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H-MEASURES AND BOUNDS ON THE EFFECTIVE PROPERTIES OF COMPOSITE MATERIALS

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Abstract: The goal of this paper is to show how the notion of H-measure, introduced by Gérard and Tartar, can be used to derive optimal bounds on the effective properties of two-phase composite materials. We re-derive the classical optimal bounds of Hashin and Shtrikman in the conductivity or elasticity setting. In an attempt to extend further this method in the context of two different conductivity properties, we obtain coupled bounds, known as Bergman bounds, which are merely sub-optimal in most cases.

1 – Introduction

The purpose of this work is to investigate the potentiality of H-measure theory in the study of effective properties of two-phase composite materials. The notion of H-measure has been introduced by Gérard [8] and Tartar [29]. It is a default measure which quantifies, in the phase space (i.e. the physical space times the Fourier space of propagation directions), the lack of compactness of weakly converging sequences. On the other hand, two-phase composite materials are the result of a fine mixture of two components with prescribed proportions. The main focus of the theory of composite materials is to characterize the range of effective (or macroscopic) properties of such mixtures when their microscopic geometrical arrangements vary. The link between these two issues (H-measures and composites) comes from the general setting of homogenization, introduced by Spagnolo [25] and Murat-Tartar [21], [22], which rigorously describes the effective

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properties of composite materials as homogenized limits obtained through weak convergence, without any assumptions of periodicity, randomness, or separation of scales.

There is a huge literature on composite materials, stemming from different communities (mathematics, physics, mechanics, engineering, ...) and we simply refer to the recent book of Milton [18] for further references. As already said, the main issue of the theory is to find the set of all possible effective (or homogenized) properties of these composite materials (this is often referred to as the G-closure problem). The usual approach is to find bounds on these effective properties, which thus define a larger set, and then try to prove the optimality of these bounds, which eventually shows that the larger set coincides with the desired G-closure set. Although the general strategy is the same, there are several methods for obtaining bounds. Since we are interested in bounds that incorporate only the phase individual properties and their volume fractions, there are mainly four different approaches: the variational method of Hashin-Shtrikman [11], the analytic representation method of Bergman [4] and Golden-Papanicolaou [9], the compensated compactness method of Tartar [27] (with its variants such as the translation method [17], or the null-lagrangian method [14]), and the H-measure method recently introduced by Tartar [29]. All these approaches, except the last ones, have been used and developed by many authors (for references, see e.g. [18]) and are quite well-known by now. Surprisingly, this is not the case with the H-measure method which, to the best of our knowledge, has only been used in the original work of Tartar [29].

There are various possible reasons for this lack of interest. First, the theory of H-measures is also helpful for small amplitude homogenization, and in the context of composite materials this is relevant for low contrast composites. Indeed, most efforts have been devoted to this last issue which is quite different from our main interest here. Second, experts have soon recognized that, when specialized to the case of periodic composite materials, the H-measure method reduces to the classical Hashin-Shtrikman method. More precisely, the H-measure of the microstructure characteristic function can be explicitly computed in the periodic setting by means of Fourier series, and it turns out to be nothing else than the so-called two-point correlation function of the microstructure. Despite this obvious connection between the two methods we believe that there is something more in the H-measure method. Of course, it is more general since it does not require any periodicity assumption as is case for the Hashin-Shtrikman method. But all the more, there is a systematic methodology in the H-measure method which is hidden in the Fourier computations of the Hashin-Shtrikman method.

Specifically, the coupled H-measure of the microstructure and of the fields is estimated by using all possible differential constraints, in the spirit of compensated compactness. In other words, the H-measure method is definitely a new point of view for deriving bounds and new ideas may emerge from it, which would not have been so clear with any previous method. Unfortunately, we have to admit that we have not been able to obtain new results with this method and that we content ourselves in giving new proofs of already known bounds.

The content and the results of this paper are as follows. In section 2 we recall the definition and main properties of H-measures. In section 3 we re-derive the well-known G-closure set of composites obtained by mixing two isotropic conductors (this result was first obtained in [14], [27]). This example was already studied by Tartar [29] in the context of H-measures. However, his argument was more general since it mixes H-measures and Young measures. We considerably simplify the exposition in the hope that it provides a clear and concise introduction to the H-measure method. The main result is Proposition 3.1 which gives optimal bounds on the trace of the homogenized conductivity tensor. We also give in Remark 3.5 some advantages of the H-measure method with respect to the compensated-compactness method as well as to the Hashin-Shtrikman variational principle.

Section 4 is the analogue of section 3 in the elasticity setting. We re-derive the trace bounds on the effective tensor of a mixture of two isotropic elastic phases which imply the famous Hashin-Shtrikman bounds on the bulk and shear moduli of an isotropic composite (see [7], [11], [17], [19]). Our main result is Proposition 4.1.

Of course, our main motivation when studying the H-measure method was to find new bounds. We therefore investigate in Section 5 the problem of finding coupled bounds for two different homogenized conductivity properties (for example, thermal and electrical conductivities) with the same microgeometry. This problem has been adressed e.g. in [3], [5], [6], [15], [16], but a complete answer is not yet known (except in 2-D). Unfortunately, using the H-measure method, we are unable to obtain the best already available bounds. Our main result, Theorem 5.1, merely yields the sub-optimal coupled bounds previously obtained by Bergman [3] in the well-ordered case, while Theorem 5.4 improves on Bergman bounds in the two-dimensional non well-ordered case (but still gives sub-optimal bounds). However, we believe this section has its own interest by showing the limitation of the H-measure method, at least in the way we use it. In some sense, our goal is more to popularize the H-measure method and its potentialities.

2 – Properties of H-measures

The notion of H-measure has been introduced by Gérard [8] and Tartar [29]. It is a default measure which quantifies, in the phase space, the lack of compactness of weakly converging sequences in $L^2(\mathbb{R}^N)$. In other words, it indicates where in the physical space, and at which frequency in the Fourier space, are the obstructions to strong convergence. As recognized by Tartar [29], this abstract tool has many important applications in the mathematical theory of composite materials. We briefly recall the necessary results on H-measures and refer to [8], [29] for complete proofs.

Let u_ϵ be a sequence of functions defined in $\mathbb{R}^N$ with values in $\mathbb{R}^p$. The components of the vector-valued function u_ϵ are denoted by $(u^i_\epsilon)_{1\le i\le p}$. We assume that u_ϵ converges weakly to 0 in $L^2(\mathbb{R}^N)^p$.

Theorem 2.1. *There exist a subsequence (still denoted by ϵ) and a family of complex-valued Radon measures $(\mu_{ij}(x,\xi))_{1\le i,j\le p}$ on $\mathbb{R}^N \times S_{N-1}$ such that, for every functions $\phi_1(x)$, $\phi_2(x)$ in $C_0(\mathbb{R}^N)$, and $\psi(\xi)$ in $C(S_{N-1})$, it satisfies*

$$\int_{\mathbb{R}^N}\int_{S_{N-1}} \phi_1(x)\,\overline{\phi}_2(x)\,\psi\Big(\frac{\xi}{|\xi|}\Big)\,\mu_{ij}(dx,d\xi) = \\ = \lim_{\epsilon\to 0}\int_{\mathbb{R}^N} \mathcal{F}(\phi_1\,u^i_\epsilon)(\xi)\;\overline{\mathcal{F}(\phi_2\,u^j_\epsilon)}(\xi)\,\psi\Big(\frac{\xi}{|\xi|}\Big)\,d\xi \;.$$

The matrix of measures $\mu = (\mu_{ij})_{1\le i,j\le p}$ is called the H-measure of the subsequence u_ϵ. It takes its values in the set of hermitian and non-negative matrices

$$\mu_{ij} = \overline{\mu}_{ji}\,, \qquad \sum_{i,j=1}^{p} \lambda_i\,\overline{\lambda}_j\,\mu_{ij} \ge 0 \quad \forall\,\lambda\in\mathbb{C}^p \;.$$

Let us explain the notations of Theorem 2.1: S_{N-1} is the unit sphere in $\mathbb{R}^N$, $C(S_{N-1})$ is the space of continuous complex-valued functions on S_{N-1}, $C_0(\mathbb{R}^N)$ that of continuous complex-valued functions decreasing to 0 at infinity in $\mathbb{R}^N$, and $\overline{z}$ denotes the complex conjugate of the complex number z. Finally, $\mathcal{F}$ is the Fourier transform operator defined in $L^2(\mathbb{R}^N)$ by

$$(\mathcal{F}\phi)\,(\xi) = \int_{\mathbb{R}^N} \phi(x)\,e^{-2i\pi(x\cdot\xi)}\,dx \;.$$

In Theorem 2.1, the role of the test functions ϕ_1 and ϕ_2 is to localize in space, while that of ψ is to localize in the directions of oscillations.

Remark 2.2. Theorem 2.1 furnishes a representation formula for the limit of quadratic objects of the sequence u_ϵ. When we take $\psi \equiv 1$, we recover the usual default measure in the physical space, i.e. $\int_{S_{N-1}} \mu_{ij}(\cdot, d\xi)$ is just the weak-* limit measure of the sequence $u^i_\epsilon \overline{u}^j_\epsilon$, which is bounded in $L^1(\mathbb{R}^N)$. Therefore, the H-measure gives a more precise representation of the compactness default, taking into account oscillation directions. □

Theorem 2.1 can be easily generalized to more general quadratic forms of u_ϵ in the context of pseudo-differential operators (see section 18.1 in [12]). Let us recall that a standard pseudo-differential operator q is defined through its symbol $(q_{ij}(x,\xi))_{1\le i,j\le p} \in C^\infty(\mathbb{R}^N \times \mathbb{R}^N)$ by

$$(qu)_i(x) = \sum_{j=1}^{p} \mathcal{F}^{-1}\Big(q_{ij}(x,\cdot)\, \mathcal{F}u_j(\cdot)\Big)(x)$$

for any smooth and compactly supported function u. In the sequel, we shall only use so-called polyhomogeneous pseudo-differential operators of order 0, i.e. whose (principal) symbol $(q_{ij}(x,\xi))_{1\le i,j\le p}$ is homogeneous of degree 0 in ξ and with compact support in x. (Remark that, being homogeneous of degree 0, the previous symbol is not smooth at $\xi = 0$. This is not a problem since any regularization by a smooth cut-off at the origin gives rise to the same pseudo-differential operator up to the addition of a smoothing operator.) Recall also that a polyhomogeneous pseudo-differential operators of order 0 is a bounded operator in $L^2(\mathbb{R}^N)^p$.

