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GENERAL EXISTENCE RESULTS FOR SECOND ORDER NONCONVEX SWEEPING PROCESS WITH UNBOUNDED PERTURBATIONS *

Messaoud Bounkhel °

Abstract: This paper is devoted to study the existence of solutions for general second order sweeping processes with perturbations of the form $\dot{x}(t) \in K(x(t))$, $\ddot{x}(t) \in -N(K(x(t)); \dot{x}(t)) + F(t, x(t), \dot{x}(t)) + G(t, x(t), \dot{x}(t))$, where K is a nonconvex set-valued mapping with compact values, F is an unbounded scalarly upper semicontinuous convex set-valued mapping, and G is an unbounded continuous non convex set-valued mapping taking their values in separable Hilbert spaces.

1 – Introduction

The existence of solutions for the second order differential inclusion

$$\text{(SDI)} \qquad \ddot{x}(t) \in G\Big(t, x(t), \dot{x}(t)\Big)$$

has been studied by many authors (see for example [1, 7, 8, 15, 17, 18, 22]). In [7], Castaing studied for the first time the existence problem for the following particular type of second order differential inclusions

$$\text{(SSP)} \qquad \ddot{x}(t) \in -N\Big(K(x(t)); \dot{x}(t)\Big) \quad \text{and} \quad \dot{x}(t) \in K(x(t)) \ ,$$

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where K is a convex set-valued mapping with compact values. Many papers (for example [7, 8, 17, 22]) studied since this particular problem. The general problem (SDI) has been treaded in several ways. For instance, the authors in [1] solved the problem when G takes the following particular type: $G(t,x(t),\dot{x}(t)) = \gamma\,\dot{x}(t) + \partial f(x(t))$, where $\gamma > 0$ and f is a lower semicontinuous convex function. Their motivations come from a mechanical problem that they called the heavy ball problem with friction. For more details we refer the reader to [1] and the references therein. In [4], the authors studied the following particular problem of (SDI)

$$\text{(SSPP1)} \qquad \ddot{x}(t) \in -N\Big(K(x(t));\dot{x}(t)\Big) + F(t,\dot{x}(t)) \;.$$

They proved several existence results when $K\colon\, H \rightrightarrows H$ is nonconvex set-valued mapping with compact values, H is a finite dimensional space, and the perturbation $F:\; [0,+\infty[\times H \rightrightarrows H$ is bounded with convex values. Their proofs are strongly based upon the fixed point theorems and some new existence results by [6] for first order sweeping processes. They also proved existence results for another particular problem of (SDI)

$$\text{(SSPP2)} \qquad \ddot{x}(t) \in -N\Big(K(x(t));\dot{x}(t)\Big) + F(t,x(t)) \;,$$

when K is a nonconvex set-valued mapping with compact values, H is a separable Hilbert space, and F is a nonconvex continuous set-valued mapping. Note that the problem (SSPP2) with memory has been studied in [15] when K is a convex set-valued mapping with compact values.

Our aim in this paper is to prove existence results for the following general problem

$$\text{(SSPMP)} \quad \ddot{x}(t) \in -N\Big(K(x(t));\dot{x}(t)\Big) + F\Big(t,x(t),\dot{x}(t)\Big) + G\Big(t,x(t),\dot{x}(t)\Big) \;,$$

where K is a nonconvex set-valued mapping with compact values, H is a separable Hilbert space, F is a scalarly upper semicontinuous convex set-valued mapping, and G is a nonconvex continuous set-valued mapping. This general problem covers all the problems studied before and mentioned above. We will call it the Second order Sweeping Process with Mixed Perturbations (in short (SSPMP)).

This paper is organized as follows. In section 2, we recall some definitions and prove some useful results that will be needed in all the paper. In Section 3 we prove our main existence theorems. We start with a general existence result

(Theorem 3.1) of (SSPMP), when K is assumed to be contained in a convex compact set in H, the perturbation F is globally scalarly upper semicontinuous with convex values, and G is nonconvex continuous, and both F and G satisfy the linear growth condition. The main difficulties we met in the paper and in particular in the proof of Theorem 3.1 is the nonconvexity of the set-valued mapping K and G. To overcome those difficulties posed by the nonconvexity of K, we use some new techniques developed by Bounkhel and Thibault in [6] for first order sweeping processes and used later by [4] for second order sweeping processes without perturbations. We adapt the techniques used in [15] to overcome the difficulties posed by the nonconvexity of G. In Theorem 3.2 and Theorem 3.3 we will be interested with the case when the assumption "K is contained in a convex compact set in H" is replaced by "K is bounded". An existence result for such case is proved under the following additional assumptions: K is anti-monotone, G satisfies a weak linear growth condition, and F is either monotone with respect to the third variable or satisfies a weak linear growth condition (see Theorems 3.2–3.3 for such condition). The proofs of those theorems are based strongly upon new properties of uniformly prox-regular sets proved in [6]. The result of Theorems 3.2–3.3 cannot be covered by Theorem 3.1 because the compactness assumption on K cannot be distorted in the proof of Theorem 3.1. In Section 4 we prove existence results for (SSPCP) (the Second Order Sweeping Process with a Convex Perturbation F) when the perturbation F is assumed to be globally measurable and only upper semicontinuous with respect to the second and the third variables. The idea of the proof is based on an approximation method. We approximate the set-valued mapping F by a sequence of globally u.s.c. set-valued mapping F_n and we study the convergence of the solutions x_n of $(\mathrm{SSPCP})_n$ associated with F_n (the existence of such solutions is ensured by our results in Section 3). In Section 5 we study the compactness and the closedness of the solution sets of (SSPCP). Section 6 is reserved for a particular case of (SSPMP) when the perturbation F is defined in terms of the subdifferential of Lipschitz functions.

2 – Preliminaries

Throughout the paper H will denote a *real separable Hilbert space.*

Let S be a closed subset of H. We denote by $d_S(\cdot)$ or $d(\cdot, S)$ the usual distance function to S, i.e., $d_S(x) := \inf\limits_{u \in S} \|x - u\|$. We need first to recall some notation and definitions that will be used in all the paper.

