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A NOTE ON THE HOLOMORPHIC INVARIANTS OF TIAN–ZHU *

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In this short note, we compute the holomorphic invariants defined by Tian and Zhu [4] on smooth hypersurfaces of CP^n. The holomorphic invariants, which generalize the famous Futaki invariants [1], are obstructions towards the existence of Kähler–Ricci solitons.

For a Kähler manifold with the first positive Chern class, the existence of the Kähler–Ricci soliton can be reduced to the existence of the solution of a nonlinear equation of Monge–Ampere type. In general, solving such an equation is highly nontrivial. Similarly to the Futaki invariants, the Tian–Zhu invariants give the obstruction *before* one needs to solve the equation. It is thus very important to compute it concretely. In this paper, in the case of hypersurfaces, we give an explicit formula.

Let $M \subset CP^n$ be a smooth hypersurface defined by a homogeneous polynomial $F=0$ of degree d. Let v and X be two holomorphic vector fields on CP^n. For the sake of simplicity, we assume that

$$v = \sum_{i=0}^n v^i Z_i \frac{\partial}{\partial Z_i} \quad \text{and} \quad X = \sum_{i=0}^n X^i Z_i \frac{\partial}{\partial Z_i} ,$$

where $[Z_0, ..., Z_n]$ is the homogeneous coordinate of CP^n, $(v^0, ..., v^n) \in C^{n+1}$, $(X^0, ..., X^n) \in C^{n+1}$. We further assume that

$$\sum_{i=0}^n v^i = 0 , \quad \sum_{i=0}^n X^i = 0 . \tag{1}$$

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If v and X are tangent vector fields of M, then there are complex numbers λ and κ such that

$$vF = \kappa F\,, \qquad XF = \lambda F\,. \tag{2}$$

Let ω be the Kähler form of the Fubini–Study metric of CP^n. Then $(n-d+1)\,\omega$ restricts to a representative of the first Chern class $c_1(M)$ of M. Thus there is a smooth function ξ on M such that

$$\operatorname{Ric}\Big((n-d+1)\omega|_M\Big) - (n-d+1)\omega|_M = \partial\overline{\partial}\xi\,.$$

For fixed holomorphic vectors X and v, the holomorphic invariant defined by Tian–Zhu [4], in our context, is

$$F_X(v) = (n-d+1)^{n-1}\int_M v\Big(\xi - (n-d+1)\theta_X\Big)\, e^{(n-d+1)\theta_X}\,\omega^{n-1}\,, \tag{3}$$

where θ_X is defined as

$$\begin{cases} i(X)\omega = \dfrac{\sqrt{-1}}{2\pi}\,\overline{\partial}\theta_X\,, \\ \displaystyle\int_M e^{(n-d+1)\theta_X}\,\omega^{n-1} = d\,. \end{cases} \tag{4}$$

The main property of the Tian–Zhu invariants is the following (cf. [4]):

Theorem 1. *Let $F_X(v)$ be the Tian–Zhu invariant. Then we have*

1. *If the Kähler-Ricci soliton exists, that is, we have*

$$\operatorname{Ric}(\omega) - \omega = L_X\omega$$

for some Kähler metric ω. Then $F_X(v) \equiv 0$.

2. *$F_X(v)$ is independent of the choice of the Kähler metric ω within the first Chern class.*

In this note, we give a "computable" expression of $F_X(v)$. Our main result is as follows:

Theorem 2. *Using the notations as above, defined the function*

$$\varphi(X) = \sum_{k=0}^{\infty} \frac{n!\,(n-d+1)^k}{(n+k)!} \sum_{\alpha_0+\cdots+\alpha_n=k} X_0^{\alpha_0}\cdots X_n^{\alpha_n}\,, \tag{5}$$

where $\alpha_0, ..., \alpha_n \in \mathbb{Z}^{n+1}$ are nonnegative integers. Let

$$\sigma(X) \;=\; \Big(-\frac{\lambda(n-d+1)}{n} + d\Big)\varphi(X) + \frac{d}{n}\sum_{i=0}^{n} X^i \,\frac{\partial\varphi(X)}{\partial X^i}\;. \tag{6}$$

Then the invariants defined by Tian–Zhu can be explicitly expressed as

$$F_X(v) \;=\; -(n-d+1)^{n-1}\, d\left(\kappa + \sum_{i=0}^{n} v^i\, \frac{\partial \log\sigma(X)}{\partial X^i}\right)\;. \tag{7}$$

Corollary 1. *The Futaki invariant for the hypersurface M is*

$$F(v) \;=\; -(n-d+1)^{n-1}\,\frac{(n+1)\,(d-1)}{n}\,\kappa\;. \;\blacksquare$$

The rest of this note is devoted to the proof Theorem 2. We define

$$\tilde\theta_X \;=\; \frac{\lambda_0|Z_0|^2+\cdots+\lambda_n|Z_n|^2}{|Z_0|^2+\cdots+|Z_n|^2}\;. \tag{8}$$

Then we have

$$i(X)\omega = \overline{\partial}\tilde\theta_X\;. \tag{9}$$

By comparing the above equation with (4), we have

$$\theta_X = \tilde\theta_X + c_X \tag{10}$$

for a constant c_X. First, we have the following lemma

Lemma 1.

$$\int_{CP^n} e^{(n-d+1)\tilde\theta_X}\,\omega^n \;=\; \varphi(X)\;,$$

where $\varphi(X)$ is defined in (5).