Theorem 2.3. *Let u_ϵ be a sequence which converges weakly to 0 in $L^2(\mathbb{R}^N)^p$. There exist a subsequence and a H-measure μ such that, for any polyhomogeneous pseudo-differential operator q of degree 0 with principal symbol $(q_{ij}(x,\xi))_{1\le i,j\le p}$, we have*

$$\lim_{\epsilon\to 0} \int_{\mathbb{R}^N} q(u_\epsilon)\cdot \overline{u}_\epsilon \, dx = \int_{\mathbb{R}^N}\int_{S_{N-1}} \sum_{i,j=1}^{p} q_{ij}(x,\xi)\, \mu_{ij}(dx,d\xi) \; .$$

Remark 2.4. Recall that $\sum_{i,j=1}^p q_{ij}\,\mu_{ij} = tr(q\mu)$ where $q\mu$ denotes the product of the $p\times p$ matrices q and μ. We shall often use the following notation of Tartar

$$\langle\langle \mu \cdot Q(x,\xi,U)\rangle\rangle = \int_{S_{N-1}} \sum_{i,j=1}^{p} q_{ij}(x,\xi)\mu_{ij}(x,d\xi),$$

and $Q(x,\xi,U) = \sum_{i,j=1}^p q_{ij}(x,\xi)\,\overline{U}_i\, U_j$ and U is just a dummy variable. □

An important property of the H-measure μ (apart from being a hermitian non-negative matrix) is its localization principle which is a generalization of the compensated compactness theory of Murat and Tartar [20], [26].

Theorem 2.5. *Let u_ϵ be a sequence which converges weakly to 0 in $L^2(\mathbb{R}^N)^p$ and defines a H-measure μ. If u_ϵ is such that, for $1 \leq m \leq m_0$,*

$$\sum_{j=1}^{p}\sum_{k=1}^{N} \frac{\partial}{\partial x_k}\left(A^m_{jk}(x)\, u^j_\epsilon\right) \;\to\; 0 \quad \text{in } H^{-1}_{\mathrm{loc}}(\Omega) \text{ strongly}$$

where the coefficients A^m_{ij} are continuous in an open set Ω of $\mathbb{R}^N$, then the H-measure satisfies

$$(1) \qquad \sum_{j=1}^{p}\sum_{k=1}^{N} A^m_{jk}(x)\, \xi_k \mu_{ji} = 0 \;\text{ in } \Omega \quad \forall\, i,m,\;\; 1 \leq i \leq p,\; 1 \leq m \leq m_0 \;.$$

For example, if $u_\epsilon = \nabla v_\epsilon$ is a gradient of some scalar sequence v_ϵ, then it satisfies $\operatorname{curl} u_\epsilon = 0$ which gives $m_0 = N(N-1)/2$ differential constraints from which we easily deduce that $\mu = \xi \otimes \xi \nu$ where ν is a scalar H-measure. As a consequence of Theorem 2.5 we obtain the following.

Theorem 2.6. *Let μ be a H-measure satisfying (1). Let $(q_{ij}(x,\xi))_{1\leq i,j\leq p}$ be a symbol homogeneous of degree 0 in ξ. Introducing a characteristic set*

$$\Lambda = \left\{ (x,\xi,U) \in \mathbb{R}^N \times S_{N-1} \times \mathbb{C}^p \text{ s.t. } \sum_{j=1}^{p}\sum_{k=1}^{N} A^m_{jk}(x)\, \xi_k\, U_j = 0,\; 1 \leq m \leq m_0 \right\} ,$$

if $\displaystyle\sum_{i,j=1}^{p} q_{ij}(x,\xi)\, \overline{U}_i\, U_j \geq 0$ *for any* $(x,\xi,U) \in \Lambda$, *then* $\displaystyle\sum_{i,j=1}^{p} q_{ij}\, \mu_{ij} \geq 0.$

Proof of Theorem 2.6: We briefly outline the algebra of the proof (see [8], [29] for a complete proof). For given (x,ξ) we denote by $\Lambda_{x,\xi}$ the set of U such that $(x,\xi,U) \in \Lambda$. By Theorem 2.5 the vectors $a^m \in \mathbb{R}^p$, for $1 \leq m \leq m_0$, of components $a^m_j = \sum_{k=1}^N A^m_{jk}(x)\, \xi_k$, for $1 \leq j \leq p$, belong to the kernel of μ. On the other hand, $\Lambda_{x,\xi}$ is just the orthogonal set of this collection of vectors a^m. Since μ is hermitian non-negative of order p, we deduce that μ belongs to the convex hull of the set $\{\lambda \otimes \overline{\lambda} \text{ s.t. } \lambda \in \Lambda_{x,\xi}\}$. In other words, there exist

$r = \mathrm{rank}(\mu) \leq p$ elements $(\lambda^1, .., \lambda^r)$ in $\Lambda_{x,\xi}$ and r positive real numbers $(\alpha_1, .., \alpha_r)$ such that

$$\mu = \sum_{i=1}^{r} \alpha_i \lambda^i \otimes \overline{\lambda^i} \quad \text{with} \quad \sum_{i=1}^{r} \alpha_i = 1 \ .$$

By assumption, we have $q\lambda \cdot \overline{\lambda} \geq 0$ for any $\lambda \in \Lambda_{x,\xi}$, and thus

$$\sum_{i,j=1}^{p} q_{ij} \, \mu_{ij} = \sum_{k=1}^{r} \alpha_k \sum_{i,j=1}^{p} q_{ij} \, \overline{\lambda_i^k} \, \lambda_j^k \ \geq \ 0 \ . \ \blacksquare$$

Combining Theorem 2.3 and Theorem 2.6, we obtain, as a corollary, a result of compensated compactness that will be the main technical ingredient in the sequel.

Corollary 2.7. *Let u_ϵ be a sequence converging weakly to u in $L^2(\mathbb{R}^N)^p$. Assume that, for $1 \leq m \leq m_0$, u_ϵ satisfies*

$$\sum_{j=1}^{p} \sum_{k=1}^{N} \frac{\partial}{\partial x_k} \left(A_{jk}^m(x) \, u_\epsilon^j \right) \ \to \ \sum_{j=1}^{p} \sum_{k=1}^{N} \frac{\partial}{\partial x_k} \left(A_{jk}^m(x) \, u^j \right) \qquad \textit{in} \ \ H^{-1}_{\mathrm{loc}}(\Omega) \ \textit{strongly}$$

where the coefficients A_{jk}^m are continuous in an open set Ω of $\mathbb{R}^N$. Let Λ be the characteristic set defined by

$$\Lambda = \left\{ (x, \xi, U) \in \mathbb{R}^N \times S_{N-1} \times \mathbb{C}^p \ \ s.t. \ \ \sum_{j=1}^{p} \sum_{k=1}^{N} A_{jk}^m \, \xi_k \, U_j = 0, \ 1 \leq m \leq m_0 \right\} .$$

Let q be a polyhomogeneous pseudo-differential operator of order 0 with hermitian principal symbol $(q_{ij}(x,\xi))_{1\leq i,j \leq p}$ such that $\sum_{i,j=1}^{p} q_{ij}(x,\xi) \, \overline{U}_i U_j \geq 0$ for any $(x, \xi, U) \in \Lambda$. Then, for any non-negative, smooth ϕ with compact support in Ω,

$$\liminf_{\epsilon \to 0} \int_{\mathbb{R}^N} \phi(x) \, q(u_\epsilon) \cdot \overline{u}_\epsilon \, dx \ \geq \int_{\mathbb{R}^N} \phi(x) \, q(u) \cdot \overline{u} \ dx \ .$$

3 – Simple bounds in conduction

This section is devoted to the computation of the so-called G-closure of two isotropic phases in the conductivity setting. This is a well-known results which goes back to [14], [27] and which can be derived by using H-measures as shown by Tartar in [29]. We specialize Tartar's argument (which is more general since it also involves Young measures) to the case at hand. Let us emphasize that our purpose here is just to warm up before we attack more difficult problems. To obtain the G-closure the idea is to derive bounds on the homogenized or effective tensor of a dielectric composite. Let us recall some notations.

We denote by $0 < \alpha \leq \beta$ the conductivities of the two isotropic components that fill a bounded open set Ω in $\mathbb{R}^N$. Let $\chi_\epsilon(x)$ be a sequence of characteristic functions of the domain occupied by phase α. We define a sequence of two-phase mixtures by the local conductivity tensor

$$A_\epsilon(x) \;=\; \chi_\epsilon(x)\,\alpha\,\mathrm{Id} + (1-\chi_\epsilon(x))\,\beta\,\mathrm{Id}\ , \tag{2}$$

where Id is the identity matrix in $\mathbb{R}^N$. For any right hand side $f \in L^2(\Omega)$, we denote by u_ϵ the unique solution in $H_0^1(\Omega)$ of

$$-\ \mathrm{div}\,(A_\epsilon \nabla u_\epsilon) = f \quad \text{ in }\ \Omega\ .$$

The main result of H-convergence (see [22]) states that, up to a subsequence, there exist a function $\theta \in L^\infty(\Omega;[0,1])$ and a tensor-valued function $A_* \in (L^\infty(\Omega))^{N^2}$ such that

$$\begin{array}{ll} \chi_\epsilon \rightharpoonup \theta & \text{weakly-* in }\ L^\infty(\Omega)\ , \\ A_\epsilon & H\text{-converges to }\ A_*\ , \end{array} \tag{3}$$

in the sense that, for any right hand side f, the solution u_ϵ satisfies

$$\begin{array}{l} u_\epsilon \rightharpoonup u_0 \quad \text{weakly in }\ H_0^1(\Omega)\ , \\ \sigma_\epsilon = A_\epsilon \nabla u_\epsilon \rightharpoonup \sigma_0 = A_* \nabla u_0 \quad \text{weakly in }\ L^2(\Omega)^N\ , \end{array}$$

where u_0 is the unique solution in $H_0^1(\Omega)$ of the homogenized problem

$$-\ \mathrm{div}(A_* \nabla u_0) = f \quad \text{ in }\ \Omega\ .$$

The function θ is the proportion of phase α and A_* is the effective tensor of the composite material obtained by the mixing process (2). A further property of H-convergence is the following convergence of energies

$$A_\epsilon \nabla u_\epsilon \cdot \nabla u_\epsilon \rightharpoonup A_* \nabla u_0 \cdot \nabla u_0 \quad \text{and} \quad A_\epsilon^{-1}\sigma_\epsilon \cdot \sigma_\epsilon \rightharpoonup A_*^{-1}\sigma_0 \cdot \sigma_0$$

in the sense of distributions in Ω. The G-closure problem is to find what is the possible range of values of A_* for given constituents α, β and proportions $\theta, 1-\theta$. Up to extracting another subsequence, there exist the so-called arithmetic and harmonic averages $\overline{A}$, $\underline{A}$ such that

$$A_\epsilon \rightharpoonup \overline{A} \ \text{ and }\ A_\epsilon^{-1} \rightharpoonup \underline{A}^{-1} \quad \text{weakly * in }\ (L^\infty(\Omega))^{N^2}.$$

It is well-known that the values of A_* must lie between the arithmetic and harmonic averages

$$\underline{A}(x) \le A_*(x) \le \overline{A}(x) \quad \text{in } \Omega . \tag{4}$$

However these bounds are not optimal and can be improved. The optimal bounds, often called the Hashin–Shtrikman bounds, are obtained in the following result.

