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THE CONVERGENCE APPROACH TO EXPONENTIABLE MAPS

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Dedicated to John Isbell

Abstract: Exponentiable maps in the category **Top** of topological spaces are characterized by an easy ultrafilter-interpolation property, in generalization of a recent result by Pisani for spaces. From this characterization we deduce that perfect (= proper and separated) maps are exponentiable, generalizing the classical result for compact Hausdorff spaces. Furthermore, in generalization of the Whitehead–Michael characterization of locally compact Hausdorff spaces, we characterize exponentiable maps of **Top** between Hausdorff spaces as restrictions of perfect maps to open subspaces.

1 – Introduction

That compact Hausdorff spaces are exponentiable in the category **Top** of topological spaces has been known since at least the 1940s (see Fox [17] and Arens [1]). Our original motivation for writing this paper was to establish the fibred version of this fact which, despite the extensive literature on exponentiability, does not seem to have been treated conclusively in previous articles.

Recall that a *space* X is *exponentiable* if it allows for the natural formation of function spaces Y^X for every other space Y; more precisely, if the functor $(-)\times X\colon \mathbf{Top}\to\mathbf{Top}$ has a right adjoint, which turns out to be equivalent to

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the preservation of quotient maps by $(-)\times X$. Exponentiable spaces were characterized topologically by Day and Kelly [13]. As Isbell [22] observed, their characterization amounts to saying that the lattice of open sets must be continuous; equivalently, these are the core-compact spaces, in the sense that every neighbourhood of a point contains a smaller one with the property that every open cover of the given neighbourhood contains a finite subcover of the smaller one. Generalizing Whitehead's result [41] for Hausdorff spaces, Brown [5] already in 1964 showed that locally compact spaces (in which every point has a base of compact neighbourhoods) are exponentiable. For Hausdorff spaces the two notions become equivalent (Michael [29]), even for sober spaces (Hofmann–Lawson [21]). There is no known constructive example of an exponentiable space that is not locally compact (Isbell [22]). For an elementary account of these results, see [16].

Trading now **Top** for the category **Top**$/Y$ of spaces X over the fixed base space Y, given by continuous maps $f\colon X\to Y$, Niefield [31], [32] gave an elegant but, when put in standard topological terminology, generally complicated topological characterization of *exponentiable maps* in **Top**, which entails the Day-Kelly result in case $Y=1$ is a one-point space. Niefield's result becomes very tractable though when f is a subspace embedding, in which case exponentiability of f means local closedness of X in Y (so that X is open in its closure $\overline{X}$ in Y), and even when f is just an injective map, as was shown by Richter [37]. Under suitable restrictions on X and Y it becomes very applicable as well; for instance, it shows that every map from a locally compact space to a locally Hausdorff space is exponentiable (Niefield [33]). However, it seems to be very cumbersome to derive from it the statement we are aiming for, namely:

Theorem A. *Every perfect map of topological spaces is exponentiable in* **Top**.

Here we call a continuous map $f\colon X\to Y$ *perfect* if it is both

- *stably closed*, so that every pullback of f is a closed map, which is equivalent to f being *proper* in the sense of Bourbaki [4], so that $f\times 1_Z\colon X\times Z\to Y\times Z$ is closed for every space Z;

  and

- *separated*, so that the diagonal Δ_X is closed in the fibred product $X\times_Y X$, which means that any distinct points x, y in X with $f(x)=f(y)$ may be separated by disjoint open neighbourhoods in X.

Thanks to the Kuratowski–Mrowka Theorem, stable closedness of $X \to Y$ for $Y = 1$ means compactness of X , while separatedness obviously amounts to Hausdorffness of X in this case. Categorically it is clear that Theorem A is the "right" map generalization of the space result of the 1940s (see [8]).

Pisani's characterization of exponentiable spaces X in **Top** is based on Barr's presentation [2] of topological spaces as relational algebras (which recently has led to much more general studies of so-called lax algebras, see [10] and [12]), and it reads as follows. Let UX be the set of ultrafilters on X, and for $\mathfrak{U} \in UUX$, let

$$\mu_X(\mathfrak{U}) = \bigcup_{\mathcal{A} \in \mathfrak{U}} \bigcap_{\mathfrak{a} \in \mathcal{A}} \mathfrak{a}$$

be the sum of the ultrafilters $\mathfrak{a}$ ($\mathfrak{a} \in \mathcal{A} \subseteq UX$, $\mathcal{A} \in \mathfrak{U}$); see [19]. Now X is exponentiable if and only if X has the *ultrafilter interpolation property*: whenever $\mu_X(\mathfrak{U}) \to x$ in X, then there is $\mathfrak{a} \in UX$ with $\mathfrak{U} \to \mathfrak{a}$ and $\mathfrak{a} \to x$ (with a naturally defined notion of convergence in UX). For simplicity we often write

$$\mathfrak{U} \Rightarrow x$$

instead of $\mu_X(\mathfrak{U}) \to x$.

It turns out that Pisani's characterization allows for a natural generalization from spaces to maps, which occured to us after seeing the Janelidze–Sobral criterion (see [24] and [7]) for triquotient maps of finite topological spaces in the sense of Michael [30]. Hence, we first looked at the category **PrSet** of *preordered sets* (= reflexive transitive graphs = sets with a reflexive, transitive binary relation $\to$) and monotone maps; here every object is exponentiable, while a map $f\colon X \to Y$ is exponentiable in **PrSet** if and only if it has the following interpolation (or convexity) property:

whenever $u \to x$ in X and $f(u) \to b \to f(x)$ in Y,
then there is a in X with $f(a) = b$ and $u \to a \to x$ in X;

$$\begin{tikzcd} X \arrow[dd, "f"'] & u \arrow[rr] \arrow[dr, dashed] \arrow[dd, dash] & & x \arrow[dd, dash] \\ & & a \arrow[ur, dashed] \arrow[d, dash] & \\ Y & f(u) \arrow[r] & b \arrow[r] & f(x) \end{tikzcd} \tag{1}$$

see the recent papers [34] and [39] which draw on the more general result of Giraud [18] in **Cat**. Writing now $\overset{\bullet}{u} \to x$ instead of $u \to x$ we obtain also a characterization of exponential maps in the (isomorphic) category of Alexandroff topological spaces (where every point has a least open neighbourhood). Now, the characterization in **Top** comes about by just appropriately replacing principal ultrafilters by arbitrary ones:

Theorem B. *A continuous map $f : X \to Y$ is exponentiable in* **Top** *if and only if f has the ultrafilter interpolation property:*

> *whenever $\mathfrak{U} \Rightarrow x$ in X and $f(\mathfrak{U}) \to \mathfrak{b}$ in UY and $\mathfrak{b} \to f(x)$ in Y, then there is $\mathfrak{a} \in UX$ with $f(\mathfrak{a}) = \mathfrak{b}$, $\mathfrak{U} \to \mathfrak{a}$ in UX, and $\mathfrak{a} \to x$ in X.*

$$
\text{(2)} \qquad
\begin{tikzcd}
X \arrow[d, "f"'] \\
Y
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathfrak{U} \arrow[rr, Rightarrow] \arrow[dr, dashed] \arrow[d, dash] & & x \arrow[d, dash] \\
 & \mathfrak{a} \arrow[ur, dashed] \arrow[d, dash] & \\
f(\mathfrak{U}) \arrow[r] & \mathfrak{b} \arrow[r] & f(x)
\end{tikzcd}
$$

The first purpose of this paper is to prove Theorem B and derive Theorem A from it.

While the derivation of Theorem A from B is easy, the proof of Theorem B is quite involved. We employ the approach first developed in [10] and work within the category **URS** whose objects are simply sets provided with an *ultra-relational structure*, i.e., any ("convergence") relation between ultrafilters on X and points in X — no further condition. Within this category, topological spaces are characterized by a reflexivity and transitivity property, just like preordered sets amongst graphs.

In Section 2 we give a summary of the main categorical and filter-theoretic notions and tools used in Section 3, which contains the proofs of Theorems A and B. Section 4 is devoted to a discussion of some of the immediate consequences of these theorems. In particular, we give refined versions and generalizations of the invariance and inverse invariance theorems of local compactness under perfect mappings, as first established by [26] and [40] and recorded in [15].