Let $f\colon H \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous (l.s.c.) function and let x be any point where f is finite. We recall that *the proximal subdifferential* $\partial^P f(x)$ is the set of all $\xi \in H$ for which there exist $\delta, \sigma > 0$ such that for all $x' \in x + \delta\,\mathbb{B}$

$$\langle \xi, x' - x\rangle \;\leq\; f(x') - f(x) + \sigma\|x' - x\|^2 \;.$$

Here $\mathbb{B}$ denotes the closed unit ball centered at the origin of H.

By convention we set $\partial^P f(x) = \emptyset$ if $f(x)$ is not finite. Note that $\partial^P f(x)$ is always convex but may be non closed.

Let S be a nonempty closed subset of H and x be a point in S. We recall (see [14]) that *the proximal normal cone* of S at x is defined by $N^P(S;x) := \partial^P \psi_S(x)$, where ψ_S denotes the indicator function of S, i.e., $\psi_S(x') = 0$ if $x' \in S$ and $+\infty$ otherwise. Note that the proximal normal cone is also given by

$$N^P(S;x) \;=\; \Big\{\xi \in H\colon\; \exists\, \alpha > 0 \;\text{ s.t. } x \in \operatorname{Proj}(x + \alpha\xi, S)\Big\}$$

where

$$\operatorname{Proj}(u,S) \;:=\; \Big\{y \in S\colon\; d_S(u) = \|u - y\|\Big\}\;.$$

Recall now that for a given $r \in\,]0,+\infty]$ a subset S is uniformly r-prox-regular (see [19]) or equivalently r-proximally smooth (see [14]) if and only if every nonzero proximal normal to S can be realized by an r-ball, this means that for all $\bar{x} \in S$ and all $0 \neq \xi \in N^P(S;\bar{x})$ one has

$$\left\langle \frac{\xi}{\|\xi\|},\, x - \bar{x}\right\rangle \;\leq\; \frac{1}{2r}\,\|x - \bar{x}\|^2\;,$$

for all $x \in S$. We make the convention $\frac{1}{r} = 0$ for $r = +\infty$. Recall that for $r = +\infty$ the uniform r-prox-regularity of S is equivalent to the convexity of S. The following proposition summarizes some important consequences of the uniform prox-regularity needed in the sequel. For the proof of these results we refer the reader to [14, 19].

Proposition 2.1. *Let S be a nonempty closed subset in H and let $r \in\,]0,+\infty]$. If the subset S is uniformly r-prox-regular then the following hold:*

(i) *For all $x \in H$ with $d_S(x) < r$, one has $\operatorname{Proj}(x,S) \neq \emptyset$;*

(ii) *The proximal subdifferential of d_S coincides with all the subdifferentials contained in the Clarke subdifferential at all points $x \in H$ satisfying $d_S(x) < r$. So, in such case, the subdifferential $\partial d_S(x) := \partial^P d_S(x) = \partial^C d_S(x)$ is a closed convex set in H.*

As a consequence of (ii) we get that for uniformly r-prox-regular sets, the proximal normal cone to S coincides with all the normal cones contained in the Clarke normal cone at all points $x \in S$, i.e., $N^P(S;x) = N^C(S;x)$. In such case, we put $N(S;x) := N^P(S;x) = N^C(S;x)$. Here $\partial^C d_S(x)$ and $N^C(S;x)$ denote respectively the Clarke subdifferential of d_S and the Clarke normal cone to S (see [14] for their definitions and properties).

In [6], the authors established a new characterization of the uniform prox-regularity in terms of the subdifferential of the distance function. We recall here a consequence of their result that will be used in the proofs of our main results.

Proposition 2.2 ([6]). *Let S be a nonempty closed subset in H and let $r \in]0,+\infty]$. Assume that S is uniformly r-prox-regular. Then*

$$(P_r) \qquad \begin{cases} \text{for all } x \in S, \quad \text{and all } \xi \in \partial d_S(x) \quad \text{one has} \\ \qquad \langle \xi, x' - x \rangle \ \leq \ \dfrac{2}{r} \, \|x' - x\|^2 + d_S(x') \ , \\ \qquad \text{for all } x' \in H \ \text{ with } \ d_S(x') \leq r \ . \end{cases}$$

Now, we recall some preliminaries concerning set-valued mappings.

($\star$) Let $K\colon X \rightrightarrows H$ be a compact-valued mapping from a normed vector space X to a Hilbert space H. We will say that K is Hausdorff-continuous (resp. Lipschitz with ratio $\lambda > 0$) if for any $x \in X$ one has

$$\lim_{x' \to x} \mathcal{H}\Big(K(x), K(x')\Big) = 0 \tag{2.2}$$

(resp. if for any $x, x' \in X$ one has

$$\mathcal{H}\Big(K(x), K(x')\Big) \leq \lambda \|x - x'\| \quad) \ .$$

Here $\mathcal{H}$ denotes the Hausdorff distance relative to the norm associated with the Hilbert space H defined by

$$\mathcal{H}(A,B) := \max\Big\{ \sup_{a \in A} d_B(a), \ \sup_{b \in B} d_A(b) \Big\} \ .$$

($\star$) Let $\Phi\colon X \rightrightarrows Y$ be a set- valued mapping defined between two topological vector spaces X and Y. We recall that Φ is upper semicontinuous (in short u.s.c.) at $\bar{x} \in \mathrm{dom}(\Phi) := \{x \in X\colon \Phi(x) \neq \emptyset\}$ if for any open $\mathcal{O}$ containing $\Phi(\bar{x})$ there exists a neighbourhood V of $\bar{x}$ such that $\Phi(V) \subset \mathcal{O}$.

We close this section with the following theorem by Bounkhel and Thibault [6]. We give the proof here for the convenience of the reader (see also [4]). It proves a closedness property of the subdifferential of the distance function associated with a set-valued mapping. Note that the statement of this theorem in [6] is given with $X = \mathbb{R}$, but the same arguments of the proof still work for any normed vector space X because the proof is based on the uniform prox-regularity of the values of the set-valued mapping and it is independent from the structure of the space X. The key of the proof is the characterization of uniformly prox-regular subsets proved in Theorem 3.1 in [6]. Another version of this result is given in [3] to study some nonconvex economic models.