Proposition 3.1. *Let A_* be the H-limit of a sequence A_ϵ defined by (2). It satisfies*

$$tr(A_* - \alpha\,\mathrm{Id})^{-1} \le \frac{N}{(1-\theta)\,(\beta-\alpha)} + \frac{\theta}{(1-\theta)\,\alpha} \tag{5}$$

and

$$tr(A_*^{-1} - \beta^{-1}\mathrm{Id})^{-1} \le \frac{N}{\theta}(\alpha^{-1} - \beta^{-1})^{-1} + \frac{(N-1)\,(1-\theta)}{N\theta}\,\beta\ . \tag{6}$$

Remark 3.2. The G-closure (at the given proportion θ) is indeed defined by inequalities (5), (6) and (4). More precisely, the G-closure is the set of all symmetric tensors A_* with eigenvalues $\lambda_1, ..., \lambda_N$ satisfying

$$\lambda_\theta^- = \left(\theta\,\alpha^{-1} + (1-\theta)\,\beta^{-1}\right)^{-1} \le \lambda_i \le \lambda_\theta^+ = \theta\,\alpha + (1-\theta)\,\beta\,, \quad \forall\, 1 \le i \le N\ ,$$

$$\sum_{i=1}^N \frac{1}{\lambda_i - \alpha} \le \frac{1}{\lambda_\theta^- - \alpha} + \frac{N-1}{\lambda_\theta^+ - \alpha}\ ,$$

$$\sum_{i=1}^N \frac{1}{\beta - \lambda_i} \le \frac{1}{\beta - \lambda_\theta^-} + \frac{N-1}{\beta - \lambda_\theta^+}\ .$$

To rigorously prove such a statement, one must show that any tensor satisfying these bounds is actually attained as an H-limit of a sequence A_ϵ defined by (2). For details we refer to [14], [27]. □

Remark 3.3. If the composite material is isotropic, the homogenized tensor A_* is equal to $a_*\mathrm{Id}$ and the bounds (5), (6) becomes $hs^-(\theta,\alpha,\beta) \le a_* \le hs^+(\theta,\alpha,\beta)$ with

$$hs^-(\theta,\alpha,\beta) = \theta\alpha + (1-\theta)\beta - \frac{\theta\,(1-\theta)\,(\beta-\alpha)^2}{(N-1)\alpha + \beta - (1-\theta)(\beta-\alpha)}\ ,$$

$$hs^+(\theta,\alpha,\beta) = \theta\alpha + (1-\theta)\beta - \frac{\theta\,(1-\theta)\,(\beta-\alpha)^2}{(N-1)\beta + \alpha + \theta(\beta-\alpha)}\ . \ □$$

Proof: Introducing constant vectors η, μ in $\mathbb{R}^N$, we derive optimal bounds on A_* by passing to the limit in the simple inequalities

$$(A_\epsilon - \alpha \,\mathrm{Id})\,(\nabla u_\epsilon + \eta) \cdot (\nabla u_\epsilon + \eta) \;\geq\; 0, \qquad \text{a.e. in } \;\Omega \;, \tag{7}$$

$$(A_\epsilon^{-1} - \beta^{-1}\,\mathrm{Id})\,(\sigma_\epsilon + \mu) \cdot (\sigma_\epsilon + \mu) \;\geq\; 0, \qquad \text{a.e. in } \;\Omega \;. \tag{8}$$

We first focus on the lower bound, namely we start from (7) which implies

$$A_\epsilon \nabla u_\epsilon \cdot \nabla u_\epsilon + (A_\epsilon - \alpha\,\mathrm{Id})\,\eta \cdot \eta - 2\,\alpha\,\nabla u_\epsilon \cdot \eta \;\geq\; \alpha\,\nabla u_\epsilon \cdot \nabla u_\epsilon - 2\,A_\epsilon \nabla u_\epsilon \cdot \eta \;. \tag{9}$$

By using homogenization theory, we can pass to the limit in the left hand side of (9). On the other hand, we use the notion of H-measure in order to pass to the limit in the right hand side. Introducing a coupled variable $V_\epsilon = (\nabla u_\epsilon, A_\epsilon)$, we recognize that the right hand side of (9) is a quadratic form $q(V_\epsilon)\cdot V_\epsilon$. Introducing $V_0 = (\nabla u_0, \overline{A})$, we have

$$q(V_\epsilon) \cdot V_\epsilon \;=\; 2\,q(V_\epsilon) \cdot V_0 - q(V_0) \cdot V_0 + q(V_\epsilon - V_0) \cdot (V_\epsilon - V_0) \;.$$

Denoting by π the H-measure of $V_\epsilon - V_0$, we thus pass to the limit in (9) by virtue of Theorem 2.3:

$$A_* \nabla u_0 \cdot \nabla u_0 + (\overline{A} - \alpha\,\mathrm{Id})\,\eta \cdot \eta - 2\,\alpha\,\nabla u_0 \cdot \eta \;\geq\; \alpha\,\nabla u_0 \cdot \nabla u_0 - 2\,\overline{A}\,\nabla u_0 \cdot \eta + X \;, \tag{10}$$

where X is the H-correction defined by

$$X \;=\; \lim_{\epsilon \to 0} q(V_\epsilon - V_0) \cdot (V_\epsilon - V_0) \;=\; \int_{S_{N-1}} tr\Big(q\,\pi(x, d\xi)\Big)$$

or equivalently, by using the notations of Remark 2.4,

$$X = \langle\langle \pi \cdot Q(U, A) \rangle\rangle \qquad \text{with } \; Q(U, A) = q(U, A) \cdot (U, A) \;.$$

The idea is now to use Theorem 2.6 in order to get a lower bound on X. Remark however that the quadratic form q, defined by

$$q(V) \cdot V = \alpha\,|U|^2 - 2\,A\,U \cdot \eta \qquad \text{with } \; V = (U, A) \in \mathbb{R}^N \times \mathbb{R}^{N^2} \;,$$

is not coercive with respect to the matrix variable A. So there is no hope to prove that q is non-negative on a characteristic set, as required by Theorem 2.6. Therefore, we apply this result to a slightly different quadratic form.

Recall that a gradient ∇u_ϵ satisfies $\mathrm{curl}\,\nabla u_\epsilon = 0$. Thus, in the spirit of Theorems 2.5 and 2.6, we introduce a characteristic set

$$\Lambda \;=\; \Big\{ (\xi, U) \in S_{N-1} \times \mathbb{C}^N ,\; \xi_i U_j - \xi_j U_i = 0 \Big\} \;,$$

and its projection

$$\Lambda_\xi = \left\{ U \in \mathbb{R}^N ,\ (\xi, U) \in \Lambda \right\} .$$

Clearly, we have $\Lambda_\xi = \mathbb{R}\xi$. We thus introduce a new quadratic form q' defined by

$$q'(U,A) \cdot (U,A) \ = \ q(U,A) \cdot (U,A) - \min_{U \in \Lambda_\xi} q(U,A) \cdot (U,A) \ .$$

By definition q' is non-negative in the characteristic set of the couple (U, A). Thus, applying Theorem 2.6 we get $tr(q'\pi) \geq 0$ which implies that

$$X = \langle\langle \pi \cdot Q(U,A) \rangle\rangle \ \geq \ \langle\langle \pi \cdot \min_{U \in \Lambda_\xi} Q(U,A) \rangle\rangle \ .$$

Introducing π_A, the H-measure of $A_\epsilon - \overline{A}$, since the quadratic form $\min_{U \in \Lambda_\xi} Q(U,A)$ depends only on A, we get

$$X \ \geq \ \langle\langle \pi \cdot \min_{U \in \Lambda_\xi} Q(U,A) \rangle\rangle \ = \ \langle\langle \pi_A \cdot \min_{U \in \Lambda_\xi} Q(U,A) \rangle\rangle \ .$$

It remains to compute the right hand side of the above inequality. We found

$$\min_{U \in \Lambda_\xi} Q(A,U) = -\frac{(A\eta \cdot \xi)^2}{\alpha\, |\xi|^2} \ .$$

On the other hand, since $A_\epsilon - \overline{A} = (\alpha - \beta)\,(\chi_\epsilon - \theta)\,\mathrm{Id}$, the H-measure π_A reduces to

$$\pi_A = (\beta - \alpha)^2\, \mathrm{Id} \otimes \mathrm{Id}\, \nu$$

where ν is the H-measure of $(\chi_\epsilon - \theta)$ which satisfies

$$\nu(\xi) \geq 0 \quad \text{and} \quad \int_{S_{N-1}} \nu(d\xi) = \theta(1-\theta) \ . \tag{11}$$

We finally obtain

$$\langle\langle \pi_A \cdot \min_{U \in \Lambda_\xi} Q(A,U) \rangle\rangle \ = \ -\frac{(\beta-\alpha)^2}{\alpha} \int_{S_{N-1}} \frac{(\eta \cdot \xi)^2}{|\xi|^2}\, \nu\,(d\xi) \ .$$

Introducing a matrix M defined by

$$M = \frac{1}{\theta(1-\theta)} \int_{S^{N-1}} \xi \otimes \xi \ \nu\,(d\xi) \ , \tag{12}$$

which is symmetric, non-negative, and has unit trace, we deduce from (10)

$$\begin{aligned}(A_* - \alpha\,\mathrm{Id})\, \nabla u_0 \cdot \nabla u_0 + (\overline{A} - \alpha\,\mathrm{Id})\, \eta \cdot \eta + 2\, (\overline{A} - \alpha\,\mathrm{Id})\, \nabla u_0 \cdot \eta \ \geq \\ \geq \ -\frac{\theta\,(1-\theta)\,(\beta-\alpha)^2}{\alpha}\, M\, \eta \cdot \eta \ .\end{aligned} \tag{13}$$

Finally, minimizing with respect to ∇u_0 (which can locally take any value in $\mathbb{R}^N$), and noticing that $\overline{A} - \alpha\,\mathrm{Id} = (1-\theta)\,(\beta-\alpha)\,\mathrm{Id}$, (13) yields

$$(A_* - \alpha\,\mathrm{Id})^{-1}\,\eta\cdot\eta \le \frac{|\eta|^2}{(1-\theta)\,(\beta-\alpha)} + \frac{\theta\,M\eta\cdot\eta}{(1-\theta)\,\alpha}\ .$$

Since η is an arbitrary vector, we get

$$(A_* - \alpha\,\mathrm{Id})^{-1} \le \frac{\mathrm{Id}}{(1-\theta)\,(\beta-\alpha)} + \frac{\theta\,M}{(1-\theta)\,\alpha}\ . \tag{14}$$

Taking the trace of (14) yields the well-known optimal lower bound. Combining (14) and the fact that the maximal eigenvalue of M is 1 allows to recover the lower harmonic mean bound.

