field $\mathbb{F}$ of cardinality q and $\mathcal{A} \subset V$ be a coordinate arrangement with respect to a basis $v_1, v_2, \ldots, v_d$ of V. Denote the dual basis of V^* by $x_1, x_2, \ldots, x_d$. Then a linear combination*

$$f = \sum c_{e_1,e_2,\ldots,e_d}\, x_1^{e_1}\, x_2^{e_2} \cdots x_d^{e_d}$$

of monomials with $e_i \le q-1$ vanishes on $\mathcal{A}$ if and only if each monomial $x_1^{e_1}\, x_2^{e_2} \cdots x_d^{e_d}$ vanishes on $\mathcal{A}$.

Proof: In order to prove the assertion for the coordinate arrangement, it is enough to prove it for a single coordinate subspace. Without loss of generality, we take W to be generated by $v_1, v_2, \ldots, v_h$, with $h \le d$. Decompose the sum as $f = f' + f''$, where f' contains the monomials in $x_1, x_2, \ldots, x_h$, i.e. with $e_i = 0$ for $i > h$. Thus f' is a polynomial function from W to $\mathbb{F}$ expressed in the basis monomials $x_1^{e_1}\, x_2^{e_2} \cdots x_h^{e_h}$ that vanishes on W. As the basis monomials are linearly independent, it follows that $f' = 0$. But any monomial in f'' contains some factor x_i with $i > h$, hence vanishes on W. ∎

We remark that the statement of the lemma is not true in the case of a general (not coordinate) arrangement. Indeed, we have the example $\mathcal{A}_m^{m+1}$ in $V := L(\mathbb{F}, m+1)_{m+1}$, where

$$d = \sum_i (-1)^i d_i \in \mathcal{O}(\mathbb{F}, m, \mathbb{F}, m+1) \subset map(V, \mathbb{F})$$

vanishes on $\mathcal{A}_m^{m+1}$, but not the single monomials d_i. Denote the dual basis of σ_I by $\langle I \rangle \in Hom(K_n^m, \mathbb{F})$. By lemma 2, it holds that a basis of $\tilde{C}^n(K_\bullet^m; \mathbb{F}) = map(K_n^m, \mathbb{F})$ is given by the monomials

$$\langle I_1 \rangle^{e_1} \langle I_2 \rangle^{e_2} \cdots \langle I_s \rangle^{e_s}$$

where $I_1, I_2, ..., I_s$ are pairwise different $(n-m)$-subsets of $[n-1] = \{0, 1, \dots, n-1\}$ and the exponents satisfy $1 \le e_i \le q-1$ with q being the cardinality of $\mathbb{F}$.

Theorem 6. *A basis of $C^n(K_\bullet^m; \mathbb{F})$ is given by the monomials as above $\langle I_1 \rangle^{e_1} \langle I_2 \rangle^{e_2} \cdots \langle I_s \rangle^{e_s}$ that satisfy $I_1 \cap I_2 \cap \ldots \cap I_s = \emptyset$. The rank $d(t)$ of the finite-dimensional $\mathbb{F}$-vectorspace*

$$\left(\mathcal{F}^{\le t} C^n(K_\bullet^m; \mathbb{F})\right) \Big/ \left(\mathcal{F}^{\le t-1} C^n(K_\bullet^m; \mathbb{F})\right)$$

is given by the identity

$$\sum_{t \ge 0} d(t)\, x^t = \sum_{s \ge 0} (x + x^2 + \ldots + x^{q-1})^s \sum_{r=0}^{n} (-1)^r \tbinom{n}{r} \left(\!\tbinom{\binom{n-r}{m}}{s}\!\right).$$

Proof: By the preceding lemma, there is a basis of monomials. Let $w = \sum_{i \in I} c_I\, \sigma_I$ be some vector in the degenerate subspace W_i, where $c_I \in \mathbb{F}$. Then

$$\Big(\langle I_1 \rangle^{e_1} \langle I_2 \rangle^{e_2} \cdots \langle I_s \rangle^{e_s}\Big)(w) = \begin{cases} 0 & \text{if there is some } I_k \text{ with } i \notin I_k \\ c_{I_1} c_{I_2} \cdots c_{I_r} & \text{if all } I_k \text{ satisfy } i \in I_k \end{cases}.$$