Theorem 2.1. *Let $r \in\,]0,+\infty]$, Ω be an open subset in a normed vector space X, and $K\colon \Omega \rightrightarrows H$ be a Hausdorff-continuous set-valued mapping with compact values. Assume that $K(z)$ is uniformly r-prox-regular for all z in Ω. Then for a given $0 < \delta < r$ the following holds:*

> *" for any $\bar{z} \in \Omega$, $\bar{x} \in K(\bar{z}) + (r-\delta)\mathbb{B}$, $x_n \to \bar{x}$, $z_n \to \bar{z}$ with $z_n \in \Omega$, (x_n is not necessarily in $K(z_n)$) and $\xi_n \in \partial d_{K(z_n)}(x_n)$ with $\xi_n \to^w \bar{\xi}$ one has $\bar{\xi} \in \partial d_{K(\bar{z})}(\bar{x})$."*

Here $\to^w$ means the weak convergence in H.

Proof: Fix $\bar{z} \in \Omega$, and $\bar{x} \in K(\bar{z}) + (r-\delta)\mathbb{B}$. As $x_n \to \bar{x}$ one gets for n sufficiently large $x_n \in \bar{x} + \frac{\delta}{4}\mathbb{B}$. On the other hand, since the subset $K(\bar{z})$ is uniformly r-prox-regular one can choose (by Proposition 2.1) a point $\bar{y} \in K(\bar{z})$ with $d_{K(\bar{z})}(\bar{x}) = \|\bar{y} - \bar{x}\|$. So, one can write by the definition of the Hausdorff distance,

$$d_{K(z_n)}(x_n) \;\le\; \mathcal{H}\Big(K(z_n), K(\bar{z})\Big) + \|x_n - \bar{y}\| \;,$$

and hence the Hausdorff-continuity of K yields for n large enough

$$d_{K(z_n)}(x_n) \;\le\; \frac{\delta}{4} + \|x_n - \bar{x}\| + \|\bar{x} - \bar{y}\| \;\le\; \frac{\delta}{4} + \frac{\delta}{4} + r - \delta \;=\; r - \frac{\delta}{2} \;<\; r \;.$$

Therefore, for any n large enough, we apply the property (P_r'') in Theorem 3.1 in [6] with $\xi_n \in \partial d_{K(z_n)}(x_n)$ to get

$$\langle \xi_n, u - x_n\rangle \;\le\; \frac{8}{r - d_{K(z_n)}(x_n)}\,\|u - x_n\|^2 + d_{K(z_n)}(u) - d_{K(z_n)}(x_n) \;, \tag{2.3}$$

for all $u \in H$ with $d_{K(z_n)}(u) < r$. This inequality still holds for all $u \in \bar{x} + \delta'\mathbb{B}$

with $0 < \delta' < \dfrac{\delta}{4}$ because for such u one has

$$d_{K(z_n)}(u) \le \|u - \bar{x}\| + \|\bar{x} - x_n\| + d_{K(z_n)}(x_n) \le \delta' + \frac{\delta}{4} + r - \frac{\delta}{2} < r \ .$$

Consequently, by the continuity of the distance function with respect to (z, x) (because of (2.2)), the inequality (2.3) gives, by letting $n \to +\infty$,

$$\langle \bar{\xi}, u - \bar{x} \rangle \le \frac{8}{r - d_{K(\bar{z})}(\bar{x})} \|u - \bar{x}\|^2 + d_{K(\bar{z})}(u) - d_{K(\bar{z})}(\bar{x}) \qquad \text{for all} \ \ u \in \bar{x} + \delta' \mathbb{B} \ .$$

This ensures that $\bar{\xi} \in \partial d_{K(\bar{z})}(\bar{x})$ and so the proof of the theorem is complete. ∎

Remark 2.1. As a direct consequence of this theorem we have the upper semicontinuity of the set-valued mapping $(z, x) \mapsto \partial d_{K(z)}(x)$ from $T \times H$ to H endowed with the weak topology, which is equivalent (see for example Proposition 1.4.1 and Theorem 1.4.2 in [2]) to the u.s.c. of the function $(z, x) \mapsto \sigma(\partial d_{K(z)}(x), p)$ for any $p \in H$. Here $\sigma(S, p)$ denotes the support function associated with S, i.e., $\sigma(S, p) := \sup_{s \in S} \langle s, p \rangle$. Following the terminology used in [10] and their references we will say that a set-valued mapping $K \colon X \rightrightarrows H$ is scalarly u.s.c. on X if and only if for every $p \in X^*$ the support functions $\sigma(K(\cdot), p)$ are u.s.c. on X. Recall that when $X = H = \mathbb{R}^n$ and K is convex-valued mapping one has the upper semicontinuity of K is equivalent to its scalar upper semicontinuity. See for instance Castaing and Valadier [11]. □

3 – Existence results for second order nonconvex sweeping processes with perturbations

In the present section and Section 4 let $r \in \,]0, +\infty]$, $x_0 \in H$, $u_0 \in K(x_0)$, $\mathcal{V}_0$ be an open neighbourhood of x_0 in H, and $K \colon \operatorname{cl}(\mathcal{V}_0) \rightrightarrows H$ be a Lipschitz set-valued mapping with ratio $\lambda > 0$ taking nonempty closed uniformly r-prox-regular values in H. Our aim in this section is to prove the local existence of (SSPMP) on $\operatorname{cl}(\mathcal{V}_0)$, that is, there exists $T > 0$, Lipschitz mappings $x \colon [0, T] \to \operatorname{cl}(\mathcal{V}_0)$ and $u \colon [0, T] \to H$ such that

$$\begin{cases} u(0) = u_0 \,, \quad u(t) \in K(x(t)), \quad \text{for all } t \in [0, T] \ ; \\ x(t) = x_0 + \displaystyle\int_0^t u(s)\, ds, \qquad \text{for all} \ \ t \in [0, T] \ ; \\ \dot{u}(t) \in -N\Big(K(x(t)); u(t)\Big) + F\Big(t, x(t), u(t)\Big) + G\Big(t, x(t), u(t)\Big), \qquad \text{a.e.} \ \ [0, T] \ . \end{cases}$$

We begin by recalling the following lemma proved in [16].