We now turn to the upper bound. Starting from (8) we get

$$A_\epsilon^{-1}\sigma_\epsilon\cdot\sigma_\epsilon + (A_\epsilon^{-1} - \beta^{-1}\,\mathrm{Id})\,\mu\cdot\mu - 2\,\beta^{-1}\sigma_\epsilon\cdot\mu \ge \beta^{-1}\sigma_\epsilon\cdot\sigma_\epsilon - 2\,A_\epsilon^{-1}\sigma_\epsilon\cdot\mu\ . \tag{15}$$

As for the lower bound, we can pass to the limit in the left hand side of (15) by using homogenization theory, and in the right hand side by using H-measures. Introducing the quadratic form q defined by

$$Q(V) = q(V)\cdot V = \beta^{-1}U\cdot U - 2A^{-1}U\cdot\mu \quad \text{with } V = (U, A^{-1})\ ,$$

we obtain the limit of (15)

$$A_*^{-1}\sigma_0\cdot\sigma_0 + (\underline{A}^{-1} - \beta^{-1}\mathrm{Id})\,\mu\cdot\mu - 2\,\beta^{-1}\sigma_0\cdot\mu \ge \beta^{-1}\sigma_0\cdot\sigma_0 - 2\,\underline{A}^{-1}\sigma_0\cdot\mu + X\ , \tag{16}$$

where X is a new H-correction defined by

$$X = \int_{S_{N-1}} tr\,(q\,\pi(x, d\xi)) = \langle\langle \pi\cdot Q(U, A^{-1})\rangle\rangle\ ,$$

where π is the H-measure of $V_\epsilon = (\sigma_\epsilon - \sigma_0, A_\epsilon^{-1} - \underline{A}^{-1})$. Since $\mathrm{div}\,\sigma_\epsilon$ remains in a compact set of $H^{-1}(\Omega)$, in the spirit of Theorem 2.5 we introduce the characteristic set

$$\Lambda' = \left\{(\xi, U) \in S_{N-1}\times\mathbb{R}^N \ s.t.\ \sum_{i=1}^N \xi_i\,U_i = 0\right\},$$

and its projection

$$\Lambda'_\xi = \left\{U \in \mathbb{R}^N,\ (\xi, U) \in \Lambda'\right\}\ .$$

As before we define a non-negative quadratic form q' by

$$q'(U,A^{-1})\cdot(U,A^{-1}) \;=\; q(U,A^{-1})\cdot(U,A^{-1}) - \min_{U\in\Lambda'_\xi} q(U,A^{-1})\cdot(U,A^{-1})\ .$$

Applying Theorem 2.6 we get $tr(q'\pi)\geq 0$. Introducing π'_A, the H-measure of $A_\epsilon^{-1}-\underline{A}^{-1}$, since the quadratic form $\min\limits_{U\in\Lambda'_\xi} Q(U,A^{-1})$ depends only on A^{-1}, we get

$$X \geq \langle\langle \pi\cdot \min_{U\in\Lambda'_\xi} Q(U,A^{-1})\rangle\rangle = \langle\langle \pi'_A\cdot \min_{U\in\Lambda'_\xi} Q(U,A^{-1})\rangle\rangle\ .$$

An easy computation gives

$$\min_{U\in\Lambda'_\xi} Q(U,A^{-1}) \;=\; \beta\Big(\frac{\xi\otimes\xi}{|\xi|^2}-\mathrm{Id}\Big)A^{-1}\mu\cdot A^{-1}\mu\ .$$

On the other hand, since $A_\epsilon^{-1}-\underline{A}^{-1} = (\alpha^{-1}-\beta^{-1})\,(\chi_\epsilon-\theta)\,\mathrm{Id}$, the H-measure π'_A reduces to

$$\pi'_A = \frac{(\beta-\alpha)^2}{(\alpha\,\beta)^2}\ \mathrm{Id}\otimes\mathrm{Id}\,\nu$$

where ν is again the H-measure of $(\chi_\epsilon-\theta)$ which satisfies (11). With the help of the matrix M defined by (12) we obtain

$$(A_*^{-1}-\beta^{-1}\,\mathrm{Id})\,\sigma_0\cdot\sigma_0 + (\underline{A}^{-1}-\beta^{-1}\,\mathrm{Id})\,\mu\cdot\mu + 2\,(\underline{A}^{-1}-\beta^{-1}\,\mathrm{Id})\,\sigma_0\cdot\mu \;\geq$$

$$\geq\; -\theta(1-\theta)\,\frac{(\beta-\alpha)^2}{\alpha^2\,\beta}\,(\mathrm{Id}-M)\,\mu\cdot\mu\ .$$

Minimizing with respect to σ_0 yields

$$(A_*^{-1}-\beta^{-1}\,\mathrm{Id})^{-1} \leq \frac{1}{\theta}\,(\alpha^{-1}-\beta^{-1})^{-1}\,\mathrm{Id} + \frac{(1-\theta)}{\theta}\,\beta(\mathrm{Id}-M)\ .$$

Taking the trace gives the well-known upper bound. ∎

Remark 3.4. The above proof works in the general setting of H-convergence, i.e. for a composite material without any geometrical assumption on its microstructure (like periodicity, ergodicity, or isotropy). In the context of periodic composite materials, this proof reduces to the original work of Hashin and Shtrikman [11]. Inequalities (7) and (8) are the usual starting points of the so-called Hashin-Shtrikman variational principles, and the H-measure ν of $(\chi_\epsilon-\theta)$ is just the so-called two-point correlation function of the microstructure. □

Remark 3.5. One advantage of the H-measure method is that the above proof of Proposition 3.1 immediately delivers optimality conditions for the sequence of microstructures χ_ϵ such that A_ϵ H-converges to an optimal homogenized tensor A_*, i.e. a tensor for which equality holds in (5) or (6). Each inequality of the proof turns out to be actually an equality which yields a necessary condition of optimality involving the H-measures of the characteristic functions and of the gradient fields. These optimality conditions are much more general than those obtained with the Hashin-Shtrikman variational principle since the latter ones are restricted to the spatially periodic case although it is well-known that most of the optimal microstructures are not periodic (like laminated composites).

Another advantage of the H-measure method with respect to the compensated compactness method is that the algebra is simpler in the following sense: in the former case the result is obtained by passing to the limit in a single conductivity equation, while in the latter case N equations must be considered simultaneously (where N is the space dimension). □

4 – Trace bounds in elasticity

In this section, we generalize the method of section 3 to derive the well-known Hashin-Shtrikman bounds on the effective tensor of an elastic composite material. Apart from the original application to small amplitude homogenization by Tartar [28], the notion of H-measure has not been used in the context of elastic composites.

We first recall some notations and definitions. We consider elastic composites obtained by mixing two isotropic phases in a bounded domain Ω of $\mathbb{R}^N$. Their Hooke's law A and B are fourth-order tensors defined by

$$(17)\qquad A = 2\,\mu_A I_4 + \left(\kappa_A - \frac{2\,\mu_A}{N}\right) I_2 \otimes I_2 \quad\text{and}\quad B = 2\,\mu_B I_4 + \left(\kappa_B - \frac{2\,\mu_B}{N}\right) I_2 \otimes I_2 \ ,$$

where μ_A, μ_B are the shear moduli and κ_A, κ_B are the bulk moduli. Here and in the sequel, I_d denotes the identity tensor of order d. The two phases are assumed to be well-ordered, namely

$$(18)\qquad 0 < \mu_A \le \mu_B\,, \quad 0 < \kappa_A \le \kappa_B \ .$$

We shall also need the Lamé coefficients defined by

$$(19)\qquad \lambda_A = \kappa_A - \frac{2\,\mu_A}{N}\,, \quad \lambda_B = \kappa_B - \frac{2\,\mu_B}{N} \ .$$

Introducing a family of characteristic functions $\chi_\epsilon(x)$, we consider a sequence of two-phase mixtures defined by its elasticity tensor

$$A_\epsilon(x) = \chi_\epsilon(x) A + (1 - \chi_\epsilon(x))\, B \ . \tag{20}$$

For any loading force $f \in L^2(\Omega)^N$, we denote by u_ϵ the unique solution in $H_0^1(\Omega)^N$ of the linearized elasticity system

$$- \operatorname{div}(A_\epsilon(x)\, e(u_\epsilon)) = f \quad \text{ in } \ \Omega \ ,$$

with $e(u_\epsilon) = (\nabla u_\epsilon + \nabla^T u_\epsilon)/2$. As in the conductivity case, the main result of homogenization (see [22], [7]) states that, up to a subsequence, there exist a limit proportion $\theta \in L^\infty(\Omega; [0,1])$ and an effective tensor $A_* \in L^\infty(\Omega)^{N^4}$ such that

$$\chi_\epsilon \rightharpoonup \theta \ \text{ weakly * in } L^\infty(\Omega) \quad \text{ and } \quad A_\epsilon \ \ H\text{-converges to } A_* \ ,$$

in the sense that, for any right hand side f, the solution u_ϵ converges weakly to u_0 in $H_0^1(\Omega)^N$, and $\sigma_\epsilon = A_\epsilon\, e(u_\epsilon)$ converges weakly to $\sigma_0 = \mathcal{A}_*\, e(u_0)$ in $L^2(\Omega)^{N^2}$, where u_0 is the unique solution in $H_0^1(\Omega)^N$ of the homogenized problem

$$- \operatorname{div}(A_*\, e(u_0)) = f \quad \text{ in } \ \Omega \ .$$

A further property of H-convergence is the convergence of energies in the sense of distributions in Ω

$$A_\epsilon e(u_\epsilon) \cdot e(u_\epsilon) \rightharpoonup A_* e(u_0) \cdot e(u_0) \quad \text{ and } \quad A_\epsilon^{-1} \sigma_\epsilon \cdot \sigma_\epsilon \rightharpoonup A_*^{-1} \sigma_0 \cdot \sigma_0 \ .$$

Contrary to the conductivity case, the G-closure problem has not yet been solved in the context of elasticity. Nevertheless, some optimal bounds (called trace bounds, see [17], [19]) are known which delivers part of the boundary (but not all of it) of the G-closure set. By using H-measure theory, we give a new proof of the following well-known theorem which was first proved by compensated compactness in [7] and [19].