Thus the monomial vanishes on W_i if and only if $i \notin \bigcap_k I_k$, and it vanishes on all $W_0, W_1, \dots, W_{n-1}$ if and only if $\bigcap_k I_k = \emptyset$. Denote the number of s-systems of $(n-m)$-subsets $\{I_1, I_2, \dots, I_s\}$ of $[n-1]$ which satisfy $I_1 \cap I_2 \cap \ldots \cap I_s = \emptyset$ by $u(n, n-m, s)$. For any such s-system $\{I_1, I_2, \dots, I_s\}$, we can form $(q-1)^s$ polynomials $\langle I_1 \rangle^{e_1} \langle I_2 \rangle^{e_2} \cdots \langle I_s \rangle^{e_s}$ by choosing exponents between 1 and $q-1$. As we obtain any normalized monomial by suitable s, I_k and e_k $(k = 1, \dots s)$, it follows

$$\sum_{s=0}^{\infty} u(n, n-m, s)\, (q-1)^s = \sum_{r=0}^{n} (-1)^r \tbinom{n}{r} q^{\binom{n-r}{m}}.$$

Of course, we have $u(n, n-m, s) = 0$ for $s > \binom{n}{n-m}$ as the maximal s-system consists of all $(n-m)$-subsets of $[n-1]$. Inserting $q = (q-1)+1$ the right hand side is equal to

$$\sum_{s=0}^{\infty} \sum_{r=0}^{n} (-1)^r \binom{n}{r} \binom{\binom{n-r}{m}}{s} (q-1)^s \ .$$

Now we use the fact that we can choose the finite field $\mathbb{F}$ arbitrary, i.e. q runs through the powers of prime numbers. Thus we can consider $q-1$ as a variable and the equation is an equality in the polynomial ring $\mathbb{Z}[q-1]$ which is satisfied for infinitely many values of $q-1$. Hence we get

$$u(n, n-m, s) \ = \ \sum_{r=0}^{n} (-1)^r \binom{n}{r} \binom{\binom{n-r}{m}}{s} .$$

Let $v(q, t, s)$ be the number of monomials of total degree t for a fixed s-system. It holds

$$v(q, t, s) \ = \ \#\Big\{ (e_1, e_2, \ldots, e_s) \mid 1 \le e_i \le q-1, \ \sum_i e_i = t \Big\}$$

which can be written in a generating series as

$$\sum_{t \ge 0} v(q, t, s)\, x^t \ = \ (x + x^2 + \ldots + x^{q-1})^s \ .$$

Clearly the dimension of the filtration quotient of degree t in the theorem is given by

$$d(t) = \sum_{s \ge 0} u(n, n-m, s)\, v(q, t, s) \ ,$$

which gives the identity for $\sum_{t \ge 0} d(t) x^t$. ∎

Corollary 3. *For $n > t\,m$, it holds*

$$\mathcal{F}^{\le t} C^n(K_\bullet^m; \mathbb{F}) \ = \mathcal{F}^{\le t} \mathcal{Z}(\mathbb{F}, m, \mathbb{F}, n) = 0 \ .$$

In particular, there are no linear cocycle operations for $n > m$, there are no quadratic cocycle operations outside the stable range (i.e., for $n > 2\,m$), there are no tertiary cocycle operations outside the metastable range (i.e., for $n > 3\,m$), and so on.

Proof: By the theorem above, a basis of monomials of filtration $\leq t$ is given by $\langle I_1\rangle^{e_1}\langle I_2\rangle^{e_2}\cdots\langle I_s\rangle^{e_s}$ with $I_1\cap I_2\cap\ldots\cap I_s=\emptyset$ and $\sum_i e_i\leq t$. Since $e_i\geq 1$, it follows $s\leq t$, and the condition of empty intersection is equivalent to $\bar{I}_1\cup\bar{I}_2\cup\ldots\cup\bar{I}_s=[n-1]$, where $\bar{I}_j$ denotes the complement of the $(n-m)$-subset I_j in $[n-1]$. Since $\#[n-1]=n$, it follows $t\cdot m\geq s\cdot m\geq n$. ∎

As the differential d is linear, it does not increase the filtration of a cochain:

$$d\colon\ \mathcal{F}^{\leq t}C^n(K^m_\bullet;\mathbb{F})\to\mathcal{F}^{\leq t}C^{n+1}(K^m_\bullet;\mathbb{F})\ .$$

Hence we have a filtered cochain complex for $K^m_\bullet\approx K(\mathbb{F},m)_\bullet$ and we can form the **associated spectral sequence** [14]

$$E_0^{p,q}=\Big(\mathcal{F}^{\leq p}C^{p+q}(K^m_\bullet;\mathbb{F})\Big)\Big/\Big(\mathcal{F}^{\leq p-1}C^{p+q}(K^m_\bullet;\mathbb{F})\Big)\implies H^{p+q}(K^m_\bullet;\mathbb{F})\ ,$$

which converges as the filtration is bounded (here, q denotes an index and not the cardinality of $\mathbb{F}$, of course). Our theorem above gives the additive structure of the $E_0^{*,*}$-term. As the cup product on the cochain level behaves additively with respect to algebraic filtration of cochain or cocycle operations, and defines the cup product in cohomology, we see that our spectral sequence has a multiplicative structure. The $E_\infty^{*,*}$-term is just the graded algebra associated to the algebraic filtration of cohomology operations which we have defined in Section 5. It would be interesting to understand this spectral sequence from a combinatorial point of view because this could lead to an independent computation of the cohomology of Eilenberg–MacLane spaces.