Lemma 3.1. *Let (X, d_X) and (Y, d_Y) be two metric spaces and let $h\colon X \to Y$ be a uniformly continuous mapping. Then for every sequence $(\epsilon_n)_{n\geq 1}$ of positive numbers there exists a strictly decreasing sequence of positive numbers $(e_n)_{n\geq 1}$ converging to* 0 *such that*

(**1**) *for any $n \geq 2$, $\frac{1}{e_{n-1}}$ and $\frac{e_{n-1}}{e_n}$ are integers ≥ 2;*

(**2**) *for any $n \geq 1$, and any $x_1, x_2 \in X$, one has*

$$d_X(x_1, x_2) \leq e_n \implies d_Y(h(x_1), h(x_2)) \leq \epsilon_n \ .$$

We prove our first main theorem in this section.

Theorem 3.1. *Let $G, F\colon [0, +\infty[\times H \times H \rightrightarrows H$ be two set-valued mappings and let $\varsigma > 0$ such that $x_0 + \varsigma\mathbb{B} \subset \mathcal{V}_0$. Assume that the following assumptions are satisfied:*

(**i**) *For all $x \in \mathrm{cl}(\mathcal{V}_0)$, $K(x) \subset \mathcal{K}_1 \subset l\,\mathbb{B}$, for some convex compact set $\mathcal{K}_1$ in H and some $l > 0$;*

(**ii**) *F is scalarly u.s.c. on $[0, \frac{\varsigma}{l}] \times \mathrm{gph}\, K$ with nonempty convex weakly compact values;*

(**iii**) *G is uniformly continuous on $[0, \frac{\varsigma}{l}] \times \alpha\mathbb{B} \times l\,\mathbb{B}$ into nonempty compact subsets of H, for $\alpha := \|x_0\| + \varsigma$;*

(**iv**) *F and G satisfy the linear growth condition, that is,*

$$F(t, x, u) \subset \rho_1(1 + \|x\| + \|u\|)\,\mathbb{B} \quad \text{and} \quad G(t, x, u) \subset \rho_2(1 + \|x\| + \|u\|)\,\mathbb{B}\ ,$$

for all $(t, x, u) \in [0, \frac{\varsigma}{l}] \times gph\, K$ for some $\rho_1, \rho_2 \geq 0$.

Then for every $T \in\,]0, \frac{\varsigma}{l}]$ there exist Lipschitz mappings $x\colon [0, T] \to \mathrm{cl}(\mathcal{V}_0)$ and $u\colon [0, T] \to H$ such that

$$u(0) = u_0\,, \quad u(t) \in K(x(t)), \quad x(t) = x_0 + \int_0^t u(s)\, ds, \quad \text{for all } t \in [0, T]\ ; \tag{3.1}$$

$$\dot{u}(t) \in -N\Big(K(x(t)); u(t)\Big) + F\Big(t, x(t), u(t)\Big) + G\Big(t, x(t), u(t)\Big), \quad \text{a.e. } [0, T]\ , \tag{3.2}$$

with $\|\dot{x}(t)\| \leq l$ and $\|\dot{u}(t)\| \leq l\lambda + 2(1 + \alpha + l)(\rho_1 + \rho_2)$ a.e. on $[0, T]$.

In other words, there is a Lipschitz solution $x\colon [0,T] \to \mathrm{cl}(\mathcal{V}_0)$ *to the Cauchy problem for the second order differential inclusion:*

$$\begin{cases} \ddot{x}(t) \in -N\Big(K(x(t)); \dot{x}(t)\Big) + F\Big(t, x(t), \dot{x}(t)\Big) + G\Big(t, x(t), \dot{x}(t)\Big), & \text{a.e. } [0,T] \ ; \\ x(0) = x_0\,, \quad \dot{x}(0) = u_0\,, \quad \dot{x}(t) \in K(x(t)), \quad \textit{for all } t \in [0,T] \ , \end{cases}$$

with $\|\dot{x}(t)\| \le l$ *and* $\|\ddot{x}(t)\| \le l\lambda + 2(\rho_1+\rho_2)(1+\alpha+l)$ *a.e. on* $[0,T]$.

Proof: We give the proof in four steps.

Step 1. *Construction of the approximants.*

Let $T \in \,]0, \frac{\varsigma}{l}]$ and put $I := [0,T]$ and $\mathcal{K} := I \times \alpha\mathbb{B} \times l\,\mathbb{B}$. Then by the assumption (iv) we have

$$\|F(t,x,u)\| \;\le\; \rho_1(1+\|x\|+\|u\|) \;\le\; \rho_1(1+\alpha+l) \;=:\; \zeta_1 \ , \tag{3.3}$$

and

$$\|G(t,x,u)\| \;\le\; \rho_2(1+\|x\|+\|u\|) \;\le\; \rho_2(1+\alpha+l) \;=:\; \zeta_2 \ , \tag{3.4}$$

for all $(t,x,u) \in \mathcal{K} \cap (I \times \mathrm{gph}\,K)$. Note that $\mathcal{K} \cap (I \times \mathrm{gph}\,K) \ne \emptyset$ because $(x_0,u_0) \in (\alpha\mathbb{B} \times l\,\mathbb{B}) \cap \mathrm{gph}\,K$.

Let $\epsilon_n = \frac{1}{2^n}$, $(n = 1,2,....)$. Then by the uniform continuity of G on the set $\mathcal{K}$ and Lemma 3.1, there is a strictly decreasing sequence of positive numbers (e_n) converging to 0 such that $e_n \le 1$, and $\frac{T}{e_{n-1}}$ and $\frac{e_{n-1}}{e_n}$ are integers ≥ 2 and the following implication holds:

$$\|(t,x,u) - (t',x',u')\| \le \eta\, e_n \implies \mathcal{H}\Big(G(t,x,u), G(t',x',u')\Big) \le \epsilon_n \ , \tag{3.5}$$

for every $(t,x,u), (t',x',u') \in \mathcal{K}$ where $\|(t,x,u)\| = |t| + \|x\| + \|u\|$ and $\eta = (1 + 3\,l + l\lambda + 2(\zeta_1+\zeta_2))$.