Finally, coming back to our discussion of exponentiable spaces, we study in Section 5 the map-version of the Whitehead–Michael characterization of exponentiable spaces as locally compact spaces, within the realm of Hausdorff spaces. Since the locally compact Hausdorff spaces are precisely the open subspaces of compact Hausdorff spaces, at the map level one would expect exponentiable separated maps to be characterized as restrictions of perfect maps to open subspaces. We succeeded proving this for maps with Hausdorff codomain:

Theorem C. *For Y a Hausdorff space, the exponentiable, separated maps $f\colon X \to Y$ in* **Top** *are precisely the composites*

$$X \xhookrightarrow{\;i\;} Z \xrightarrow{\;p\;} Y$$

with an open embedding i and a perfect map p.

We conjecture, however, that the assumption on Y may be dropped.

In this paper we neither discuss any of the many localic or topos-theoretic aspects of the theme of this paper, nor do we elaborate here on the presentation of exponentiable spaces as lax Eilenberg-Moore algebras, but refer the Reader to [32], [34] and to [35], respectively.

2 – Preparations

2.1 (*The ultrafilter monad*). The assignment $X \mapsto UX$ defines a functor $U\colon \mathbf{Set} \to \mathbf{Set}$; for a mapping $f\colon X \to Y$ in **Set**, $Uf\colon UX \to UY$ assigns to $\mathfrak{a} \in UX$ the (ultra)filter $f(\mathfrak{a})$, generated by $\{f(A) \mid A \in \mathfrak{a}\}$. This functor preserves coproducts (disjoint unions), and it is terminal with this property: for any coproduct-preserving functor $F\colon \mathbf{Set} \to \mathbf{Set}$ there is a unique natural transformation $F \to U$ (see [3]). Therefore U carries a unique monad structure (which was first discussed in [27]; its Eilenberg–Moore algebras are precisely the compact Hausdorff spaces — see also [28]). Hence, there are natural maps

$$\eta_X\colon X \to UX\,, \qquad \mu_X\colon UUX \to UX$$

satisfying the monad conditions

$$\mu_X \cdot \eta_{UX} = 1_X = \mu_X \cdot U\eta_X\,, \qquad \mu_X \cdot \mu_{UX} = \mu_X \cdot U\mu_X\,.$$

$\eta_X(x) = \dot{x}$ and μ_X defined as in the Introduction. Hence, for $\mathfrak{U} \in UUX$, a typical set in $\mu_X(\mathfrak{U}) \in UX$ has the form

$$\bigcup_{\mathfrak{a} \in \mathcal{A}} A_{\mathfrak{a}}$$

for some $\mathcal{A} \in \mathfrak{U}$ and with all $A_{\mathfrak{a}} \in \mathfrak{a}$; alternatively, a subset $A \subseteq X$ lies in $\mu_X(\mathfrak{U})$ precisely when the set

$$A^\sharp = \Big\{\mathfrak{a} \in UX \mid A \in \mathfrak{a}\Big\}$$

lies in $\mathfrak{U}$.

2.2 (*Extension of U to* Rel(**Set**), see [35]). Let Rel(**Set**) be the category whose objects are sets, while a morphism $\rho\colon X \nrightarrow Y$ is a relation $\rho \subseteq X\times Y$ from X to Y and composition is as usual:

$$\sigma \cdot \rho = \Big\{(x,z) \mid \exists y\colon\ (x,y) \in \rho \ \text{ and } \ (y,z) \in \sigma\Big\}\ .$$

Hence, **Set** is a non-full subcategory of Rel(**Set**). Now U can be extended to a functor $U\colon \mathrm{Rel}(\mathbf{Set}) \to \mathrm{Rel}(\mathbf{Set})$ when for $\rho\colon X \nrightarrow Y$ we define $U\rho\colon UX \nrightarrow UY$ by

$$(\mathfrak{a},\mathfrak{b}) \in U\rho \qquad \text{iff} \quad \rho^{\mathrm{op}}(B) \in \mathfrak{a} \ \text{ for all } \ B \in \mathfrak{b}\ .$$

(For $A \subseteq X$ we write $\rho(A) = \{y \mid \exists x \in A\colon\ (x,y) \in \rho\}$, and $\rho^{\mathrm{op}} \subseteq Y\times X$ is the relation opposite to ρ). Furthermore, if $\rho \subseteq \sigma\colon\ X \nrightarrow Y$, then also $U\rho \subseteq U\sigma$.

2.3 (*Ultrarelational structures, grizzly spaces*). By an ultrarelational structure on a set X we mean a relation $\rho\colon\ UX \nrightarrow X$; we write

$$\mathfrak{a} \xrightarrow{\rho} x \ \text{ or } \ \mathfrak{a} \to x \qquad \text{if } \ (\mathfrak{a},x) \in \rho\ .$$

A map $f\colon\ (X,\rho) \to (Y,\sigma)$ of such (very general) structures is continuous if

$$\mathfrak{a} \xrightarrow{\rho} x \ \text{ in } X \qquad \text{implies} \qquad f(\mathfrak{a}) \xrightarrow{\sigma} f(x) \ \text{ in } Y\ .$$

This defines the category **URS**, the objects of which are also called *grizzly spaces*. The relational extension of U yields for a grizzly space (X,ρ) a grizzly space $(UX, U\rho)$; hence, there is a functor

$$U\colon\ \mathbf{URS} \to \mathbf{URS}$$

(since $f\cdot \rho \subseteq \sigma \cdot Uf$ implies $Uf \cdot U\rho \subseteq U\sigma \cdot UUf$, by 2.2).

Explicitly, the ultrarelational structure of UX is given by

$$\mathfrak{U} \to \mathfrak{a} \ \Leftrightarrow \ \downarrow\! A \in \mathfrak{U} \qquad \text{for all } \ A \in \mathfrak{a}\ ,$$

where $\downarrow\! A = \rho^{\mathrm{op}}(A) = \{\mathfrak{c} \in UX \mid \exists\, x \in A\colon\ \mathfrak{c} \to x\}$. One easily shows:

$$\mathfrak{U} \to \mathfrak{a} \ \Leftrightarrow \ \uparrow\! \mathcal{A} \in \mathfrak{a} \qquad \text{for all } \ \mathcal{A} \in \mathfrak{U}\ ,$$

where $\uparrow\! \mathcal{A} = \rho(\mathcal{A}) = \{x \in X \mid \exists\, \mathfrak{c} \in \mathcal{A}\colon\ \mathfrak{c} \to x\}$.

2.4 (*Topological spaces amongst grizzly spaces*). Via the usual notion of (ultra)filter convergence, the category **Top** is fully embedded into **URS**, and it is essentially known how to recognize topological spaces inside **URS**: *a grizzly space X is topological if and only if*

(**1**) $\eta_X(x) \to x$ for all $x \in X$,

(**2**) whenever $\mathfrak{U} \to \mathfrak{a}$ in UX and $\mathfrak{a} \to x$ in X, then $\mathfrak{U} \Rightarrow x$ in X (that is: $\mu_X(\mathfrak{U}) \to x$ in X).

Proofs of this fact are normally given within the realm of *pseudo-topological spaces* (those $X \in$ **URS** satisfying (1), see [36]) or of *pretopological spaces*, i.e., those $X \in$ **URS** satisfying (1) and

($\mathbf{1\frac{1}{2}}$) $\mathfrak{a} \to x$ whenever $\bigcap_{\mathfrak{b} \to x} \mathfrak{b} \subseteq \mathfrak{a}$ (see [35]).

For a categorical analysis of the first two in the chain of bireflective embeddings

$$\mathbf{Top} \to \mathbf{PrTop} \to \mathbf{PsTop} \to \mathbf{URS}\ ,$$

see also [20].

2.5 (*Prime Filter Theorem*, see [25]). Recall that a *filter* of a 0-1-lattice is an up-closed subset $F \subseteq L$ which is a sub-semilattice of $(L, \wedge, 1)$; it is *prime* if $0 \notin F$, and if $a \vee b \in F$ implies either $a \in F$ or $b \in F$; the lattice-dual notion is (*prime*) *ideal*. Now, if I is an ideal of L and F a filter disjoint from I, then there is a filter U of L which is maximal amongst those containing F and disjoint from I. Moreover, if L is distributive, any such filter U is prime.