Proposition 4.1. *Let A_* be the H-limit of a sequence A_ϵ defined by (20). Then, if ν is the scalar H-measure of $\chi_\epsilon - \theta$, we have*

$$(1-\theta)\,(A_* - A)^{-1} \le (B - A)^{-1} - \theta\, T \ , \tag{21}$$

and

$$\theta (A_*^{-1} - B^{-1})^{-1} \le (A^{-1} - B^{-1})^{-1} - (1-\theta)\, T' \ , \tag{22}$$

where, for any symmetric matrix η, T is defined by

$$T\eta \cdot \eta \;=\; \frac{1}{\theta(1-\theta)} \int_{S_{N-1}} \left(\frac{\lambda_A + \mu_A}{\mu_A(\lambda_A + 2\mu_A)} \frac{(\eta\,\xi \cdot \xi)^2}{|\xi|^4} - \frac{1}{\mu_A} \frac{|\eta\,\xi|^2}{|\xi|^2} \right) \nu(d\xi)$$

and T' is defined by

$$\begin{aligned} T'\eta \cdot \eta \;=\; & \frac{1}{\theta(1-\theta)} \int_{S_{N-1}} \Bigg(2\,\mu_B\,|\eta|^2 - 4\,\mu_B \frac{|\eta\,\xi|^2}{|\xi|^2} + 2\,\mu_B \frac{(\eta\,\xi \cdot \xi)^2}{|\xi|^4} \\ & - \frac{2\,\mu_B\,\lambda_B}{(\lambda_B + 2\,\mu_B)} \left(\frac{\eta\,\xi \cdot \xi}{|\xi|^2} - tr(\eta) \right)^2 \Bigg) \nu(d\xi) \ . \end{aligned}$$

Remark 4.2. If A_* is isotropic, namely $A_* = 2\,\mu_* I_4 + (\kappa_* - 2\,\mu_*/N)\, I_2 \otimes I_2$, (21) and (22) yield the well-known Hashin-Shtrikman bounds [11] for its bulk and shear moduli

$$\frac{1-\theta}{\kappa_* - \kappa_A} \le \frac{1}{\kappa_B - \kappa_A} + \frac{\theta}{2\,\mu_A + \lambda_A} \ , \tag{23}$$

$$\frac{\theta}{\kappa_B - \kappa_*} \le \frac{1}{\kappa_B - \kappa_A} + \frac{1-\theta}{2\,\mu_B + \lambda_B} \tag{24}$$

and

$$\frac{1-\theta}{2\,(\mu_* - \mu_A)} \le \frac{1}{2\,(\mu_B - \mu_A)} + \frac{\theta\,(N-1)\,(\kappa_A + 2\,\mu_A)}{(N^2+N-2)\,\mu_A(2\,\mu_A + \lambda_A)} \ , \tag{25}$$

$$\frac{\theta}{2\,(\mu_B - \mu_*)} \le \frac{1}{2\,(\mu_B - \mu_A)} + \frac{(1-\theta)\,(N-1)\,(\kappa_B + 2\,\mu_B)}{(N^2+N-2)\,\mu_B(2\,\mu_B + \lambda_B)} \ . \tag{26}$$

Furthermore, as proved in [7], these lower and upper bounds are simultaneously attained by an isotropic rank-p sequential laminate with $p = N(N+1)/2$. Unfortunately, these bounds do not characterize all possible isotropic homogenized tensor A_* obtained by mixing phases A and B in proportion θ and $1-\theta$. □

Proof: As in the conductivity case we start from the following simple inequalities that hold for any symmetric matrices η, τ in $\mathbb{R}^{N^2}$

$$(A_\epsilon - A)\,(e(u_\epsilon) + \eta) \cdot (e(u_\epsilon) + \eta) \;\ge\; 0 \quad \text{a.e. in } \ \Omega \ , \tag{27}$$

$$(A_\epsilon^{-1} - B^{-1})\,(\sigma_\epsilon + \tau) \cdot (\sigma_\epsilon + \tau) \;\ge\; 0 \quad \text{a.e. in } \ \Omega \ . \tag{28}$$

Lower bound: Developing (27) yields

$$A_\epsilon e(u_\epsilon) \cdot e(u_\epsilon) + (A_\epsilon - A)\,\eta \cdot \eta - 2\,A e(u_\epsilon) \cdot \eta \;\ge\; A e(u_\epsilon) \cdot e(u_\epsilon) - 2\,A_\epsilon e(u_\epsilon) \cdot \eta \ . \tag{29}$$

Homogenization theory allows to pass to the limit in the left hand side, and H-measure theory in the right hand side, of (29). Denoting by $\overline{A} = \theta A + (1-\theta) B$ the weak limit (or arithmetic mean) of A_ϵ, we set $V_\epsilon = (A_\epsilon, e(u_\epsilon))$ and $V_0 = (\overline{A}, e(u_0))$. The right hand side of (29) is a quadratic form $q(V_\epsilon)\cdot V_\epsilon$ which we rewrite

$$q(V_\epsilon)\cdot V_\epsilon \;=\; 2\,q(V_\epsilon)\cdot V_0 - q(V_0)\cdot V_0 + q(V_\epsilon - V_0)\cdot(V_\epsilon - V_0)\ .$$

Denoting by π the H-measure of $V_\epsilon - V_0$, by virtue of Theorem 2.3 we pass to the limit in (29)

$$(30)\qquad A_* e(u_0)\cdot e(u_0) + (\overline{A} - A)\,\eta\cdot\eta - 2Ae(u_0)\cdot\eta \;\geq\; Ae(u_0)\cdot e(u_0) - 2\overline{A}e(u_0)\cdot\eta + X\ ,$$

where X is the H-correction

$$X \;=\; \lim_{\epsilon\to 0} q(V_\epsilon - V_0)\cdot(V_\epsilon - V_0) \;=\; \int_{S_{N-1}} tr(q\,\pi(x, d\xi)) \;=\; \langle\langle \pi\cdot Q(\mathcal{A}, U)\rangle\rangle\ ,$$

with the notation of Remark (2.4) and $Q(\mathcal{A},U) = q(\mathcal{A},U)\cdot(\mathcal{A},U)$. Noting that $e(u_\epsilon)$, as a strain tensor, satisfies:

$$\frac{\partial e(u_\epsilon)_{jl}}{\partial x_i\,\partial x_k} + \frac{\partial e(u_\epsilon)_{ik}}{\partial x_j\,\partial x_l} - \frac{\partial e(u_\epsilon)_{jk}}{\partial x_i\,\partial x_l} - \frac{\partial e(u_\epsilon)_{il}}{\partial x_j\,\partial x_k} \;=\; 0\ ,$$

we introduce the corresponding characteristic set

$$\Lambda \;=\; \left\{(\xi, U)\in S_{N-1}\times\mathbb{R}^{N^2},\ \exists\,\zeta\ \ s.t.\ \ U = \zeta\odot\xi\right\}\ ,$$

where $\zeta\odot\xi = (\zeta\otimes\xi + \xi\otimes\zeta)/2$, and its projection

$$\Lambda_\xi \;=\; \left\{U\in\mathbb{R}^{N^2},\ (\xi, U)\in\Lambda\right\}\ .$$

As in section 3 we introduce a new quadratic form

$$q'(\mathcal{A},U)\cdot(\mathcal{A},U) \;=\; q(\mathcal{A},U)\cdot(\mathcal{A},U) - \min_{U\in\Lambda_\xi} q(\mathcal{A},U)\cdot(\mathcal{A},U)$$

which is positive on Λ_ξ. By application of Theorem 2.6 we obtain that $tr(q'\pi)\geq 0$ which implies

$$(31)\qquad X \;=\; \langle\langle\pi\cdot Q(\mathcal{A},U)\rangle\rangle \;\geq\; \langle\langle\pi_A\cdot\min_{U\in\Lambda_\xi} Q(\mathcal{A},U)\rangle\rangle\ ,$$

where π_A is the H-measure of $A_\epsilon - \overline{A}$. Since $A_\epsilon - \overline{A} = (A-B)(\chi_\epsilon - \theta)$, we easily find that $\pi_A = (B-A)\otimes(B-A)\,\nu$ where ν is the scalar H-measure of $\chi_\epsilon - \theta$ which satisfies (11).

An easy computation gives

$$\min_{U\in\Lambda_\xi} Q(\mathcal{A},U) = \frac{\lambda_A+\mu_A}{\mu_A(\lambda_A+2\,\mu_A)}\,\frac{(\mathcal{A}\,\eta\,\xi\cdot\xi)^2}{|\xi|^4} - \frac{1}{\mu_A}\,\frac{|\mathcal{A}\,\eta\,\xi|^2}{|\xi|^2}\ . \tag{32}$$

Thus we obtain

$$\langle\langle \pi_A\cdot \min_{U\in\Lambda_\xi} Q(\mathcal{A},U)\rangle\rangle = \\ = \int_{S_{N-1}}\left(\frac{\lambda_A+\mu_A}{\mu_A(\lambda_A+2\,\mu_A)}\,\frac{\Big((B-A)\,\eta\,\xi\cdot\xi\Big)^2}{|\xi|^4} - \frac{1}{\mu_A}\,\frac{|(B-A)\,\eta\,\xi|^2}{|\xi|^2}\right)\nu(d\xi)\ .$$

Introducing the symmetric fourth-order tensor T defined, for every symmetric matrix $\eta'\in\mathbb{R}^{N^2}$, by

$$T\eta'\cdot\eta' = \frac{1}{\theta(1-\theta)}\int_{S_{N-1}}\left(\frac{\lambda_A+\mu_A}{\mu_A(\lambda_A+2\,\mu_A)}\,\frac{(\eta'\,\xi\cdot\xi)^2}{|\xi|^4} - \frac{1}{\mu_A}\,\frac{|\eta'\,\xi|^2}{|\xi|^2}\right)\nu(d\xi)\ ,$$

we deduce from (30)

$$A_*e(u_0)\cdot e(u_0) + (\overline{A}-A)\eta\cdot\eta - 2\,Ae(u_0)\cdot\eta - Ae(u_0)\cdot e(u_0) + 2\,\overline{A}e(u_0)\cdot\eta \ \geq \\ \geq\ \theta(1-\theta)\,T(B-A)\,\eta\cdot(B-A)\,\eta\ . \tag{33}$$

Finally, minimizing with respect to $e(u_0)$ (which can locally take any value in the set of symmetric matrices), (33) yields

$$(1-\theta)\,(A_*-A)^{-1}\,\eta\cdot\eta \leq (B-A)^{-1}\,\eta\cdot\eta - \theta\,T\,\eta\cdot\eta\ . \tag{34}$$

Remark that

$$T:\ \ I_2\otimes I_2 = \frac{1}{\lambda_A+2\,\mu_A}$$

and

$$T:\ \ \left(I_4 - \frac{1}{N}\,I_2\otimes I_2\right) = \frac{(N-1)\,(\kappa_A+2\,\mu_A)}{2\,\mu_A(\lambda_A+2\,\mu_A)}$$

which imply the desired lower Hashin-Shtrikman bounds if A_* is isotropic.

Upper bound: Developing (28) gives

$$A_\epsilon^{-1}\sigma_\epsilon\cdot\sigma_\epsilon + (A_\epsilon^{-1}-B^{-1})\,\tau\cdot\tau - 2\,B^{-1}\sigma_\epsilon\cdot\tau \ \geq\ B^{-1}\sigma_\epsilon\cdot\sigma_\epsilon - 2\,A_\epsilon^{-1}\sigma_\epsilon\cdot\tau\ . \tag{35}$$

As for the lower bound, we pass to limit in the left hand side of (35) by using homogenization theory and in the right hand side by using H-measures. Introducing another quadratic form

$$Q(V) = q(V)\cdot V = B^{-1}U\cdot U - 2\,\mathcal{A}^{-1}U\cdot\tau \quad \text{ with }\ V = (\mathcal{A}^{-1},U)$$

and denoting by $\underline{A}^{-1}$ the weak limit of A_ϵ^{-1} we obtain the limit of (35)

$$A_*^{-1}\sigma_0\cdot\sigma_0 + (\underline{A}^{-1}-B^{-1})\,\tau\cdot\tau - 2\,B^{-1}\sigma_0\cdot\tau \;\geq\; B^{-1}\sigma_0\cdot\sigma_0 - 2\,\underline{A}^{-1}\sigma_0\cdot\tau + X' , \tag{36}$$

where X' is a new H-correction defined by

$$X' = \int_{S_{N-1}} tr\Big(q\pi(x,d\xi)\Big) \;=\; \langle\langle \pi \cdot Q(\mathcal{A}^{-1},U)\rangle\rangle \ ,$$

with π' the H-measure of $V_\epsilon - V_0$, $(V_\epsilon = (\mathcal{A}_\epsilon^{-1},\sigma_\epsilon),\, V_0 = (\underline{A}^{-1},\sigma_0))$. Since $\operatorname{div}\sigma_\epsilon$ remains in a compact set of $H^{-1}(\Omega)^N$, as in Theorem 2.5, we introduce the characteristic set

$$\Lambda' = \Big\{(\xi,U)\in S_{N-1}\times\mathbb{R}^{N^2},\ U\xi=0\Big\}$$

and its projection

$$\Lambda'_\xi = \Big\{U\in\mathbb{R}^{N^2},\ (\xi,U)\in\Lambda'\Big\} \ .$$

We again define a non-negative quadratic form q' on Λ'_ξ by

$$q'(\mathcal{A}^{-1},U)\cdot(\mathcal{A}^{-1},U) \;=\; q(\mathcal{A}^{-1},U)\cdot(\mathcal{A}^{-1},U) - \min_{U\in\Lambda'_\xi} q(\mathcal{A}^{-1},U)\cdot(\mathcal{A}^{-1},U) \ .$$