As the sequence $e_n \to 0^+$, one can fix a positive integer n_0 such that

$$(\lambda l + \zeta_1 + \zeta_2)\, e_{n_0} \;\le\; \frac{r}{2} \ . \tag{3.6}$$

For each $n \ge n_0$, we consider the partition of I given by

$$P_n \;=\; \left\{ t_{n,i} = i\, e_n \colon \; i = 0,1,...,\; \mu_n = \frac{T}{e_n} \right\} \ . \tag{3.7}$$

We recall (see [16]) some important properties of the sequence of partitions $(P_n)_n$ needed in the sequel.

(Pr_1) $P_n \subset P_{n+1}$, for all $n \geq n_0$;

(Pr_2) For every $n \geq n_0$ and for every $t_{n,i} \in P_n \backslash P_1$ there exists a unique couple (m,j) of positive integers depending on $t_{n,i}$, such that $n_0 \leq m < n$, $t_{n,i} \notin P_s$ for every $s \leq m$, $t_{n,i} \in P_s$ for every $s > m$, $0 \leq j < \mu_m$ and $t_{m,j} < t_{n,i} < t_{m,j+1}$.

Put $I_{n,i} := [t_{n,i}, t_{n,i+1}[$, for all $i = 0, \dots, \mu_n - 1$ and $I_{n,\mu_n} := \{T\}$. For every $n \geq n_0$ we define the following approximating mappings on each interval $I_{n,i}$ as

$$\tag{3.8} \begin{cases} u_n(t) := u_{n,i} \ , \\ x_n(t) = x_0 + \displaystyle\int_0^t u_n(s)\, ds \ , \\ f_n(t) := f_{n,i} \ \in \ F\Big(t_{n,i}, x_n(t_{n,i}), u_{n,i}\Big) \ , \quad \text{and} \\ g_n(t) := g_{n,i} \ \in \ G\Big(t_{n,i}, x_n(t_{n,i}), u_{n,i}\Big) \ , \end{cases}$$

where $u_{n,0} = u_0$ and for all $i = 0, \dots, \mu_n - 1$, the point $u_{n,i+1}$ is given by

$$\tag{3.9} u_{n,i+1} \ \in \ \mathrm{Proj}\Big(u_{n,i} + e_n(f_{n,i} + g_{n,i}), K(x_n(t_{n,i+1}))\Big) \ .$$

Although the absence of the convexity of the images of K, we have the last equality is well defined. Indeed, as

$$x_n(t_{n,1}) \ = \ x_0 + \int_0^{t_{n,1}} u_n(s)\, ds \ \in \ x_0 + t_{n,1} l\mathbb{B} \subset x_0 + \varsigma\mathbb{B} \subset \mathcal{V}_0 \ ,$$

then by the Lipschitz property of K and the relations (i), (3.3), (3.4), (3.8), and (3.9) we get for $x := x_n(t_{n,1})$

$$\begin{aligned} d_{K(x_n(t_{n,1}))}\Big(u_{n,0} + e_n(f_{n,0} + g_{n,0})\Big) &\leq \mathcal{H}\Big(K(x_n(t_{n,0})), K(x_n(t_{n,1}))\Big) + e_n \|f_{n,0} + g_{n,0}\| \\ &\leq \lambda \|x_n(t_{n,0}) - x_n(t_{n,1})\| + e_n(\zeta_1 + \zeta_2) \\ &\leq \lambda(t_{n,1} - t_{n,0})\, \|u_{n,0}\| + e_n(\zeta_1 + \zeta_2) \\ &\leq (l\lambda + \zeta_1 + \zeta_2)\, e_{n_0} \ \leq \ \frac{r}{2} \ < \ r \end{aligned}$$

and hence as K has uniformly r-prox-regular values, by Proposition 2.1, one can choose a point $u_{n,1} \in \mathrm{Proj}(u_{n,0} + e_n(f_{n,0} + g_{n,0}), K(x_n(t_{n,1})))$. Similarly, we can define, by induction, the points $(u_{n,i})_{(0 \leq i \leq \mu_n)}$, $(f_{n,i})_{(0 \leq i \leq \mu_n)}$ and $(g_{n,i})_{(0 \leq i \leq \mu_n)}$.

Let us define $\theta_n(t) := t_{n,i}$, if $t \in I_{n,i}$. Then, the definition of $x_n(\cdot)$ and $u_n(\cdot)$ and the assumption (i) yield for all $t \in I$,

$$\tag{3.10} u_n(t) \in K(x_n(\theta_n(t))) \subset \mathcal{K}_1 \subset l\,\mathbb{B} \ .$$

So, all the mappings $x_n(\cdot)$ are Lipschitz with ratio l and they are also equi-bounded, with $\|x_n\|_\infty \le \|x_0\| + l\,T$. Here and thereby $\|x\|_\infty := \sup_{t\in I} \|x(t)\|$. Observe also that for all $n \ge n_0$ and all $t \in I$ one has

$$x_n(t) \in \alpha\mathbb{B} \cap \mathcal{V}_0 \,. \tag{3.11}$$

Indeed, the definition of $x_n(\cdot)$ and $u_n(\cdot)$ ensure that, for all $t \in I$,

$$x_n(t) = x_0 + \int_0^t u_n(s)\,ds \in x_0 + tl\,\mathbb{B} \subset x_0 + \varsigma\mathbb{B} \subset \alpha\mathbb{B} \cap \mathcal{V}_0 \,,$$

and hence $K(x_n(t))$ is well defined for all $t \in I$.

Now we define the piecewise affine approximants

$$v_n(t) := u_{n,i} + e_n^{-1}(t - t_{n,i})\,(u_{n,i+1} - u_{n,i}), \quad \text{if } t \in I_{n,i} \,. \tag{3.12}$$

Observe that $v_n(\theta_n(t)) = u_{n,i}$, for all $i = 0, \ldots, \mu_n$ and so by (3.9), (3.11), and the assumption (ii), one has $v_n(\theta_n(t)) \in K(x_n(t_{n,i})) = K(x_n(\theta_n(t))) \subset l\,\mathbb{B}$. Then by (3.3),(3.4), (3.8), (3.11), and the last relation we obtain for all $t \in I$ and all $n \ge n_0$

$$\begin{cases} f_n(t) \in F\Big(\theta_n(t), x_n(\theta_n(t)), v_n(\theta_n(t))\Big) \cap \zeta_1\mathbb{B} \quad \text{and} \\ g_n(t) \in G\Big(\theta_n(t), x_n(\theta_n(t)), v_n(\theta_n(t))\Big) \cap \zeta_2\mathbb{B} \,. \end{cases} \tag{3.13}$$