Proof: This follows from the expansion

$$e^{(n-d+1)\tilde\theta_X} = \sum_{k=0}^{\infty}\frac{(n-d+1)^k}{k!}\,\tilde\theta_X^k\;,$$

and the elementary Calculus. $\blacksquare$

Lemma 2. *Using the same notation as above, we have*

$$F_X(v) \;=\; (n-d+1)^{n-1}\Big(-\kappa d - \int_M (n-d+1)\,\theta_v\, e^{(n-d+1)\theta_X}\,\omega^{n-1}\Big)\ .$$

Proof: By [3, Theorem 4.1], we have

$$\operatorname{div} v + v(\xi) + (n-d+1)\theta_v = -\kappa\ ,$$

where θ_v is the function on CP^n defined by

$$\theta_v = \frac{v_0|Z_0|^2+\cdots+v_n|Z_n|^2}{|Z_0|^2+\cdots+|Z_n|^2}\ ,$$

and κ is defined in (2). Then (3) becomes

$$\begin{aligned} F_X(v) \;&=\; (n-d+1)^{n-1} \\ &\cdot\Big(\int_M\Big(-\kappa-\operatorname{div} v-(n-d+1)\theta_v-(n-d+1)\,v(\theta_X)\Big)\,e^{(n-d+1)\theta_X}\,\omega^{n-1}\Big). \end{aligned} \tag{11}$$

We also have

$$\operatorname{div}\,(e^{(n-d+1)\theta_X}v) \;=\; e^{(n-d+1)\theta_X}\Big(\operatorname{div} v + (n-d+1)\,v(\theta_X)\Big)\ . \tag{12}$$

The lemma follows from (4), (11), (12) and the divergence theorem. ∎

The following key lemma transfers the integration on M to the integrations on CP^n.

Lemma 3.

$$(n-d+1)\int_M \theta_v\, e^{(n-d+1)\theta_X}\,\omega^{n-1} \;=\; d\sum_{i=0}^{n} v^i\,\frac{\partial\log\sigma}{\partial X^i}\ , \tag{13}$$

where $\sigma(X)$ is defined in (6).

Proof: Let

$$\eta \;=\; \log\frac{|F|^2}{\Big(|Z_0|^2+\cdots+|Z_n|^2\Big)^d}\ . \tag{14}$$

Then η is a smooth function on CP^n outside M. We have the following identity:

$$\begin{aligned} &\overline{\partial}\Big(e^{(n-d+1)\theta_X}\,\partial\eta\wedge\omega^{n-1}\Big)-\frac{n-d+1}{n}\,i(X)\,\Big(e^{(n-d+1)\theta_X}\,\partial\eta\wedge\omega^{n}\Big) \;= \\ &\quad=\; -e^{(n-d+1)\theta_X}\,\partial\overline{\partial}\eta\wedge\omega^{n-1}-\frac{n-d+1}{n}\,e^{(n-d+1)\theta_X}\,(\lambda-d\tilde{\theta}_X)\,\omega^n\ . \end{aligned} \tag{15}$$

Since on CP^n, there are no $(2n+1)$ forms, the left hand side of the above equation is the divergence of some vector field. Integrate the equation on both side and use the divergence theorem, we have

$$\int_{CP^n} e^{(n-d+1)\theta_X}\,\partial\overline{\partial}\eta\wedge\omega^{n-1} = -\frac{n-d+1}{n}\int_{CP^n}(\lambda - d\tilde{\theta}_X)\,e^{(n-d+1)\theta_X}\,\omega^n\ . \tag{16}$$

By [2, page 388], in the sense of currents, we have

$$\partial\overline{\partial}\eta = [M] - d\omega\ . \tag{17}$$

Thus from (16),

$$\begin{aligned}\int_M e^{(n-d+1)\theta_X}\,\omega^{n-1} &= \left(-\frac{\lambda(n-d+1)}{n} + d\right)\int_{CP^n} e^{(n-d+1)\theta_X}\,\omega^n \\ &\quad + \frac{d(n-d+1)}{n}\int_{CP^n}\tilde{\theta}_X e^{(n-d+1)\theta_X}\,\omega^n\ .\end{aligned} \tag{18}$$

From Lemma 1, we have

$$\sum_{i=0}^{n} X^i\,\frac{\partial\varphi(X)}{\partial X^i} = (n-d+1)\int_{CP^n}\tilde{\theta}_X\,e^{(n-d+1)\tilde{\theta}_X}\,\omega^n\ . \tag{19}$$

By (10), (18) and (19)

$$\int_M e^{(n-d+1)\theta_X}\,\omega^{n-1} = \sigma(X)\,e^{cX}\ . \tag{20}$$

From the above equation, we have

$$(n-d+1)\int_M \theta_v\,e^{(n-d+1)\theta_X}\,\omega^{n-1} = \sum_{i=0}^{n} v^i\,\frac{\partial\sigma(X)}{\partial X^i}\,e^{cX}\ . \tag{21}$$

On the other hand, from (20), we have

$$d = \sigma(X)\,e^{cX}\ , \tag{22}$$

by (4). Lemma 3 follows from (21) and (22). ∎

Theorem 2 follows from Lemma 2 and Lemma 3.

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