Applying Theorem 2.6, we get $tr(q'\pi)\geq 0$. Introducing π'_A, the H-measure associated to $\mathcal{A}_\epsilon^{-1}$, since the quadratic form $\min\limits_{U\in\Lambda'_\xi} Q(\mathcal{A}^{-1},U)$ depends only on $\mathcal{A}^{-1}$, we get

$$X' \;\geq\; \langle\langle \pi\cdot\min_{U\in\Lambda'_\xi} Q(\mathcal{A}^{-1},U)\rangle\rangle \;=\; \langle\langle \pi'_A\cdot\min_{U\in\Lambda'_\xi} Q(\mathcal{A}^{-1},U)\rangle\rangle \ .$$

An easy computation gives

$$\begin{aligned}\min_{\sigma\in\Lambda'_\xi} Q(\mathcal{A}^{-1},\sigma) \;=\; & 2\,\mu_B\,|\mathcal{A}^{-1}\tau|^2 - 4\,\mu_B\,\frac{|\mathcal{A}^{-1}\,\tau\,\xi|^2}{|\xi|^2} + 2\,\mu_B\,\frac{(\mathcal{A}^{-1}\tau\xi\cdot\xi)^2}{|\xi|^4}\\ & - \frac{2\,\mu_B\,\lambda_B}{(\lambda_B+2\,\mu_B)}\left(\frac{\mathcal{A}^{-1}\,\tau\,\xi\cdot\xi}{|\xi|^2} - tr(\mathcal{A}^{-1}\tau)\right)^2 .\end{aligned} \tag{37}$$

On the other hand, since $\mathcal{A}_\epsilon^{-1} - \underline{A}^{-1} = (A^{-1}-B^{-1})\,(\chi_\epsilon - \theta)$, the H-measure π'_A reduces to

$$\pi'_A \;=\; (A^{-1}-B^{-1})\otimes(A^{-1}-B^{-1})\,\nu \ ,$$

where ν is again the H-measure of $\chi_\epsilon - \theta$ which satisfies (11). From (37) we finally obtain

$$\langle\langle \pi'_A \cdot \min_{\sigma\in\Lambda_\xi} Q(\mathcal{A}^{-1}, U)\rangle\rangle \; =$$

$$= \int_{S_{N-1}} \Bigg(2\,\mu_B |(A^{-1}-B^{-1})\tau|^2 - 4\,\mu_B \frac{|(A^{-1}-B^{-1})\,\tau\,\xi|^2}{|\xi|^2} + 2\,\mu_B \frac{\Big((A^{-1}-B^{-1})\,\tau\,\xi\cdot\xi\Big)^2}{|\xi|^4}$$

$$- \frac{2\,\mu_B\,\lambda_B}{(\lambda_B + 2\,\mu_B)} \left(\frac{(A^{-1}-B^{-1})\,\tau\,\xi\cdot\xi}{|\xi|^2} - tr\Big((A^{-1}-B^{-1})\,\tau\Big)\right)^2 \Bigg) \nu(d\xi) \; .$$

We introduce the tensor T' such that for every $\tau' \in \mathbb{R}^{N^2}$

$$T'\tau'\cdot\tau' \; = \; \frac{1}{\theta\,(1-\theta)} \int_{S_{N-1}} \Bigg(2\,\mu_B|\tau'|^2 - 4\,\mu_B \frac{|\tau'\xi|^2}{|\xi|^2} + 2\,\mu_B \frac{(\tau'\xi\cdot\xi)^2}{|\xi|^4}$$

$$- \frac{2\,\mu_B\,\lambda_B}{(\lambda_B + 2\,\mu_B)} \left(\frac{\tau'\xi\cdot\xi}{|\xi|^2} - tr(\tau') \right)^2 \Bigg) \nu(d\xi) \; .$$

Then, minimizing with respect to σ_0, and recalling that $(\underline{A}^{-1} - B^{-1}) = \theta(A^{-1} - B^{-1})$, (36) becomes

$$-\,\theta(A_*^{-1}-B^{-1})^{-1}\,\tau'\cdot\tau' + (A^{-1}-B^{-1})^{-1}\,\tau'\cdot\tau' \;\geq\; (1-\theta)\,T'\tau'\cdot\tau' \; ,$$

with $\tau' = (A^{-1}-B^{-1})\,\tau$. ∎

Remark 4.3. In the non well-ordered case, i.e. in the case

$$\mu_A \leq \mu_B \quad \text{ and } \quad \kappa_B \leq \kappa_A \; ,$$

we obtain the Walpole bounds [30], [7], [19]. Following the method used in the previous proof, we simply replace A in (27) by $C = 2\,\mu_A I_4 + (\kappa_B - \frac{2\,\mu_A}{N})\, I_2 \otimes I_2$ and B in (28) by $D = 2\,\mu_B I_4 + (\kappa_A - \frac{2\,\mu_B}{N})\, I_2 \otimes I_2$. The algebra is the same and the Walpole bounds are given by the inequalities of Remark 4.2 in which κ_A and κ_B are exchanged and λ_A and λ_B are respectively replaced by $\lambda_C = \kappa_B - \frac{2\,\mu_A}{N}$ and $\lambda_D = \kappa_A - \frac{2\,\mu_B}{N}$. □

5 – Coupled bounds in conduction

In order to test the potentiality of the H-measure method, we now address a problem for which a complete answer is yet unknown, namely the case of composite materials that have two different physical properties. For example, one can think of a material which has both a thermal and an electrical conductivity. Of course, there are other possible models like dielectric permittivity, magnetic permeability, or diffusivity, as long as the corresponding state equations are scalar conductivity problems.

As before, these composite materials are obtained by mixing two phases which are assumed to be isotropic for each considered property. The first phase has a thermal conductivity α and an electrical conductivity γ, while the second phase has thermal conductivity β and electrical conductivity δ. A composite material is characterized by a sequence of characteristic functions χ_ϵ for the first phase, which defines two different conductivity tensors

$$A_\epsilon = \alpha\,\chi_\epsilon\,\mathrm{Id} + \beta(1-\chi_\epsilon)\,\mathrm{Id} \quad \text{and} \quad B_\epsilon = \gamma\,\chi_\epsilon\,\mathrm{Id} + \delta(1-\chi_\epsilon)\,\mathrm{Id}\ .$$

According to the homogenization theory of H-convergence, as in (3), for a bounded domain Ω in $\mathbb{R}^N$, there exist a subsequence ϵ, a function $\theta \in L^\infty(\Omega;[0,1])$, and two tensor-valued functions A_* and $B_* \in L^\infty(\Omega)^{N^2}$ such that

$$\begin{aligned} \chi_\epsilon \rightharpoonup \theta &\quad \text{weakly-* in } L^\infty(\Omega)\ ,\\ A_\epsilon &\quad H\text{-converges to } A_*\ ,\\ B_\epsilon &\quad H\text{-converges to } B_*\ . \end{aligned}$$

The couple (A_*, B_*) denotes the two effective conductivity properties of the same composite material. Of course, each tensor A_* or B_* belongs to its corresponding G-closure set as defined in Proposition 3.1. Remark that the two tensors A_ϵ and B_ϵ share the same microstructure which implies that their homogenized limits A_* and B_* are coupled and not totally independent. The goal of this section is to derive bounds on the possible range of the couple (A_*, B_*) which is actually strictly smaller than the product of the two G-closure sets of A_* and B_*. This type of problem has been addressed in many works, see e.g. [4], [5], [6], [15] and references therein. The best bounds have been obtained by Milton in [15]: they are optimal in 2-D and partly optimal in higher space dimensions. The complete G-closure has been obtained by Cherkaev and Gibiansky in 2-D [5]. Therefore, in 3-D there is still something to be found (to be precise, it is conjectured that the upper bound in Figure 1 can be improved). Unfortunately, the H-measure method (or the way we use it) does not help for this open problem. Nevertheless, we feel that this section is instructive with respect to the limitations of the method.

We now introduce some notations. For two right hand sides $f, g \in L^2(\Omega)$, we denote by u_ϵ and v_ϵ the unique solutions in $H^1_0(\Omega)$ of

$$-\operatorname{div} A_\epsilon \nabla u_\epsilon = f \quad \text{and} \quad -\operatorname{div} B_\epsilon \nabla v_\epsilon = g \ \text{ in } \Omega \, .$$

We denote by u_0 and v_0 the weak limits of u_ϵ and v_ϵ in $H^1_0(\Omega)$ which are solutions of

$$-\operatorname{div} A_* \nabla u_0 = f \quad \text{and} \quad -\operatorname{div} B_* \nabla v_0 = g \ \text{ in } \Omega \, .$$

In the sequel of this section, for simplicity, we restrict ourselves to isotropic effective tensors A_* and B_* which are therefore identified with positive real numbers. Our method works also in the anisotropic case but the resulting algebra is more tedious.

5.1. The well-ordered case

We first assume that the two phases have well-ordered properties, namely $0 < \alpha \leq \beta$ and $0 < \gamma \leq \delta$. In the following proposition, by using once again H-measures, we recover coupled bounds obtained by Bergman [3] with the analytical representation method (these bounds are sub-optimal as discussed in Remark 5.2 below).