Now we check that the mappings v_n are equi-Lipschitz with ratio $l\lambda + 2(\zeta_1 + \zeta_2)$. Indeed, by (3.9) and the Lipschitz property of K one has

$$\begin{aligned} \|u_{n,i+1} - u_{n,i}\| &\le \Big\|u_{n,i+1} - \Big(u_{n,i} + e_n(f_{n,i} + g_{n,i})\Big)\Big\| + e_n\,\|f_{n,i} + g_{n,i}\| \\ &\le d_{K(x_n(t_{n,i+1}))}\Big(u_{n,i} + e_n(f_{n,i} + g_{n,i})\Big) + (\zeta_1 + \zeta_2)\,e_n \\ &\le \mathcal{H}\Big(K(x_n(t_{n,i})), K(x_n(t_{n,i+1}))\Big) + 2\,(\zeta_1 + \zeta_2)\,e_n \\ &\le \Big(l\lambda + 2\,(\zeta_1 + \zeta_2)\Big)\,e_n \,, \end{aligned} \tag{3.14}$$

and hence,

$$\|v_n(t) - v_n(s)\| = e_n^{-1}|t - s|\,\|u_{n,i+1} - u_{n,i}\| \le \Big(l\lambda + 2\,(\zeta_1 + \zeta_2)\Big)\,|t - s| \,.$$

It is also clear, by the definitions of $u_n(\cdot)$ and $v_n(\cdot)$, that

$$\|v_n(t) - u_n(t)\| \leq e_n^{-1}|t - t_{n,i}|\,\|u_{n,i+1} - u_{n,i}\| \leq \Big(l\lambda + 2\,(\zeta_1 + \zeta_2)\Big)\,e_n\ , \tag{3.15}$$

and hence

$$\|v_n - u_n\|_\infty \to 0\ .$$

Let us define, $\nu_n(t) := t_{n,i+1}$ if $t \in I_{n,i}$ and $i = 0, \ldots, \mu_n - 1$. The definition of $v_n(\cdot)$ given by (3.12) and the relation (3.9) yield

$$v_n(\nu_n(t)) \in K(x_n(\nu_n(t))), \qquad \text{for all}\ \ t \in I_{n,i}\ \ (i = 0, \ldots, \mu_n - 1)\ , \tag{3.16}$$

and for all $t \in I \backslash \{t_{n,i} \colon i = 0, .., \mu_n\}$ one has

$$\dot v_n(t) = e_n^{-1}(u_{n,i+1} - u_{n,i})\ . \tag{3.17}$$

So, we get for all $t \in I \backslash \{t_{n,i} \colon i = 0, .., \mu_n\}$

$$\begin{aligned} e_n\Big(\dot v_n(t) - (f_n(t) + g_n(t))\Big) &= u_{n,i+1} - \Big(u_{n,i} + e_n(f_{n,i} + g_{n,i})\Big) \\ &\in \operatorname{Proj}\Big(u_{n,i} + e_n(f_{n,i} + g_{n,i}), K(x_n(t_{n,i+1}))\Big) - \Big(u_{n,i} + e_n(f_{n,i} + g_{n,i})\Big)\ . \end{aligned}$$

Then, the properties of the proximal normal cone to subsets, ensure that we have for all $t \in I \backslash \{t_{n,i} \colon i = 0, .., \mu_n\}$

$$\begin{aligned} \dot v_n(t) - (f_n(t) + g_n(t)) &\in -N\Big(K(x_n(t_{n,i+1}); u_{n,i+1})\Big) \\ &= -N\Big(K(x_n(\nu_n(t)); v_n(\nu_n(t)))\Big)\ . \end{aligned} \tag{3.18}$$

On the other hand, by (3.14) and (3.17), it is clear that

$$\|\dot v_n(t)\| \leq \Big(l\lambda + 2(\zeta_1 + \zeta_2)\Big)\ . \tag{3.19}$$

Put $\delta := (l\lambda + 3(\zeta_1 + \zeta_2))$. Therefore, the relations (3.13), (3.18) and (3.19), and Theorem 4.1 in [6] entail for all $t \in I \backslash \{t_{n,i} \colon i = 0, .., \mu_n\}$

$$\dot v_n(t) - (f_n(t) + g_n(t)) \in -\delta\,\partial d_{K(x_n(\nu_n(t)))}(v_n(\nu_n(t)))\ . \tag{3.20}$$

Step 2. *Uniform convergence of both sequences $x_n(\cdot)$ and $v_n(\cdot)$.*

Since $e_n^{-1}(t - t_{n,i}) \leq 1$, for all $t \in I_{n,i}$ and $u_{n,i}, u_{n,i+1} \in \mathcal{K}_1$, and $\mathcal{K}_1$ is a convex set in H one gets for all $t \in I$,

$$v_n(t) = u_{n,i} + e_n^{-1}(t - t_{n,i})\,(u_{n,i+1} - u_{n,i}) \in \mathcal{K}_1\ .$$

Thus for every $t \in I$, the set $\{v_n(t) \colon n \geq n_0\}$ is relatively strongly compact in H. Therefore, the estimate (3.19) and Theorem 0.4.4 in [2] ensure that there exists a Lipschitz mapping $u \colon I \to H$ with ratio $l\lambda + 2(\zeta_1 + \zeta_2)$ such that:

(v_n) converges uniformly to u on I;

$(\dot{v}_n)$ weakly converges to $\dot{u}$ in $L^1(I, H)$.

Now we define the Lipschitz mapping $x \colon I \to H$ as

$$x(t) = x_0 + \int_0^t u(s)\,ds, \qquad \text{for all } \; t \in I \; . \tag{3.21}$$

Then by the definition of x_n one obtains for all $t \in I$,

$$\|x_n(t) - x(t)\| \;=\; \left\| \int_0^t (u_n(s) - u(s))\,ds \right\| \;\leq\; T\,\|u_n - u\|_\infty$$

and so by (3.15) we get

$$\|x_n - x\|_\infty \leq T\|u_n - v_n\|_\infty + T\|v_n - u\|_\infty \;\to\; 0 \qquad \text{as} \;\; n \to \infty \; . \tag{3.22}$$

This completes the second step.