2.6 (*Extension Lemma*, see [35]). *Let $\mathfrak{U}$ be an ultrafilter on UX and $\mathfrak{f}$ a filter on a grizzly space X such that $\downarrow F \in \mathfrak{U}$ for all $F \in \mathfrak{f}$. Then there is an ultrafilter $\mathfrak{a}$ on X containing $\mathfrak{f}$ with $\downarrow A \in \mathfrak{U}$ for all $A \in \mathfrak{a}$, hence $\mathfrak{U} \to \mathfrak{a}$ in UX.*

The proof is an application of the Prime Filter Theorem to the ideal $\mathfrak{i} = \{B \subseteq X \mid\ \downarrow B \notin \mathfrak{U}\}$ in the lattice PX of all subsets of X.

2.7 (*Exponentiability of maps via partial products*, see [14]). By definition, a morphism $f\colon X \to Y$ in a finitely-complete category $\mathbf{X}$ is *exponentiable* if the functor "pulling back along f"

$$- \times_Y X\colon \mathbf{X}/Y \longrightarrow \mathbf{X}/Y$$

has a right adjoint. This is equivalent to the existence of the partial products $P = P(f, Z)$, for each object Z in $\mathbf{X}$, which are universally defined by a diagram

(3)
$$\begin{tikzcd} Z & P\times_Y X \arrow[l, "e"'] \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & X \arrow[d, "f"] \\ & P \arrow[r, "p"] & Y \end{tikzcd}$$

such that every diagram

(4)
$$\begin{tikzcd} Z & Q\times_Y X \arrow[l, "d"'] \arrow[r, "\tilde\pi_2"] \arrow[d, "\tilde\pi_1"'] & X \arrow[d, "f"] \\ & Q \arrow[r, "q"] & Y \end{tikzcd}$$

factors as $p\cdot t = q$ and $e\cdot(t\times 1_X) = d$, by a unique morphism $t\colon Q\to P$.

Considering $Q = 1$ the terminal object one sees that, in $\mathbf{X} = \mathbf{URS}$, P should have underlying set

$$P = \Big\{(y,\alpha)\mid y\in Y,\ \alpha\colon f^{-1}y\to Z \text{ continuous}\Big\} \quad \text{with projection } p\colon P\to Y\ ,$$

so that the pullback with f has underlying set

$$P\times_Y X = \Big\{(\alpha,x)\mid x\in X,\ \alpha\colon f^{-1}f(x)\to Z \text{ continuous}\Big\}$$
$$\text{with evaluation map } e\colon P\times_Y X\to Z\ .$$

2.8 (*Canonical structures in* **URS**). **URS** is a topological category over **Set**. Hence, given any morphisms $p\colon P\to Y$, $f\colon X\to Y$ in **URS**, their pullback is formed by providing the set $P\times_Y X$ with the ultrarelational structure given by

$$\mathfrak{c}\to(u,x)\ \ :\Leftrightarrow\ \ \pi_1(\mathfrak{c})\to u \text{ and } \pi_2(\mathfrak{c})\to x\ , \qquad (*)$$

for all $u\in P$, $x\in X$ with $p(u)=f(x)$, and $\mathfrak{c}\in U(P\times_Y X)$.

Suppose now that we are given $f\colon X\to Y$ and Z in **URS** and $p\colon P\to Y$ and $e\colon P\times_Y X\to Z$ in **Set**. We shall call an ultrarelational structure ρ on P *admissible* if it makes both p and e continuous, where of course the structure of $P\times_Y X$ is induced by ρ via $(*)$. The point is that *there always exists a largest (w.r.t. $\subseteq$) admissible ultrarelational structure on P, given by*

$$\mathfrak{b}\to u\ \ :\Leftrightarrow\ \begin{cases} p(\mathfrak{b})\to p(u) \text{ and } e(\mathfrak{c})\to e(u,x) \\ \text{whenever } \mathfrak{c}\in U(P\times_Y X) \text{ and } x\in X \\ \text{with } \pi_1(\mathfrak{c})=\mathfrak{b},\ f(x)=p(u) \text{ and } \pi_2(\mathfrak{c})\to x\ , \end{cases} \qquad (**)$$

for all $u\in P$ and $\mathfrak{b}\in UP$.

2.9 (*Generation of ultrafilters on pullbacks*). Consider the pullback diagram of (3) in **Set** and $\mathfrak{b} \in UP$, $\mathfrak{a} \in UX$ with $p(\mathfrak{b}) = f(\mathfrak{a})$. Then there is an ultrafilter $\mathfrak{c}$ on $P \times_Y X$ with $\pi_1(\mathfrak{c}) = \mathfrak{b}$ and $\pi_2(\mathfrak{c}) = \mathfrak{a}$. Indeed, for all $B \in \mathfrak{b}$ and $A \in \mathfrak{a}$ there is $B' \in \mathfrak{b}$ with $p(B') \subseteq f(A)$ and then $A' \in \mathfrak{a}$ with $f(A \cap A') \subseteq f(A') \subseteq p(B \cap B')$, which shows $B \times_Y A = (B \times A) \cap (P \times_Y X) \neq \emptyset$. Hence, there is an ultrafilter $\mathfrak{c}$ containing the filterbase

$$\mathfrak{b} \times_Y \mathfrak{a} = \Big\{ B \times_Y A \mid B \in \mathfrak{b},\ A \in \mathfrak{a} \Big\}$$

and therefore $\pi_1^{-1}(\mathfrak{b}) \cup \pi_2^{-1}(\mathfrak{a})$, and any such ultrafilter has the desired properties.

2.10 (*Local cartesian closedness of* **PsTop**). For $f\colon X \to Y$ and Z in **PsTop**, one forms the partial product $P = P(f, Z)$ as in 2.7 and provides it with the largest admissible ultrarelational structure as in 2.8. First we make sure that P is a pseudotopological space and show that for $(y, \alpha) \in P$, $\mathfrak{e} := \eta_P(y, \alpha)$ converges to (y, α). By naturality of η, clearly $p(\mathfrak{e}) \to y$ since $Y \in$ **PsTop**. According to $(**)$ we must show $e(\mathfrak{f}) \to \alpha(x)$ whenever $x \in f^{-1}y$ and $\mathfrak{f} \in U(P \times_Y X)$ satisfies $\pi_1(\mathfrak{f}) = \mathfrak{e}$ and $\pi_2(\mathfrak{f}) \to x$. But $\pi_2(\mathfrak{f})$ defines an ultrafilter $\mathfrak{x}$ on $f^{-1}y$ since

$$f(\pi_2(\mathfrak{f})) = p(\pi_1(\mathfrak{f})) = p(\mathfrak{e}) \to y\ ,$$

and we obtain $\alpha(\mathfrak{x}) = e(\mathfrak{f})$. Indeed, for every $A \in \mathfrak{x}$ and $F \in \mathfrak{f}$, the hypotheses on $\mathfrak{f}$ give

$$F \cap \pi_1^{-1}(y, \alpha) \cap \pi_2^{-1}(A) \neq \emptyset\ ,$$

so that there is $a \in A$ with $(a, \alpha) \in F$, hence

$$e(a, \alpha) \in e(F) \cap \alpha(A) \neq \emptyset\ .$$

Now, with the continuity of α we readily conclude from $\mathfrak{x} \to x$

$$e(\mathfrak{f}) = \alpha(\mathfrak{x}) \to \alpha(x) = e(\alpha, x)\ .$$

This concludes the proof of $\mathfrak{e} \to (y, \alpha)$, hence of $P \in$ **PsTop**.

Given diagram (4) in **PsTop**, it remains to be shown that the unique **Set**-map $t\colon Q \to P$ with $p \cdot t = q$ and $e \cdot (t \times 1_X) = d$ is continuous([1]). For that it suffices to see that the final structure ρ on P with respect to the map t is admissible.