7 – Tables

We close with some tables of dimensions of filtration quotients, i.e., of the initial term of the spectral sequence in low dimensions for the prime $q=2$ which we computed using the previous formulae. Here, the entries in the first tables are the numbers $rk(E_0^{p,n-p}(K^m_\bullet;\mathbb{Z}/2))$, which are given by

$$\sum_{r=0}^{n}(-1)^r\tbinom{n}{r}\binom{\binom{n-r}{m}}{p}\ .$$

They sum up over p (i.e., vertically) to

$$rk(C^n(K^m_\bullet;\mathbb{Z}/2))=\sum_{r=0}^{n}(-1)^r\tbinom{n}{r}\,2^{\binom{n-r}{m}}\ .$$

The second tables denote $H^*(K^m;\mathbb{Z}/2)$, i.e., the limit term of the spectral sequence, which is known by the classical results of Serre. In the corresponding dimensions we have indicated the additive generators which are formed by certain products of admissible monomials in Steenrod squares applied to the fundamental class. Here, we also indicated the 'naive filtration' of these elements, i.e. the filtration defined by assigning a monomial $Sq^{a_1}Sq^{a_2}...Sq^{a_r}\iota_m$ the filtration 2^r and assigning to a product the sum of filtrations of its factors. This naive filtration clearly may be larger than the filtration defined above, thus the second tables may differ from the E_∞-term of the spectral sequence with respect to the vertical coordinate. It would be interesting to know if there is some (explicit) example where the naive filtration is larger than our filtration. From the tables we see that the numbers $rk(E_0^{p,n-p}(K_\bullet^m;\mathbb{Z}/2))$ grow much faster than $rk(H^*(K_\bullet^m;\mathbb{Z}/2))$, which means that there are a lot of non-trivial differentials in the spectral sequence. The case $m=1$ (with $q=2$, as this is not true for other cefficients than $\mathbb{Z}/2$) is an exception to this phenomenon insofar as there all differentials are zero, $E_0=E_\infty$, because $rk(C^n(K_\bullet^1;\mathbb{Z}/2))=\sum_{r=0}^n(-1)^r\binom{n}{r}2^{\binom{n-r}{1}}=(2-1)^n=1$.

The case $m=1$, $E_0^{**}(K_\bullet^1;\mathbb{Z}/2)$:

6							1
5						1	
4					1		
3				1			
2			1				
1		1					
0	1						
p/n	0	1	2	3	4	5	6

6							ι^6
5						ι^5	
4					ι^4		
3				ι^3			
2			ι^2				
1		ι					
0	1						
p/n	0	1	2	3	4	5	6

The case $m = 2$, $E_0^{**}(K_\bullet^2; \mathbb{Z}/2)$:

p/n	0	1	2	3	4	5	6	7	8
6					1	205	3760	23604	71078
5					6	222	1581	4410	5880
4					15	135	330	315	105
3				1	16	30	15		
2				3	3				
1			1						
0	1								

p/n	0	1	2	3	4	5	6	7	8
6									$Sq^1\iota Sq^2Sq^1\iota$
5								$\iota Sq^2Sq^1\iota$	$\iota(Sq^1\iota)^2$
4						$Sq^2Sq^1\iota$	$(Sq^1\iota)^2$	$\iota^2Sq^1\iota$	ι^4
3						$\iota Sq^1\iota$	ι^3		
2				$Sq^1\iota$	ι^2				
1			ι						
0	1								

The case $m = 3$, $E_0^{**}(K_\bullet^3; \mathbb{Z}/2)$:

p/n	0	1	2	3	4	5	6	7	8
6						210	37500	1356250	20556676
5						252	13992	221396	1642760
4					1	205	3600	22820	72380
3					4	100	480	945	840
2					6	15	10		
1				1					
0	1								

p/n	0	1	2	3	4	5	6	7	8
6									
5									
4							$Sq^2Sq^1\iota$	$Sq^3Sq^1\iota$	$(Sq^1\iota)^2$
3								$\iota Sq^1\iota$	$\iota Sq^2\iota$
2					$Sq^1\iota$	$Sq^2\iota$	ι^2		
1				ι					
0	1								