Step 3. *Relative strong compactness of* (g_n).

The points $(g_{n,i})_{i=,0\ldots,\mu_n}$ defining the step function $g_n(\cdot)$ was chosen arbitrarily in our construction. Nevertheless, by using the uniform continuity of the set-valued mapping G over $\mathcal{K}$ and the techniques of [16] (see also [15, 23]), the sequence $g_n(\cdot)$ can be constructed *relatively strongly compact for the uniform convergence* in the space of bounded functions. The construction of the sequence $g_n(\cdot)$ is similar to the one presented in [15, 23]. We give it here for the completeness and for the reader's convenience.

To prove the relative strong compactness for the uniform convergence in the space of bounded functions we will use a very useful compactness criterion proved in Theorem 0.4.5 in [2]. First we need to prove that for all $t \in I$, the set $\{g_n(t) \colon n \geq n_0\}$ is relatively strongly compact in H. By the definition of $\theta_n(\cdot)$ we have for all $t \in I$ and all $n \geq n_0$ $|\theta_n(t) - t| \leq e_n$. Then $(x_n \circ \theta_n)$ and $(v_n \circ \theta_n)$ converge uniformly on I to x and u respectively. Now, by (3.13) and the continuity of G on $I \times \operatorname{gph} K$ one has

$$d_{G(t,x(t),u(t))}(g_n(t)) \leq \mathcal{H}\Big(G\Big(\theta_n(t), x_n(\theta_n(t)), v_n(\theta_n(t))\Big)\Big) \;, \\ G(t, x(t), u(t)) \to 0 \qquad \text{as} \;\; n \to \infty \; .$$

This implies the relative strong compactness of the set $\{g_n(t)\colon\ n \geq n_0\}$ in H for all $t \in I$ because $G(t, x(t), u(t))$ is a strongly compact set in H. Now, we have to show that the sequence is *an equioscillating* family of bounded functions in the sense of [2]. Recall that a family $\mathcal{F}$ of bounded mappings $x\colon I \to H$ is equioscillating if for every $\epsilon > 0$, there exists a finite partition of I into subintervals J_j $(j = 0, \ldots, m)$ such that for all $x \in \mathcal{F}$ and all $j = 0, \ldots, m$ one has $\omega_{J_j}(x) \leq \epsilon$, where $\omega_J(x)$ denotes the oscillation of x in J defined by

$$\omega_J(x) := \sup\Big\{\|x(s) - x(t)\|\colon\ s, t \in J\Big\}\ . \tag{3.23}$$

Fix any $\epsilon > 0$ and let $m_0 \geq n_0$ such that $4\,\epsilon_{m_0} \leq \epsilon$. Consider the finite partition $J_j := [t_{m_0,j}, t_{m_0,j+1}[$ $(j = 0, \ldots, \mu_{m_0} - 1)$ of I. We shall prove that

$$\omega_{J_j}(g_n) \leq \epsilon, \qquad \text{for all}\ \ n \geq n_0\ \ \text{and all}\ \ j = 0, \ldots, \mu_{m_0} - 1\ . \tag{3.24}$$

For that purpose, we have to choose $g_{n,i}$ in (3.8) in such way that the following condition holds for every $n \geq n_0$ and $i = 0, \ldots, \mu_{m_0} - 1$:

$$\begin{aligned} &\|g_n(t_{n,i}) - g_n(t_{n,i-1})\| \leq \epsilon_n, \qquad \text{if}\ \ t_{n,i} \in P_1\ , \\ &\|g_n(t_{n,i}) - g_n(t_{m,p})\| \leq \epsilon_m, \qquad \text{if}\ \ t_{n,i} \notin P_1\ , \end{aligned} \tag{3.25}$$

where (m, p) is the unique pair of integers assigned to $t_{n,i}$ such that $m < n$, $t_{n,i} \notin P_j$ for $j \leq m$, $t_{n,i} \in P_j$ for $j > m$ and $t_{m,p} < t_{n,i} < t_{m,p+1}$. For $i = 0$ we take $g_{n,0} \in G(0, x_0, u_0)$. By induction we assume that $g_{n,j} \in G(t_{n,j}, x_n(t_{n,j}), u_{n,j})$ have been defined for all $j \in \{0, \ldots, i-1\}$.

If $t_{n,i} \in P_1$, it suffices to take $g_{n,i} \in G(t_{n,i}, x_n(t_{n,i}), u_{n,i})$ such that:

$$\|g_{n,i} - g_{n,i-1}\| \ \leq\ \mathcal{H}\Big(G(t_{n,i}, x_n(t_{n,i}), u_{n,i}), G(t_{n,i-1}, x_n(t_{n,i-1}), u_{n,i-1})\Big)\ .$$

Indeed, by virtue of (3.10), (3.14), and (3.19) we have

$$\begin{aligned} \|(t_{n,i}, x_n(t_{n,i}), u_{n,i}) - (t_{n,i-1}, x_n(t_{n,i-1}), u_{n,i-1})\| \ &\leq\ \Big(1 + l + l\lambda + 2(\zeta_1 + \zeta_2)\Big)\, e_n \\ &\leq\ \eta\, e_n\ , \end{aligned}$$

which in combining with (3.6) gives

$$\|g_n(t_{n,i}) - g_n(t_{n,i-1})\| \ =\ \|g_{n,i} - g_{n,i-1}\| \ \leq\ \epsilon_n\ .$$

If $t_{n,i} \notin P_1$, then $t_{m,p} \in P_n$(because $m < n$) and so there is a unique integer $q < i$ such that $t_{m,p} = t_{n,q}$. Hence $t_{n,i} - t_{n,q} = t_{n,i} - t_{m,p} < t_{m,p+1} - t_{m,p} \leq e_m$. This with (3.10) and (3.19) imply

$$\begin{aligned} \|(t_{n,i}, x_n(t_{n,i}), u_{n,i}) - (t_{n,q}, x_n(t_{n,q}), u_{n,q})\| \ &\leq\ \Big(1 + l + l\lambda + 2(\zeta_1 + \zeta_2)\Big)\, e_m \\ &\leq\ \eta\, e_m\ , \end{aligned}$$

which together with (3.6) yield

$$\mathcal{H}\Big(G(t_{n,i}, x_n(t_{n,i}), u_{n,i}), G(t_{n,q}, x_n(t_{n,q}), u_{n,q})\Big) \le \epsilon_m \ .$$

Since $g_n(t_{m,p}) = g_n(t_{n,q}) = g_{n,q} \in G(t_{n,q}, x_n(t_{n,q}), u_{n,q})$, we may choose $g_{n,i} \in G(t_{n,i}, x_n(t_{n,i}), u_{n,i})$ such that

$$\|g_n(t_{n,i}) - g_n(t_{m,p})\| = \|g_{n,i} - g_{n,q}\| \le \epsilon_m \ ,$$

which is the second inequality in (3.25).