([1]) Note that, in order for t to take values in P, one really needs pseudo-topological spaces, and not just grizzly spaces. In fact, **URS** fails to be locally cartesian closed, as erroneously claimed in an early version of this paper.

Hence, let $\mathfrak{b} \xrightarrow{\rho} u$ in P, which means $\mathfrak{d} \to v$ in Q for some $\mathfrak{d}$, v with $t(\mathfrak{d}) = \mathfrak{b}$, $t(v) = u$. We must verify $\mathfrak{b} \to u$ in the sense of $(**)$. Since q is continuous, $p(\mathfrak{b}) = p \cdot t(\mathfrak{d}) = q(\mathfrak{d}) \to q(v) = p(u)$. Let $\mathfrak{c} \in U(P \times_y X)$, $x \in X$ with $\pi_1(\mathfrak{c}) = \mathfrak{b}$, $f(x) = p(u)$ and $\pi_2(\mathfrak{c}) \to x$, and consider the pullback diagram

(5)
$$\begin{tikzcd}
Q \times_Y X \arrow[r, "t \times 1_X"] \arrow[d, "\tilde{\pi}_1"'] & P \times_Y X \arrow[d, "\pi_1"] \\
Q \arrow[r, "t"] & P
\end{tikzcd}$$

Once we have found $\mathfrak{e} \in U(Q \times_Y X)$ with $\tilde{\pi}_1(\mathfrak{e}) = \mathfrak{d}$ and $(t \times 1_X)(\mathfrak{e}) = \mathfrak{c}$, we conclude $\mathfrak{e} \to (v, x)$ in Q (since $\tilde{\pi}_1(\mathfrak{e}) = \mathfrak{d} \to v$ and $\tilde{\pi}_2(\mathfrak{e}) = \tilde{\pi}_2((t \times 1_X)(\mathfrak{e})) = \pi_2(\mathfrak{c}) \to x$), which implies

$$e(\mathfrak{c}) = e\Big((t \times 1_X)(\mathfrak{e})\Big) = d(\mathfrak{e}) \to d(v, x) = e(u, x) \ ,$$

by continuity of d. For the existence of $\mathfrak{e}$, since $\pi_1(\mathfrak{c}) = t(\mathfrak{d})$, one can just use the pullback (5) in **Set** and apply 2.9.

Hence, every morphism in **PsTop** is exponentiable, i.e., **PsTop** is locally cartesian closed.

2.11 (*Coincidence of partial products in* **Top** *and in* **PsTop**, see [6]). For $f\colon X \to Y$ and Z in **Top**, f *exponentiable in* **Top**, one may on the one hand form the partial product $P_{\mathbf{Top}}(f, Z)$ in **Top**, and on the other hand, like for any morphism in **PsTop**, the partial product $P_{\mathbf{PsTop}} = P(f, Z)$ in **PsTop**. But there is no need to distinguish between these two objects: see Theorem 2.1 of [6].

2.12 (*Perfect and open maps in* **URS**). We call a map $f\colon X \to Y$ in **URS**

– *proper* if for all $\mathfrak{a} \in UX$ and $y \in Y$ with $f(\mathfrak{a}) \to y$ there is $x \in X$ with $\mathfrak{a} \to x$ and $f(x) = y$,

(6)
$$\begin{tikzcd}
X \arrow[d, "f"'] & & \mathfrak{a} \arrow[r, dashed] \arrow[d, dash] & x \arrow[d, dash] \\
Y & & f(\mathfrak{a}) \arrow[r] & y
\end{tikzcd}$$

– *separated* if for all $\mathfrak{a} \in UX$ and $x_1,\ x_2 \in X$ with $\mathfrak{a} \to x_1,\ \mathfrak{a} \to x_2$ and $f(x_1) = f(x_2)$ one has $x_1 = x_2$,

– *perfect* if it is proper and separated,

and

– *open* if for all $\mathfrak{b} \in UY$ and $x \in X$ with $\mathfrak{b} \to f(x)$ there is $\mathfrak{a} \in UX$ with $\mathfrak{a} \to x$ and $f(\mathfrak{a}) = \mathfrak{b}$.

For $f\colon X \to Y$ in **Top**, these notions characterize the corresponding properties mentioned in the Introduction in terms of ultrafilter convergence (see [4] and [10]).

3 – The proofs of Theorems A and B

3.1 (*The ultrafilter interpolation property is sufficient for exponentiability in* **Top**). Let $f\colon X \to Y$ and Z be in **Top** and construct their partial product diagram (3) in **URS** as in 2.10. It then suffices to show $P \in$ **Top**, via 2.4. Hence, we consider $u \in P$, $\mathfrak{b} \in UP$, and $\mathfrak{V} \in UUP$ with $\mathfrak{V} \to \mathfrak{b}$ and $\mathfrak{b} \to u$ and must verify $\mathfrak{V} \Rightarrow u$, that is: $\mu_P(\mathfrak{V}) \to u$, using $(**)$ of 2.8.

First, by continuity of p and of Up one has $p(\mathfrak{V}) \to p(\mathfrak{b})$ and $p(\mathfrak{b}) \to p(u)$, hence

$$p(\mu_P(\mathfrak{V})) = \mu_X(p(\mathfrak{V})) \to p(u)$$

by naturality of μ and topologicity of Y.

Next, we consider $\mathfrak{c} \in U(P \times_Y X)$ and $x \in X$ with $\pi_1(\mathfrak{c}) = \mu_P(\mathfrak{V})$, $f(x) = p(u)$ and $\pi_2(\mathfrak{c}) \to x$ and must show $e(\mathfrak{c}) \to e(u, x)$, which we shall do in three steps.

Step 1: We construct $\mathfrak{W} \in UU(P \times_Y X)$ with $\pi_1(\mathfrak{W}) = \mathfrak{V}$ and $\mu_{P\times_Y X}(\mathfrak{W}) = \mathfrak{c}$. For that, for each $C \in \mathfrak{c}$, let

$$C^* := \left\{\mathfrak{d} \in U(P \times_Y X) \mid C \in \mathfrak{d}\right\},$$

and observe that $\{C^* \mid C \in \mathfrak{c}\}$ is a filterbase in $U(P \times_Y X)$. This system may be enlarged by the elements of $(U\pi_1)^{-1}(\mathfrak{V})$. Indeed, for every $C \in \mathfrak{c}$ and $\mathcal{V} \in \mathfrak{V}$, the definition of $\mu_P(\mathfrak{V}) = \pi_1(\mathfrak{c})$ gives $\mathcal{V}' \in \mathfrak{V}$ with $\pi_1(C) \in \mathfrak{v}'$ for all $\mathfrak{v}' \in \mathcal{V}'$. Hence, for any chosen $\mathfrak{v}' \in \mathcal{V} \cap \mathcal{V}'$ we have $\pi_1(C) \in \mathfrak{v}'$ and find an ultrafilter $\mathfrak{d} \supseteq \{C\} \cup \pi_1^{-1}(\mathfrak{v}')$. Then $\mathfrak{d} \in C^* \cap (U\pi_1)^{-1}(\mathcal{V}) \neq \emptyset$. Now any ultrafilter

$$\mathfrak{W} \supseteq \{C^* | \ C \in \mathfrak{c}\} \cup (U\pi_1)^{-1}(\mathfrak{V})$$

has the desired properties.

Step 2: We put $\mathfrak{U} := \pi_2(\mathfrak{W})$ and obtain $\mathfrak{U} \Rightarrow x$ since

$$\mu_X(\mathfrak{U}) = \mu_X(\pi_2(\mathfrak{W})) = \pi_2(\mu_{P\times_Y X}(\mathfrak{W})) = \pi_2(\mathfrak{c}) \to x \ .$$

Furthermore, since $p(\mathfrak{V}) \to p(\mathfrak{b}) \to p(u)$ with

$$f(\mathfrak{U}) = f(\pi_2(\mathfrak{W})) = p(\pi_1(\mathfrak{W})) = p(\mathfrak{V}) \quad \text{ and } \ f(x) = p(u) \ ,$$

the ultrafilter interpolation property of f gives $\mathfrak{a} \in UX$ with $f(\mathfrak{a}) = p(\mathfrak{b})$ and $\mathfrak{U} \to \mathfrak{a} \to x$.