Theorem 5.1. *Let $\theta \in [0,1]$ be the volume fraction of the first phase with properties (α, γ). We assume that the effective properties A_* and B_*, as well as the second order moment matrix of the H-measure ν of $\chi_\epsilon - \theta$, are isotropic. If $1 \leq \dfrac{\beta}{\alpha} \leq \dfrac{\delta}{\gamma}$, then*

$$f^-(A_*) \leq B_* \leq f^+(A_*)$$

where the curve $f^+(A_*)$ is an hyperbola (see Figure 1) parametrized by ξ in the interval $(1-\theta)/N \leq \xi \leq 1-\theta/N$

$$\begin{aligned} A_* &= \theta\,\alpha + (1-\theta)\,\beta - \frac{\theta\,(1-\theta)\,(\beta-\alpha)^2}{N\Big(\xi\,\alpha + (1-\xi)\,\beta\Big)} \\ f^+(A_*) &= \theta\,\gamma + (1-\theta)\,\delta - \frac{\theta\,(1-\theta)\,(\delta-\gamma)^2}{N\Big(\xi\,\gamma + (1-\xi)\,\delta\Big)} \end{aligned} \tag{38}$$

and the curve $f^-(A_*)$ is also an hyperbola parametrized by η in the interval $(1-\theta)(N-1)/N \le \eta \le 1-\theta(N-1)/N$

$$(39)\qquad \begin{aligned} A_*^{-1} &= \frac{\theta}{\alpha} + \frac{1-\theta}{\beta} - \frac{(N-1)\,\theta(1-\theta)\,(\alpha^{-1}-\beta^{-1})^2}{N\Big(\eta\,\alpha^{-1} + (1-\eta)\,\beta^{-1}\Big)} \\ f^-(A_*)^{-1} &= \frac{\theta}{\gamma} + \frac{1-\theta}{\delta} - \frac{(N-1)\,\theta(1-\theta)\,(\gamma^{-1}-\delta^{-1})^2}{N\Big(\eta\,\gamma^{-1} + (1-\eta)\,\delta^{-1}\Big)}\ . \end{aligned}$$

If $1 \le \dfrac{\delta}{\gamma} \le \dfrac{\beta}{\alpha}$, *then*

$$f^+(A_*) \le B_* \le f^-(A_*)$$

with the same definitions (39) and (38) of f^- and f^+.

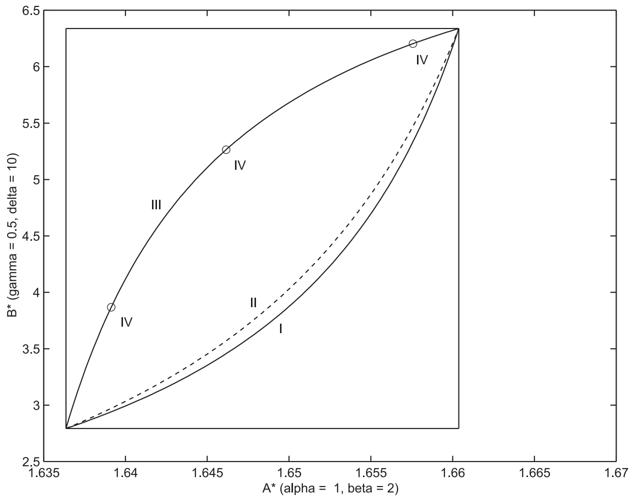

Fig. 1 : Coupled bounds in dimension $N = 3$. Proportion $\theta = 0.3$.
Curves I and III: lower and upper Bergman bounds obtained by using H-measures.
Curve II: Milton optimal lower bound attained by laminates.
Points IV: optimal points obtained by a Schulgasser construction.

Remark 5.2. For the extremal values of η and ξ, one recovers in (39) and (38) the upper and lower Hashin-Shtrikman bounds on an isotropic composite. In other words, these two curves pass through the Hashin-Shtrikman points $(A_*, B_*) = (hs^+(\theta,\alpha,\beta), hs^+(\theta,\gamma,\delta))$, $\quad (A_*, B_*) = (hs^-(\theta,\alpha,\beta), hs^-(\theta,\gamma,\delta))$, where hs^+ and hs^- are the upper and lower Hashin-Shtrikman bounds on an isotropic composite defined in Remark 3.3.

The bound $f^-(A_*)$ is not optimal and has been improved by Milton [15] (his optimal bound is attained by two nested rank-N laminations).

In dimension $N \geq 3$, the other bound $f^+(A_*)$ was shown by Milton [16] to be optimal for five values of ξ, namely $(1-\theta)/N$, $(1-\theta)/(N-1)$, $(1-\theta)$, $1-\theta/(N-1)$, $1-\theta/N$ (using a construction of Schulgasser which does not make sense if $N \leq 2$ [24]). See Figure 1. □

Proof: The main idea is to couple a lower bound for A_ϵ and an upper bound for B_ϵ (or equivalently a lower bound for B_ϵ^{-1}). Let a, b, c be three non-negative real numbers such that the following tensor is non-negative

$$\begin{pmatrix} A_\epsilon - a\mathrm{Id} & c\mathrm{Id} \\ c\mathrm{Id} & B_\epsilon^{-1} - b^{-1}\mathrm{Id} \end{pmatrix} \geq 0 \ . \tag{40}$$

The above condition of positivity is equivalent to

$$a \leq \alpha\,, \quad b \geq \delta\,, \quad (\alpha - a)\,(\gamma^{-1} - b^{-1}) \geq c^2\,, \quad (\beta - a)\,(\delta^{-1} - b^{-1}) \geq c^2\ . \tag{41}$$

For given c, we choose a, b in order to saturate the last two inequalities in (41). Computing a and b in terms of c yields

$$0 \leq a = \frac{1}{2}\left(\alpha + \beta - \sqrt{(\beta-\alpha)^2 + 4\,c^2\,\frac{\beta - \alpha}{\gamma^{-1} - \delta^{-1}}}\right) \leq \alpha\ , \tag{42}$$

and

$$0 \leq b^{-1} = \frac{1}{2}\left(\gamma^{-1} + \delta^{-1} - \sqrt{(\gamma^{-1} - \delta^{-1})^2 + 4\,c^2\,\frac{\gamma^{-1} - \delta^{-1}}{\beta - \alpha}}\right) \leq \gamma^{-1}\ , \tag{43}$$

if and only if

$$0 \leq c \leq c_{\text{lim}} = \min\left(c_1 = \sqrt{\frac{\gamma^{-1} - \delta^{-1}}{\alpha^{-1} - \beta^{-1}}},\ c_2 = \sqrt{\frac{\alpha - \beta}{\gamma - \delta}}\right)\,. \tag{44}$$

In the sequel, we choose $c = c_{\text{lim}}$ which gives the best results. Following the method used in the proof of Proposition 3.1 (for details, see the extended version [1] of the present paper), we pass to the limit in inequality (40) and we obtain:

$$\begin{pmatrix} A_* - p & r \\ r & B_*^{-1} - q^{-1} \end{pmatrix} \geq 0\ , \tag{45}$$

where p, q, r depend explicitely on $\theta, \alpha, \beta, \gamma, \delta$. Assuming that A_* and B_* are isotropic, the coupled bound (45) can easily be interpreted. Indeed, in such a case it says that a 2×2 matrix is non-negative which is true if and only if its trace and determinant are non-negative. The trace of (45) will give no new information since it does not involve the coupling factor c. On the other hand, the determinant of (45) yields an upper bound which is an hyperbola in the plane (A_*, B_*)

$$B_* \leq F^+_\theta(A_*) = \frac{1}{q^{-1} + \dfrac{r^2}{A_* - p}} . \tag{46}$$

Finally, if we change A_ϵ in A_ϵ^{-1} and B_ϵ^{-1} in B_ϵ in the tensor (40), or equivalently if we invert the roles of (α, β) and (γ, δ), the same argument yields a lower bound

$$A_* \leq F^-_\theta(B_*) = \frac{1}{\tilde{q}^{-1} + \dfrac{\tilde{r}^2}{B_* - \tilde{p}}} \tag{47}$$

where the coefficients $(\tilde{p}, \tilde{q}, \tilde{r})$ are the same functions of $\theta, \alpha, \beta, \gamma, \delta$ than the previous coefficients (p, q, r) except that (α, β) has been exchanged with (γ, δ).

By doing some algebra, we prove that (46) and (47) coincide with the bounds obtained by Bergman [4]. This rigorously shows that the two bounds actually pass through the Hashin-Shtrikman points. If $1 \leq \beta/\alpha \leq \delta/\gamma$, after some tedious but simple algebra, the upper bound (46) is equivalent to

$$\frac{\beta}{\beta - \alpha} - \frac{\theta\,(1-\theta)\,(\beta - \alpha)}{N(\overline{A} - A_*)} \leq \frac{\delta}{\delta - \gamma} - \frac{\theta\,(1-\theta)\,(\delta - \gamma)}{N(\overline{B} - B_*)} , \tag{48}$$

while the lower bound (47) is

$$\begin{aligned} \frac{\beta^{-1}}{\alpha^{-1} - \beta^{-1}} - \frac{(N-1)\,\theta(1-\theta)\,(\alpha^{-1} - \beta^{-1})}{N(\underline{A}^{-1} - A_*^{-1})} &\leq \\ \leq \frac{\delta^{-1}}{\gamma^{-1} - \delta^{-1}} - \frac{(N-1)\,\theta(1-\theta)\,(\gamma^{-1} - \delta^{-1})}{N(\underline{B}^{-1} - B_*^{-1})} &. \end{aligned} \tag{49}$$

If $1 \leq \delta/\gamma \leq \beta/\alpha$, another tedious computation shows that the upper bound (46) is now equivalent to (49), while the lower bound (47) is equivalent to (48). Therefore, the bounds are the same than in the case $1 \leq \beta/\alpha \leq \delta/\gamma$, except that the upper bound becomes the lower one and vice-versa. Finally, it is easily seen that the curve defined by equality in (48) can be parametrized by (38). Similarly, equality in (49) is parametrized by (39). ∎

Remark 5.3. As any other method for computing bounds, the H-measure method is subject to some arbitrary choices and computational difficulties. For example, the construction of the tensor in inequality (40) in which we passed to the limit is totally arbitrary. We tried to used others (matrices of) quadratic functionals (or other coupling terms instead of $c\mathrm{Id}$ which gave non trivial contributions in the minimization of Q) but either the resulting bound was worse or was too complicated to be simplified and useful. □

5.2. The non well-ordered two-dimensional case

We now assume that the two phases are not well-ordered, namely $0 < \alpha \leq \beta$ and $0 < \delta \leq \gamma$. We restrict ourselves to the two-dimensional setting. We obtain coupled bounds which improve those obtained by Bergman [3] but are less tight than the optimal ones of Milton [15], Cherkaev and Gibianski [5] (attained by laminates).

Theorem 5.4. *Let $\theta \in [0,1]$ be the volume fraction of the first phase with properties (α,γ). We assume that the effective properties A_* and B_*, as well as the second order moment matrix of the H-measure ν of $\chi_\epsilon - \theta$, are isotropic. If $1 \leq \dfrac{\beta}{\alpha} \leq \dfrac{\gamma}{\delta}$, then*

$$g^-(A_*) \leq B_* \leq g^+(A_*)$$

where the curve $g^+(A_*)$ is an hyperbola parametrized by ξ in the interval $(1-\theta)/2 \leq \xi \leq 1-\theta/2$

$$\begin{aligned} A_*^{-1} &= \frac{\theta}{\alpha} + \frac{1-\theta}{\beta} - \frac{\theta\,(1-\theta)\,(\alpha^{-1}-\beta^{-1})^2}{2\,(\xi\,\alpha^{-1} + (1-\xi)\,\beta^{-1})} \\ g^+(A_*) &= \theta\,\gamma + (1-\theta)\,\delta - \frac{\theta\,(1-\theta)\,(\delta-\gamma)^2}{2\,(\xi\,\gamma + (1-\xi)\,\delta)} \end{aligned} \tag{50}$$

and the curve $g^-(A_*)$ is also an hyperbola defined by the same parameter ξ

$$\begin{aligned} A_* &= \theta\,\alpha + (1-\theta)\,\beta - \frac{\theta\,(1-\theta)\,(\beta-\alpha)^2}{2\,(\xi\,\alpha + (1-\xi)\,\beta)} \\ g^-(A_*)^{-1} &= \frac{\theta}{\gamma} + \frac{1-\theta}{\delta} - \frac{\theta\,(1-\theta)\,(\gamma^{-1}-\delta^{-1})^2}{2\,(\xi\,\gamma^{-1} + (1-\xi)\,\delta^{-1})}\,. \end{aligned} \tag{51}$$

If $1 \leq \dfrac{\gamma}{\delta} \leq \dfrac{\beta}{\alpha}$, *then*

$$g^+(A_*) \leq B_* \leq g^-(A_*)$$

with the same definitions (51) and (50) of g^- *and* g^+.