The case $m = 4$, $E_0^{**}(K_\bullet^4; \mathbb{Z}/2)$:

p/n	0	1	2	3	4	5	6	7	8
6							5005	1588125	$\approx 1.18 \cdot 10^{21}$
5						1	2997	303632	9589986
4						5	1335	42910	535955
3						10	395	3570	14560
2						10	45	70	35
1					1				
0	1								

p/n	0	1	2	3	4	5	6	7	8
6									
5									
4								$Sq^2Sq^1\iota$	$Sq^3Sq^1\iota$
3									
2						$Sq^1\iota$	$Sq^2\iota$	$Sq^3\iota$	ι^2
1					ι				
0	1								

The case $m = 5$, $E_0^{**}(K_\bullet^5; \mathbb{Z}/2)$:

p/n	0	1	2	3	4	5	6	7	8
6							1	54257	32034352
5							6	20307	3657192
4							15	5880	319830
3							20	1190	17640
2							15	105	280
1						1			
0	1								

p/n	0	1	2	3	4	5	6	7	8
6									
5									
4									$Sq^2Sq^1\iota$
3									
2							$Sq^1\iota$	$Sq^2\iota$	$Sq^3\iota$
1						ι			
0	1								

We also note that we can compute from our results the number of simplicial maps beetwen Eilenberg–MacLane spaces (i.e., the number of closed cocycle operations). In fact, the short exact sequences $0 \to Z^n \to C^n \to B^{n+1} \to 0$ and $0 \to B^n \to Z^n \to H^n \to 0$ give

$$\begin{aligned} rk(mor(K_\bullet^m, K_\bullet^n)) &= rk(Z^n(K_\bullet^m)) \\ &= rk(H^n(K_\bullet^m)) + rk(C^{n-1}(K_\bullet^m)) - rk(Z^{n-1}(K_\bullet^m)) \end{aligned}$$

which can be used to recursively compute these numbers. We have

$rk(C^n(K_\bullet^m)) =$

6	0	1	27449	1042642	32596	1
5	0	1	768	958	26	1
4	0	1	41	11	1	0
3	0	1	4	1	0	0
2	0	1	1	0	0	0
1	0	1	0	0	0	0
0	2	1	1	1	1	1
n/m	0	1	2	3	4	5

$rk[K_\bullet^m, K_\bullet^n] =$

6	0	1	2	2	1	1
5	0	1	2	1	1	1
4	0	1	1	1	1	0
3	0	1	1	1	0	0
2	0	1	1	0	0	0
1	0	1	0	0	0	0
0	2	1	1	1	1	1
n/m	0	1	2	3	4	5

which yields

$rk(mor(K_\bullet^m, K_\bullet^n)) =$

6	0	1	731	949	26	1
5	0	1	39	11	1	1
4	0	1	4	1	1	0
3	0	1	1	1	0	0
2	0	1	1	0	0	0
1	0	1	0	0	0	0
0	2	1	1	1	1	1
n/m	0	1	2	3	4	5

We see in particular the 'rigidity phenomenon'

$$\begin{aligned} Z^m(K(R,m)_\bullet;R') &= mor(K(R,m)_\bullet, K(R',m)_\bullet) \\ &= H^m(K(R,m)_\bullet;R') = [K(R,m)_\bullet, K(R',m)_\bullet] = Hom(R,R') \end{aligned}$$

which holds for all $m > 0$ and arbitrary coefficients R, R', and

$$\begin{aligned} C^n(K^1_\bullet;\mathbb{Z}/2) \;=\; Z^n(K^1_\bullet;\mathbb{Z}/2) \;&=\; mor(K^1_\bullet, K^n_\bullet) \\ &=\; H^n(K^1_\bullet;\mathbb{Z}/2) \;=\; [K^1_\bullet, K^n_\bullet] \;=\; \mathbb{Z}/2\cdot\iota^n \end{aligned}$$

which holds for all n and coefficients $\mathbb{Z}/2$. There is a further rigidity phenomenon for the Bockstein operator with coefficients $\mathbb{Z}/2$:

$$\begin{aligned} Z^{m+1}(K^m_\bullet;\mathbb{Z}/2) \;=\; mor(K^m_\bullet, K^{m+1}_\bullet) \;&=\; [K^m_\bullet, K^{m+1}_\bullet] \\ &=\; H^{m+1}(K^m_\bullet;\mathbb{Z}/2) \;=\; \mathbb{Z}/2\cdot Sq^1\iota\ , \end{aligned}$$

which follows from the recursive formula as $rk(C^m(K^m_\bullet)) = rk(Z^m(K^m_\bullet))$.

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