Step 3: We construct $\mathfrak{d} \in U(P \times_Y X)$ with $\mathfrak{W} \to \mathfrak{d}$ and $\pi_1(\mathfrak{d}) = \mathfrak{b}$, $\pi_2(\mathfrak{d}) = \mathfrak{a}$. Indeed, since $\mathfrak{V} \to \mathfrak{b}$ and $\mathfrak{U} \to \mathfrak{a}$, with $\pi_1(\mathfrak{W}) = \mathfrak{V}$ and $\pi_2(\mathfrak{W}) = \mathfrak{U}$ one obtains

$$\downarrow \left(\pi_1^{-1}(B) \cap \pi_2^{-1}(A)\right) = \pi_1^{-1}(\downarrow B) \cap \pi_2^{-1}(\downarrow A)) \ \in \mathfrak{W}$$

for all $B \in \mathfrak{b}$ and $A \in \mathfrak{a}$. Hence, an application of the Extension Lemma 2.6 to the filter generated by the sets $\pi_1^{-1}(B) \cap \pi_2^{-1}(A)$ gives an ultrafilter $\mathfrak{d}$ with the desired properties.

Finally, since $\mathfrak{b} \to u$ and $\mathfrak{a} \to x$, we have $\mathfrak{d} \to (u, x)$, hence $e(\mathfrak{W}) \to e(\mathfrak{d}) \to e(u, x)$ and $e(\mathfrak{W}) \Rightarrow e(u, x)$ in the topological space Z. Consequently,

$$e(\mathfrak{c}) = e(\mu_{P\times_Y X}(\mathfrak{W})) = \mu_Z(e(\mathfrak{W})) \to e(u, x) \ ,$$

which finishes the proof of the "if" part of Theorem B.

3.2 Proposition *(Preservation of properness by U). For every proper map $f\colon X \to Y$ in* **URS**, *also $Uf\colon UX \to UY$ is proper.*

Proof: For $\mathfrak{U} \in UUX$ and $\mathfrak{b} \in UY$ with $f(\mathfrak{U}) \to \mathfrak{b}$ we must find $\mathfrak{a} \in UX$ with $\mathfrak{U} \to \mathfrak{a}$ and $f(\mathfrak{a}) = \mathfrak{b}$. By the Extension Lemma it would suffice to show $\downarrow f^{-1}(B) \in \mathfrak{U}$ for all $B \in \mathfrak{b}$. In fact, the set $\downarrow f^{-1}(B)$ intersects each $\mathcal{U} \in \mathfrak{U}$: since $f(\mathfrak{U}) \to \mathfrak{b}$ we have $\downarrow B \cap f(\mathcal{U}) \neq \emptyset$, so that there are $\mathfrak{a} \in \mathcal{U}$, $y \in B$ with $f(\mathfrak{a}) \to y$; by hypothesis, then there is $x \in f^{-1}y$ with $\mathfrak{a} \to x$, hence $\mathfrak{a} \in \mathcal{U} \cap \downarrow f^{-1}(B)$. ∎

3.3 (*Perfect maps in* **Top** *satisfy the ultrafilter interpolation property*). Let $f\colon X \to Y$ in **URS** with X in **Top** be perfect, and consider $\mathfrak{U} \in UUX$, $\mathfrak{b} \in UY$, $x \in X$ with $\mathfrak{U} \Rightarrow x$ and $f(\mathfrak{U}) \to \mathfrak{b} \to f(x)$. Since Uf is proper by 3.2, there is $\mathfrak{a} \in UX$ with $\mathfrak{U} \to \mathfrak{a}$ and $f(\mathfrak{a}) = \mathfrak{b}$, and since f is proper, there is $x' \in X$ with $\mathfrak{a} \to x'$ and $f(x') = f(x)$. Topologicity of X shows $\mathfrak{U} \Rightarrow x'$, hence $\mu_X(\mathfrak{U}) \to x'$ and, by hypothesis, $\mu_X(\mathfrak{U}) \to x$. Since f is separated, $x = x'$ follows.

Hence, Theorem A follows from (the "if" part of) Theorem B.

3.4 (*The ultrafilter interpolation property is necessary for exponentiability in* **Top**). Let $f\colon X \to Y$ be exponentiable in **Top**, and consider $\mathfrak{U} \in UUX$, $\mathfrak{v} \in UY$, $x_0 \in X$ with $\mathfrak{U} \Rightarrow x_0$ and $f(\mathfrak{U}) \to \mathfrak{v} \to f(x_0) =: y_0$. We must find $\mathfrak{a}_0 \in UX$ with $f(\mathfrak{a}_0) = \mathfrak{v}$ and $\mathfrak{U} \to \mathfrak{a}_0 \to x_0$.

Step 1: With $Z = \$ = \{0 \to 1\}$ the Sierpiński space, we form the partial product $P = P(f,\$)$ in **URS** as in 2.10. Since f is exponentiable in **Top**, by 2.11 we have $P \in$ **Top**.

Step 2: Our first goal is now to find $\mathfrak{b} \in UP$ and α_0 such that $\mathfrak{b} \to (y_0, \alpha_0)$ in P with $p(\mathfrak{b}) = \mathfrak{v}$. To this end, for all $\mathcal{A} \in \mathfrak{U}$ and $V \in \mathfrak{v}$, let

$$B(V,\mathcal{A}) := \Big\{(y,\alpha) \in P \mid\ y \in V \wedge \forall x \in f^{-1}y :\ \Big(\alpha(x) = 1 \ \Rightarrow\ x \in {\uparrow}\mathcal{A}\Big)\Big\}\ .$$

These sets form a filterbase on P, since

$$B(V,\mathcal{A}) \cap B(V',\mathcal{A}') \ \supseteq\ B(V \cap V', \mathcal{A} \cap \mathcal{A}')\ .$$

Hence, we can choose an ultrafilter $\mathfrak{b}$ containing them, which necessarily must satisfy $p(\mathfrak{b}) = \mathfrak{v}$.

Having any such $\mathfrak{b}$ we may define $\alpha_0\colon f^{-1}y_0 \to \$$ by

$$\alpha_0(x) = 1 \quad :\Leftrightarrow \quad \exists\, \mathfrak{a} \in UX:\ \ \mathfrak{a} \to x,\ f(\mathfrak{a}) = \mathfrak{v},\ e(\mathfrak{b} \times_Y \mathfrak{a}) \subseteq \dot{1}\ \ ,$$

where $\mathfrak{b} \times_Y \mathfrak{a}$ is as in 2.9, and $\dot{1} = \eta_S(1)$. In other words, $\alpha_0(x) = 1$ holds true precisely when $x \in f^{-1}y_0$ is an adherence point in X of the filter generated by

$$f^{-1}(\mathfrak{v}) \cup \pi_2\Big(\pi_1^{-1}(\mathfrak{b}) \cup e^{-1}(\,\dot{1}\,)\Big)\ .$$

Since this is a closed set in X, α_0 is continuous. Hence, $(y_0, \alpha_0) \in P$.

We must show $\mathfrak{b} \to (y_0, \alpha_0)$, using $(**)$ of 2.8. Since $p(\mathfrak{b}) = \mathfrak{v} \to y_0$ by hypothesis, we consider $\mathfrak{c} \in U(P \times_Y X)$, $x \in X$ with $\pi_1(\mathfrak{c}) = \mathfrak{b}$, $f(x) = y_0$ and $\mathfrak{a} := \pi_2(\mathfrak{c}) \to x$, hence $f(\mathfrak{a}) = p(\mathfrak{b}) = \mathfrak{v}$. If $e(\mathfrak{b} \times_Y \mathfrak{a}) \subseteq \dot{1}$, then $\alpha_0(x) = 1$, and trivially $e(\mathfrak{c}) \to 1$; if $e(\mathfrak{b} \times_Y \mathfrak{a}) \subseteq \dot{0}$, then $e(\mathfrak{c}) = \dot{0}$ and $e(\mathfrak{c}) \to 0$ and $e(\mathfrak{c}) \to 1$. Hence, always $e(\mathfrak{c}) \to e(\alpha_0, x)$.