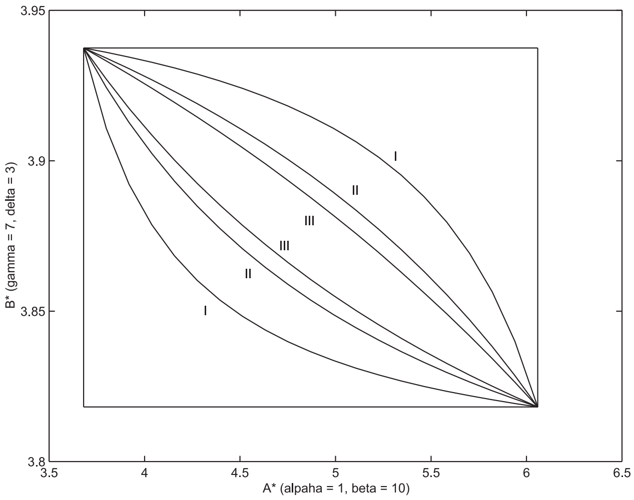

Fig. 2: Coupled bounds in the 2-D non well-ordered case. Proportion $\theta = 0.3$.
Curve I: Bergman bounds.
Curve II: bounds obtained by using H-measures.
Curve III: Milton optimal bounds (attained by laminates).

Corollary 5.5. *Let* A_* *be a composite mixture of two phases* α *and* β *and* B_* *be the composite mixture of phases (proportional to)* β *and* α. *In other words let* $\alpha, \beta, \gamma, \delta$ *satisfy*

$$\beta/\alpha \;=\; \gamma/\delta\;. \tag{52}$$

Then the lower and upper coupled bounds coincide and the G*-closure reduces to a single curve (see Figure 3). In particular, we recover the well-known Dykne-Keller phase interchange equality*

$$A_* B_* \;=\; \alpha\,\gamma \;=\; \beta\,\delta\;. \tag{53}$$

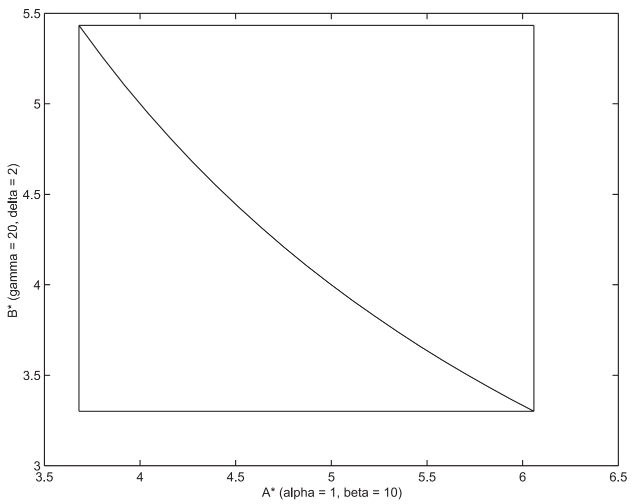

Fig. 3 : G-closure curve in the case $\beta/\alpha = \gamma/\delta$. Proportion $\theta = 0.3$.

Remark 5.6. In the two dimensional case, we recover the bounds of Theorem 5.1 from those of Theorem 5.4 by using the phase interchange equality (53). Indeed, let A_*, B_* and C_* be three composite materials respectively associated to the three pairs of isotropic phases (α, β), (γ, δ) and (δ, γ) with $\alpha \leq \beta$ and $\gamma \geq \delta$. The couple (A_*, B_*) is non well-ordered, but (A_*, C_*) is actually well-ordered. We have

$$C_* B_* = \delta\,\gamma\ ,$$
$$g^-(A_*) \leq B_* \leq g^+(A_*) \quad \text{and} \quad f^-(A_*) \leq C_* \leq f^+(A_*)\ .$$

A tedious computation shows that $\gamma\,\delta = g^-(A_*)\,f^+(A_*) = g^+(A_*)\,f^-(A_*)$. □

Proof of Theorem 5.4: We denote by R the orthogonal rotation matrix of $\mathbb{R}^2$, i.e.

$$R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The main idea is to couple two lower bounds for A_ϵ and B_ϵ. Let a, b, c be three non-negative real numbers such that the following tensor is non-negative

$$\begin{pmatrix} A_\epsilon - a\,\mathrm{Id} & c\,R \\ c\,R^t & B_\epsilon - b\,\mathrm{Id} \end{pmatrix} \geq 0\ . \tag{54}$$

The above condition of positivity is equivalent to

$$a \leq \alpha\,, \quad b \leq \delta\,, \quad (\alpha - a)\,(\gamma - b) \geq c^2\,, \quad (\beta - a)\,(\delta - b) \geq c^2\ . \tag{55}$$

For given c, we choose a, b in order to saturate the last two inequalities in (55). Computing a and b in terms of c yields

$$0 \le a = \frac{1}{2}\left(\alpha+\beta-\sqrt{(\beta-\alpha)^2+4\,c^2\,\frac{\beta-\alpha}{\gamma-\delta}}\right) \le \alpha\ , \tag{56}$$

$$0 \le b = \frac{1}{2}\left(\gamma+\delta-\sqrt{(\gamma-\delta)^2+4\,c^2\,\frac{\gamma-\delta}{\beta-\alpha}}\right) \le \delta\ , \tag{57}$$

if and only if

$$0 \le c \le c_{\lim} = \min\left(c_1 = \sqrt{\frac{\gamma-\delta}{\alpha^{-1}-\beta^{-1}}},\, c_2 = \sqrt{\frac{\beta-\alpha}{\delta^{-1}-\gamma^{-1}}}\right). \tag{58}$$

In the sequel we choose $c = c_{\lim}$ which gives the best results. The positivity condition (54) now leads to

$$\begin{pmatrix} A_* - p' & r' \\ r' & B_* - q' \end{pmatrix} \ge 0\ . \tag{59}$$

The sign of $A_* - p'$ is always positive. Thus, taking the determinant of (59), we deduce a lower bound in the plane (A_*, B_*)

$$B_* \ge F_\theta'^{-}(A_*) = q' + \frac{r'^2}{A_* - p'}\ . \tag{60}$$

Instead of starting from two coupled lower bounds for A_ϵ and B_ϵ as in (54), if we start from two coupled lower bounds for A_ϵ^{-1} and B_ϵ^{-1}, we obtain a symmetric upper bound. In 2-D, since by application of the rotation matrix R a gradient becomes a divergence-free vector, this is equivalent to change $(\alpha, \beta, \gamma, \delta)$ in $(1/\alpha, 1/\beta, 1/\gamma, 1/\delta)$ in the lower bound to get the upper bound

$$B_*^{-1} \ge F_\theta'^{+}(A_*^{-1}) = \tilde{q}' + \frac{\tilde{r}'^2}{A_*^{-1} - \tilde{p}'}\ , \tag{61}$$

where the coefficients satisfy

$$(\tilde{p}', \tilde{q}', \tilde{r}')\,(\theta, \alpha, \beta, \gamma, \delta) = (p', q', r')\,(\theta,\, 1/\alpha,\, 1/\beta,\, 1/\gamma,\, 1/\delta)\ .$$

If $1 \le \dfrac{\beta}{\alpha} \le \dfrac{\gamma}{\delta}$, after some computation the lower bound (60) is equivalent to

$$\frac{\delta^{-1}}{\delta^{-1}-\gamma^{-1}} - \frac{\theta\,(1-\theta)\,(\delta^{-1}-\gamma^{-1})}{2\,(\underline{B}^{-1}-B_*^{-1})} \ge \frac{\beta}{\beta-\alpha} - \frac{\theta\,(1-\theta)\,(\beta-\alpha)}{2(\overline{A}-A_*)} \tag{62}$$

while the upper bound (61) is given by

$$(63)\qquad \frac{\delta}{\gamma-\delta}-\frac{\theta\,(1-\theta)\,(\gamma-\delta)}{2\,(\overline{B}-B_*)}\geq\frac{\beta^{-1}}{\alpha^{-1}-\beta^{-1}}-\frac{\theta\,(1-\theta)\,(\alpha^{-1}-\beta^{-1})}{2\,(\underline{A}^{-1}-A_*^{-1})}\ .$$

If $1\leq\dfrac{\gamma}{\delta}\leq\dfrac{\beta}{\alpha}$, (60) is now equivalent to (63) while (61) is equivalent to (62). As for the well ordered case, the curve defined by (62) (respectively (63)) can be parametrized by (51) (respectively (50)). This proves in particular that the lower bound (60), as well as the upper bound (61), always pass through the Walpole points.

Remark 5.7. When $1\leq\beta/\alpha\leq\gamma/\delta$, an easy comparison between (48) and (63), and between (49) and (62) shows that the bounds obtained by using H-measures improve the bounds obtained by Bergman. The same is true when $1\leq\gamma/\delta\leq\beta/\alpha$. □

Proof of Corollary 5.5: If equality (52) holds true, taking $c=c_{\lim}=\sqrt{\alpha\,\beta}$ yields $a=b=0$ and $p'=q'=0$ and $r'=c$ (see [1]). Taking the determinant of the lower bound (59) gives an inequality in (53). The converse inequality is obtained by considering the upper bound. ∎

Remark 5.8. In higher dimensions (i.e. in dimension $N\geq 3$) the same analysis for the lower bound can be performed by replacing the rotation R by a combination of plane rotations for which the orthogonality of U and V rotated still holds true. However, the main difference is that $R^2\neq -\mathrm{Id}$ and the inverse of R is more complicated than in dimension $N=2$. Unfortunately, this yields a different and worse bound. For example, in the setting of Corollary 5.5, i.e. $\beta/\alpha=\gamma/\delta$, we obtain the following phase interchange inequality (first obtained by Schulgasser)

$$A_*B_*\geq\alpha\,\gamma=\beta\,\delta\ ,$$

which is worse than that conjectured by Milton [15] and proved in [2], [23]

$$\frac{A_*B_*}{\alpha\,\beta}+(N-2)\,\frac{A_*+B_*}{\alpha+\beta}\ \geq\ N-1\ .$$

Therefore, we must admit that we do not know how to extend the 2-D result in higher space dimension in order to derive bounds that pass through the Hashin-Shtrikman points. □

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