Step 3: For any $\mathfrak{a} \in UX$ such that there is $x \in f^{-1}y_0$ with $\mathfrak{a} \to x$, $f(\mathfrak{a}) = \mathfrak{v}$ and $e(\mathfrak{b} \times_Y \mathfrak{a}) \subseteq \dot{1}$, we shall show that $\mathfrak{U} \to \mathfrak{a}$. For that is suffices to verify that each $A \in \mathfrak{a}$ intersects all sets ${\uparrow}\mathcal{A}$, $\mathcal{A} \in \mathfrak{U}$ (see 2.3). Indeed, by hypothesis one has

$$1 \ \in\ e\Big(B(Y,\mathcal{A}) \times_Y A\Big)\ ,$$

so that there is $x \in A$ and $(y,\alpha) \in B(Y,\mathcal{A})$ with $f(y)=x$, $\alpha(x)=1$, where the latter equation means $x \in \uparrow\mathcal{A}$ by definition of $B(Y,\mathcal{A})$. Hence, $A \cap \uparrow\mathcal{A} \neq \emptyset$.

To complete the proof of Theorem B, it would now suffice to show $\alpha_0(x_0)=1$, by definition of α_0. This would be accomplished once we have found $\mathfrak{V} \in UUP$ with

$$(\diamond) \qquad p(\mathfrak{V}) = f(\mathfrak{U})\,, \qquad \mathfrak{V} \to \mathfrak{b} \quad \text{and} \quad e(\mu_P(\mathfrak{V}) \times_Y \mu_X(\mathfrak{U})) \subseteq \dot{1}\ .$$

Indeed, since $p(\mu_P(\mathfrak{V})) = f(\mu_X(\mathfrak{U}))$ one would then have an ultrafilter $\mathfrak{d} \supseteq \mu_P(\mathfrak{V}) \times_Y \mu_X(\mathfrak{U})$ with $\mathfrak{d} \to (\alpha_0, x_0)$ in $P \times_Y X$, since $\pi_1(\mathfrak{d}) = \mu_P(\mathfrak{V}) \to (y_0,\alpha_0)$ by topologicity of P, and since $\pi_2(\mathfrak{d}) = \mu_X(\mathfrak{U}) \to x_0$ by hypothesis. Hence $e(\mathfrak{d}) \to \alpha_0(x_0)$; but $e(\mathfrak{d}) = \dot{1}$, hence $\alpha_0(x_0) \neq 0$.

Step 4: In order to obtain $\mathfrak{V}$ as in $(\diamond)$ we construct $\mathfrak{W} \in UU(P \times_Y X)$ with

$$(\diamond\diamond) \qquad \begin{gathered} \pi_2(\mathfrak{W}) = \mathfrak{U}\,, \qquad \pi_1^{-1}(\downarrow B(V,\mathcal{A})) \in \mathfrak{W} \quad \text{for all } V \in \mathfrak{v} \text{ and } \mathcal{A} \in \mathfrak{U}\,, \\ \text{and} \quad \left\{ \mathfrak{d} \in U(P \times_Y X) \mid e^{-1}1 \in \mathfrak{d} \right\} \in \mathfrak{W}\ . \end{gathered}$$

One can then put $\mathfrak{V} := \pi_1(\mathfrak{W})$ and has $p(\mathfrak{V}) = f(\pi_2(\mathfrak{W})) = f(\mathfrak{U})$. Since $\downarrow B(V,\mathcal{A}) \in \mathfrak{V}$ for all $V \in \mathfrak{v}$, $\mathcal{A} \in \mathfrak{U}$, by 2.6 we can modify our choice of $\mathfrak{b}$ in Step 2 such that $\downarrow B \in \mathfrak{V}$ for all $B \in \mathfrak{b}$, hence $\mathfrak{V} \to \mathfrak{b}$. Finally, since $\mu_{P\times_Y X}(\mathfrak{W}) \supseteq \mu_P(\mathfrak{V}) \times_Y \mu_X(\mathfrak{U})$, and since $\mathcal{C} \in \mathfrak{W}$ with $\mathcal{C} := \{\mathfrak{d} \mid e^{-1}1 \in \mathfrak{d}\}$ gives $e^{-1}1 \in \mu_{P\times_Y X}(\mathfrak{W})$, also $e(\mu_P(\mathfrak{V}) \times_Y \mu_X(\mathfrak{U})) \subseteq \dot{1}$ holds true.

Hence we are left with having to find $\mathfrak{W}$ satisfying the conditions $(\diamond\diamond)$. For that, it suffices to show that for all $\mathcal{A}, \mathcal{B} \in \mathfrak{U}$ and $V \in \mathfrak{v}$ the intersection

$$\pi_2^{-1}(\mathcal{B}) \cap \pi_1^{-1}(\downarrow B(V,\mathcal{A})) \cap \mathcal{C}$$

is not empty; hence, we must find $\mathfrak{d} \in U(P \times_Y X)$ with $\pi_2(\mathfrak{d}) \in \mathcal{B}$, $\pi_1(\mathfrak{d}) \to (y,\alpha)$ for some $(y,\alpha) \in B(V,\mathcal{A})$, and $e^{-1}1 \in \mathfrak{d}$.

To this end, we first note that, since $f(\mathfrak{U}) \to \mathfrak{v}$ we have $\downarrow V \in f(\mathfrak{U})$ and therefore $f(\mathcal{B} \cap \mathcal{A}) \cap \downarrow V \neq \emptyset$, which means that there are $\mathfrak{a} \in \mathcal{B} \cap \mathcal{A}$ and $y \in V$ with $f(\mathfrak{a}) \to y$. Let now $\mathfrak{a}^*$ be the filter on P generated by the sets $A^* = \{(f(a), \chi_a) \mid a \in A\}$, with $\chi_a(x)=1$ if and only if $x \in \mathrm{cl}\{a\}$. Then $p(\mathfrak{a}^*) = f(\mathfrak{a})$. We claim that

$$\mathfrak{a}^* \to (y, \gamma_{\mathfrak{a}}) \in B(V,\mathcal{A})\ ,$$

with $\gamma_{\mathfrak{a}}(x) = 1$ being defined by ($\gamma_a(x) = 1$ if and only if $\mathfrak{a} \to x$).

To prove that $\mathfrak{a}^* \to (y,\gamma_{\mathfrak{a}})$ we use condition $(**)$ of 2.8: $p(\mathfrak{a}^*) = f(\mathfrak{a}) \to y$ holds true; for $\mathfrak{c} \in U(P \times_Y X)$, $x \in X$ with $\pi_1(\mathfrak{c}) = \mathfrak{a}^*$, $f(x) = p(y,\gamma_{\mathfrak{a}})$ and

$\pi_2(\mathfrak{c}) \to x$, we need to check that $e(\mathfrak{c}) \to e(\gamma_{\mathfrak{a}}, x)$. If $e(\mathfrak{c}) = \dot{0}$ then trivially $e(\mathfrak{c}) \to e(\gamma_{\mathfrak{a}}, x)$, since $\dot{0}$ converges to both 0 and 1. Assume now that $e(\mathfrak{c}) = \dot{1}$. Since $\pi_2(\mathfrak{c}) \to x$ (hence $\pi_2(\mathfrak{c})$ contains the filter of neighbourhoods $\Omega(x)$ of x), $\pi_1(\mathfrak{c}) = \mathfrak{a}^*$ and $\mathfrak{c} \supseteq \pi_1(\mathfrak{c}) \times_Y \pi_2(\mathfrak{c})$, then, for all $O \in \Omega(x)$ and for all $A \in \mathfrak{a}$, $O \times_Y A^* \in \mathfrak{c}$. Therefore $1 \in e(O \times_Y A^*)$, which means that there exist $x' \in O$ and $a \in A$ such that $\chi_a(x') = 1$, i.e. $x' \in \mathrm{cl}\{a\}$, which implies that $a \in O$ and, consequently, $O \cap A \neq \emptyset$. This means that $\mathfrak{a} \to x$, hence $e(\mathfrak{c}) \to e(\gamma_a, x) = 1$.

Now we can finish the proof by noting that $(\mathfrak{a}^* \times_Y \mathfrak{a}) \cup e^{-1}1$ is a filterbase, since for all $A, B \in \mathfrak{a}$ and $a \in A \cap B$, one has $(\chi_a, a) \in A^* \times_Y B$ with $e(\chi_a, a) = \chi_a(a) = 1$. Any ultrafilter $\mathfrak{d}$ containing this base has the desired properties. ∎

4 – Invariance of local compactness under perfect maps

It is well known that, for a perfect surjective map $f\colon X \to Y$ with X Hausdorff, also Y is Hausdorff, and that in this case X is locally compact if and only if Y is locally compact (see [15]). Here we show that the separation conditions on X and Y can be relaxed considerably:

4.1 Proposition. *Let $f\colon X \to Y$ in* **Top** *be proper.*

(1) *If X is locally compact, Y sober, and f surjective, then Y is locally compact.*

(2) *If Y is locally compact, X sober, and f separated, then X is locally compact.*

In fact, in conjunction with Theorem A and the Hofmann–Lawson result [21], these assertions follow from statements (1), (2) of the following Proposition, which in turn follow from statements (3), (4):

4.2 Proposition. *Let $f\colon X \to Y$ and $g\colon Y \to Z$ be in* **Top**.

(1) *If X is exponentiable and if f is proper and surjective, then also Y is exponentiable.*

(2) *If Y and f are exponentiable, so is X.*

(3) *If $g \cdot f$ is exponentiable and if f is proper and surjective, then g is exponentiable.*

(4) *If f and g are exponentiable, so is $g \cdot f$.*

Proof: (3) We use Theorem B and 2.12, and consider $\mathfrak{V} \in UUY$, $\mathfrak{c} \in UZ$ and $y \in Y$ with $\mathfrak{V} \Rightarrow y$ and $g(\mathfrak{V}) \to \mathfrak{c} \to g(y)$. Since f is surjective, there is an ultrafilter $\mathfrak{U} \in UUX$ with $f(\mathfrak{U}) = \mathfrak{V}$, and since f is proper and $f(\mu_X(\mathfrak{U})) = \mu_Y(\mathfrak{V})$, there is $x \in X$ with $\mu_X(\mathfrak{U}) \to x$ and $f(x) = y$. Now exponentiability of $g \cdot f$ gives $\mathfrak{a} \in UX$ with $g(f(\mathfrak{a})) = \mathfrak{c}$ and $\mathfrak{U} \to \mathfrak{a} \to x$, which implies $\mathfrak{V} \to f(\mathfrak{a}) \to y$.

(4) is well known (and trivial), see [31]. ∎

Remark. Proper surjective maps are biquotient maps, i.e., pullback-stable quotient maps (see [29]). As was noted by the anonymous referee (as well as in the recent paper [9]), statements (1) and (3) of 4.2 can be generalized considerably by trading "proper and surjective" for "biquotient". The proof of this generalization is in fact purely categorical if one uses the well-known fact (see [31]) that a map $f\colon X \to Y$ is exponentiable in **Top** if and only if the pullback $X \times_Y Z \to X$ along f of any quotient map $Z \to Y$ is again a quotient map. □

For the sake of completeness, we list here some further rules which, unlike 4.2 (1), (3), can be obtained purely categorically, just using the fact that the class of exponentiable morphisms contains all isomorphisms, is closed under composition and stable under pullback. Recall that a space X is *locally Hausdorff* (cf. [33]) if the diagonal Δ_X is locally closed in $X \times X$; more generally, a map $f\colon X \to Y$ is *locally separated* if the diagonal Δ_X is locally closed in $X \times_Y X$, which simply means that every point in X has a neighbourhood U such that $f_{|U}$ is separated. Equivalently: the diagonal map $X \to X \times_Y X$ is exponentiable. Note that local Hausdorffness implies soberness.

4.3 Proposition. *Let $f\colon X \to Y$, $g\colon Y \to Z$ and $p\colon P \to Y$ be in* **Top**.

(**1**) *If X is exponentiable and Y is locally Hausdorff, then f is exponentiable (see [33]).*

(**2**) *If $g \cdot f$ is exponentiable and g locally separated, then f is exponentiable.*

(**3**) *If f and P are exponentiable, so is $P \times_Y X$; in particular, the fibres $f^{-1}y$ ($y \in Y$) of the exponentiable map f are exponentiable spaces.*

(**4**) *The full subcategory of exponentiable and locally Hausdorff spaces in* **Top** *is closed under finite limits. It is contained in the full subcategory of sober locally compact spaces.*

Proof: (1) Factor f (in any finitely-complete category) as

$$X \xrightarrow{\langle 1_X, f\rangle} X \times Y \xrightarrow{\quad p_Y \quad} Y$$

where both factors are exponentiable (see [8], [38]).

(2) Apply the categorical version of (1) to $\mathbf{Top}/Z$ in lieu of $\mathbf{Top}$.

(3), (4) follow from [38], Corollary 3.4(3) and Proposition 3.6, respectively. ∎

We also mention that Theorem B as well as Proposition 4.3(1) make it easy to provide:

4.4 (*Example of an exponentiable map which is not proper*). While every finite space is compact, locally compact and exponentiable, exponentiable maps between finite spaces have obviously (locally) compact fibres, but may fail to be closed, hence they may fail to be proper: simply consider $X = \{a \to b,\ a \to b',\ b' \to c\}$, $Y = \{0 \to 1 \to 2\}$, and $f\colon X \to Y$ with $f(a) = 0$, $f(b) = f(b') = 1$, and $f(c) = 2$. Then f is exponentiable but not proper.

5 – Characterization of separated exponentiable maps

5.1 In what follows, we freely restrict and extend ultrafilters along subsets without change of notation, just forming inverse images and images along inclusion maps. Hence, for a subset $Z \subseteq X$ and $\mathfrak{a} \in UX$ with $Z \in \mathfrak{a}$, we regard $\mathfrak{a}$ also as an ultrafilter on Z; and any $\mathfrak{b} \in UZ$ is also regarded as an ultrafilter on X.

We will also use the idempotent hull cl^∞ of the natural closure cl in **URS** defined by

$$\mathrm{cl}\, A = \Big\{x \in X \mid\ x \in A \ \text{ or } \ \exists\, \mathfrak{a} \in UX\colon\ (A \in \mathfrak{a} \ \wedge\ \mathfrak{a} \to x)\Big\}\ ,$$

for every subset A of X. Thus $\mathrm{cl}^\infty(A)$ is the least subset of X containing A as well as every limit point of an ultrafilter to which it belongs.

5.2 (*Factorization in* **URS**). Let $f\colon X \to Y$ be a continuous map of grizzly spaces, and let

$$Y_0 := \Big\{y \in Y \mid \exists\, \mathfrak{a} \in UX\colon\ \Big(f(\mathfrak{a}) \to y\ \wedge\ \nexists\, x \in f^{-1}y\colon\ \mathfrak{a} \to x\Big)\Big\}\ .$$

With X^* the (disjoint) union of X and $Y_1 := \mathrm{cl}^\infty Y_0$, one obtains a factorization

$$\begin{tikzcd} X \arrow[rr, hook, "i"] \arrow[dr, "f"'] & & X^* \arrow[dl, "p"] \\ & Y & \end{tikzcd} \tag{7}$$

where p maps points of Y_1 identically. The maps i and p become continuous if we make X^* a grizzly space by declaring $\mathfrak{a} \to z$ in X^* whenever one of the following cases applies:

1. $X \in \mathfrak{a}$, $z \in X$, and $\mathfrak{a} \to z$ in X;

2. $X \in \mathfrak{a}$, $z \in Y_1$, $f(\mathfrak{a}) \to z$ in Y, and $\nexists x \in f^{-1}z$ with $\mathfrak{a} \to x$ in X;

3. $Y_1 \in \mathfrak{a}$, $z \in Y_1$, and $\mathfrak{a} \to z$ in Y_1 (as a subspace of Y).

5.3 Proposition.

(1) i is an open cl^∞-dense embedding.

(2) p is proper.

(3) With f also p is separated.

Proof: (1) If $\mathfrak{a} \to x$ in X^* with $x \in X$, then necessarily $X \in \mathfrak{a}$, and we have $\mathfrak{a} \to x$ in X. Moreover, in X^*, $\mathrm{cl}(X) = X \cup Y_0$, hence $\mathrm{cl}^\infty(X) = X^*$.

(2) For $\mathfrak{a} \in UX^*$, suppose $p(\mathfrak{a}) \to y$ in Y, and let first $X \in \mathfrak{a}$. If there is no $x \in f^{-1}y$ with $\mathfrak{a} \to x$ in X, then $y \in Y_0$ (since $f(\mathfrak{a}) = p(\mathfrak{a})$), and one has $\mathfrak{a} \to y$ in X^*. In case $Y_1 \in \mathfrak{a}$ we have $\mathfrak{a} \to y$ in Y (since p maps Y_1 identically), hence $y \in \mathrm{cl}\, Y_1 = Y_1$; by definition, this means $\mathfrak{a} \to y$ in X^*.

(3) Consider $\mathfrak{a} \to z$, $\mathfrak{a} \to z'$ in X^* with $p(z) = p(z')$, and let first $X \in \mathfrak{a}$. If both $z, z' \in X$, then $z = z'$ follows from separatedness of f, and if both $z, z' \in Y_1$, then trivially $z = z'$; the case $z \in X$ and $z' \in Y_1$ cannot occur, according to the definition of the structure of X^*. For $Y_1 \in \mathfrak{a}$ we necessarily have the trivial case $z, z' \in Y_1$ again. ∎

We point out that, since cl is idempotent when restricted to **Top**, if Y is a topological space then X^* is simply $X \cup \mathrm{cl}(Y_0)$.

5.4 Proposition. For $f\colon X \to Y$ exponentiable in **Top**, each of the following conditions implies the next:

(i) X and Y are Hausdorff spaces;

(ii) whenever $\mathfrak{U} \to \mathfrak{a}$ and $\mathfrak{U} \to \mathfrak{a}'$ in UX^* with $p(\mathfrak{a}) = p(\mathfrak{a}')$, then $\uparrow \mathfrak{a} = \uparrow \mathfrak{a}'$;

(iii) X^* is a topological space.

Proof: (ii)$\Rightarrow$(iii): For $\mathfrak{U} \to \mathfrak{a} \to z$ in X^* we must show $\mu_{X^*}(\mathfrak{U}) \to z$. Continuity of p gives $p(\mathfrak{U}) \to p(\mathfrak{a}) \to p(z)$ in Y, hence $\mu_Y(p(\mathfrak{U})) \to p(z)$ in the topological space Y.

Case 1: $X \in \mathfrak{a}$. Then the set

$$\downarrow X = \left\{\mathfrak{c} \in UX^* \mid \exists x \in X\colon \mathfrak{c} \to x\right\}$$

lies in $\mathfrak{U}$ and consists entirely of ultrafilters on X. Hence,

$$X^\sharp = \left\{\mathfrak{c} \in UX^* \mid X \in \mathfrak{c}\right\}$$

lies in $\mathfrak{U}$, so that $\mathfrak{U}$ can be considered as an ultrafilter on UX, and we have $X \in \mu_{X^*}(\mathfrak{U})$, and $\mu_X(\mathfrak{U})$ is the restriction of $\mu_{X^*}(\mathfrak{U})$.

Now, if $z \in X$, topologicity of X gives $\mu_X(\mathfrak{U}) \to z$ and therefore $\mu_{X^*}(\mathfrak{U}) \to z$. If $z \in Y_1$, since $p(\mu_{X^*}(\mathfrak{U})) = \mu_Y(p(\mathfrak{U})) \to p(z) = z$, in order to have $\mu_{X^*}(\mathfrak{U}) \to z$ it suffices to show that there is no $x \in f^{-1}z$ with $\mu_{X^*}(\mathfrak{U}) \to x$. Assuming the contrary, we may apply the ultrafilter interpolation property of f to obtain an ultrafilter $\mathfrak{a}'$ on X (and therefore on X^*) with $\mathfrak{U} \to \mathfrak{a}' \to x$ and $f(\mathfrak{a}') = p(\mathfrak{a})$, hence $p(\mathfrak{a}') = p(\mathfrak{a})$. From (ii) we then have $\mathfrak{a} \to x$, which contradicts $\mathfrak{a} \to z \in Y_1$.

Case 2: $Y_1 \in \mathfrak{a}$, hence necessarily $z \in Y_1$. If $(Y_1)^\sharp \in \mathfrak{U}$, then $\mu_{X^*}(\mathfrak{U}) \to z$ by topologicity of Y_1 (just as in Case 1 for $z \in X$). If $(Y_1)^\sharp \notin \mathfrak{U}$, $UX^* \backslash (Y_1)^\sharp = X^\sharp \in \mathfrak{U}$, so that $X \in \mu_{X^*}(\mathfrak{U})$ as above, and we can conclude the proof precisely as in the second half of Case 1.

(i)$\Rightarrow$(ii): Consider $\mathfrak{U} \to \mathfrak{a}$, $\mathfrak{U} \to \mathfrak{a}'$ in X^* with $p(\mathfrak{a}) = p(\mathfrak{a}')$. We first claim that $X \in \mathfrak{a}$ if and only if $X \in \mathfrak{a}'$. Indeed, if $X \in \mathfrak{a}$, then $\downarrow X \in \mathfrak{U}$ from $\mathfrak{U} \to \mathfrak{a}$ and $\uparrow\downarrow X \in \mathfrak{a}'$ from $\mathfrak{U} \to \mathfrak{a}'$; but $\uparrow\downarrow X \subseteq X$, since for $z \in \uparrow\downarrow X$ one has $\mathfrak{c} \in UX^*$ with $\mathfrak{c} \to z$ in X^* and $\mathfrak{c} \to x$ for some $x \in X$, whence $p(z) = p(x) = f(x)$ when Y is Hausdorff, which makes $z \in Y_1$ impossible. Consequently, $X \in \mathfrak{a}'$.

Now, let $z \in \uparrow\mathfrak{a}$. If $X \in \mathfrak{a}$, $z \in X$, then $z \in \uparrow\mathfrak{a}'$, as follows: for every $A \in \Omega(x)$ one has $A \in \mathfrak{a}$, hence $\downarrow A \in \mathfrak{U}$ and then $\uparrow\downarrow A \in \mathfrak{a}'$; but as above one has $\uparrow\downarrow A \subseteq A$ since X and Y are Hausdorff. If $X \in \mathfrak{a}$, $z \in Y$, then $p(\mathfrak{a}) = f(\mathfrak{a}) \to z$ in Y, hence $p(\mathfrak{a}') = f(\mathfrak{a}') \to z$; if there were $x \in f^{-1}z$ with $\mathfrak{a}' \to x$ in X, then also $\mathfrak{a} \to x$ in X as above, in contradiction to $z \in \uparrow \mathfrak{a}$. Hence $z \in \uparrow \mathfrak{a}'$. If $Y_1 \in \mathfrak{a}$, $z \in Y_1$, Hausdorffness of Y implies $z \in \uparrow \mathfrak{a}'$ as above. ∎

5.5 (*Proof of Theorem C*). Let first $f\colon X \to Y$ be exponentiable and separated, with Y Hausdorff. Then also X is Hausdorff, and from 5.4 we obtain the factorization

$$f = (X \overset{i}{\hookrightarrow} X^* \xrightarrow{p} Y)$$

in **Top** which, by 5.3, has the desired properties. Conversely, open embeddings are trivially separated and locally closed and therefore exponentiable (see [31]), and so are perfect maps, by Theorem A. Furthermore, exponentiable and separated maps are closed under composition. ∎

5.6 Remark. James ([23], p. 58) gives the construction of the *fibrewise Alexandroff compactification*, which provides for every continuous map $f : X \to Y$ a factorization

$$f = (X \overset{j}{\hookrightarrow} X^+ \xrightarrow{q} Y)$$

with an open embedding j and a proper map q. However, even for X and Y Hausdorff, q need not be separated; it is so, if X is also locally compact. □

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