Next, we prove that (3.24) holds.

If $n \le m_0$, then $\frac{e_n}{e_{m_0}}$ is an integer and every J_j is contained in some interval $[t_{n,k}, t_{n,k+1}[$ in which g_n is constant. Thus (3.24) is trivial in this case:

$$\omega_{J_j}(g_n) = 0, \quad \text{for all } \ j = 0, \dots, \mu_{m_0} \ \text{ and all } \ n \le m_0 \ .$$

Let $n > m_0$. As $\frac{e_{m_0}}{e_n}$ is an integer, then $2\,e_n \le e_{m_0}$. By property (Pr_1), it follows that $t_{m_0,j}, t_{m_0,j+1} \in P_n$. Thus, there exist ϱ, ϑ such that $0 \le \vartheta < \varrho$, $t_{m_0,j} = t_{n,\vartheta}$ and $t_{m_0,j+1} = t_{n,\varrho}$. The values of the mapping g_n on $J_j = [t_{m_0,j}, t_{m_0,j+1}[\ = [t_{n,\vartheta}, t_{n,\varrho}[$ are $g_n(t_{n,s}) = g_{n,s}$, with $\vartheta < s < \varrho$. So we shall prove that, for all $\vartheta < s < \varrho$,

$$\|g_n(t_{n,s}) - g_n(t_{m_0,j})\| \le 2\,\epsilon_{m_0} \ , \tag{3.26}$$

and so $\|g_n(t) - g_n(t_{m_0,j})\| \le 2\,\epsilon_{m_0}$, for all $t \in J_j$ and all $n > m_0$. Then it will follow that, for all t and s in J_j,

$$\|g_n(t) - g_n(s)\| \le \|g_n(t) - g_n(t_{m_0,j})\| + \|g_n(t_{m_0,j}) - g_n(s)\| \le 4\,\epsilon_{m_0} \le \epsilon \ .$$

Hence $\omega_{J_j}(g_n) \le \epsilon$, and (3.24) holds.

Let $t_{n,s} \in P_n$ such that $\vartheta < s < \varrho$. Then $t_{n,s} \notin P_{m_0}$ and consequently $t_{n,s} \notin P_1$. Now by property Pr_2, there exists a unique couple (m_1, p_1) such that $m_1 < n$, $t_{n,s} \in P_{m_1+1} \backslash P_{m_1}$ and $t_{m_1,p_1} < t_{n,s} < t_{m_1,p_1+1}$, with $p_1 < \mu_{m_1}$. By virtue of the second inequality in (3.25), we obtain that

$$\|g_n(t_{n,s}) - g_n(t_{m_1,p_1})\| \le \epsilon_{m_1} \ . \tag{3.27}$$

Using the same techniques in [16, 15, 23] we can show that $t_{m_0,j} \le t_{m_1,p_1}$.

If $t_{m_0,j} = t_{m_1,p_1}$, then (3.26) is true, by (3.27) and the fact that $m_1 \ge m_0$ implies $\epsilon_{m_1} \le \epsilon_{m_0}$.

If $t_{m_0,j} < t_{m_1,p_1}$, then since $t_{m_1,p_1} < t_{m_0,j+1}$ it follows that $t_{m_1,p_1} \notin P_{m_0}$ and so $t_{m_1,p_1} \notin P_1$. Then, by Pr_2, there is a unique couple (m_2, p_2) such that $m_2 < m_1$,

$t_{n,s} \in P_{m_2+1} \backslash P_{m_2}$ and $t_{m_2,p_2} < t_{m_1,p_1} < t_{m_2,p_2+1}$, with $p_2 < \mu_{m_2}$. Again by virtue of the second inequality in (3.25), we obtain that

$$\|g_n(t_{m_2,p_2}) - g_n(t_{m_1,p_1})\| \leq \epsilon_{m_2} \ , \tag{3.28}$$

because $t_{m_1,p_1} \in P_n$ ($m_1 < n$ implies $P_{m_1} \subset P_n$). As mentioned above for the couple (m_1, p_1), it is not hard to check that $t_{m_0,j} \leq t_{m_2,p_2}$. If $t_{m_0,j} = t_{m_2,p_2}$, then (3.26) follows by summing (3.27) and (3.28), since $\epsilon_{m_1} + \epsilon_{m_2} \leq \epsilon_{m_0}$ (because $m_1, m_2 \geq m_0$). The case If $t_{m_0,j} < t_{m_2,p_2}$ is treated as above.

The inductive procedure is now clear: There exists a finite sequence $\{(m_i, p_i)\}$, $i = 0, \ldots, k$ such that $m_0 \leq m_k < m_{k-1} < \ldots < m_1 < n$, $t_{m_k,p_k} = t_{m_0,j}$, $t_{m_i,p_i} \in P_{m_i} \subset P_n$ for all i and

$$\|g_n(t_{m_i,p_i}) - g_n(t_{m_{i+1},p_{i+1}})\| \leq \epsilon_{m_{i+1}}, \quad \text{for } \ i = 0, \ldots, k-1 \ .$$

Consequently, by applying these inequalities, (3.27), and the triangle inequality, we obtain

$$\|g_n(t_{n,s}) - g_n(t_{m_1,p_1})\| \ \leq \ \epsilon_{m_1} + \epsilon_{m_2} + \ldots + \epsilon_{m_k} \ \leq \ 2\,\epsilon_{m_0} \ .$$

Thus completing the proof of (3.26) and so we get the relative strong compactness for the uniform convergence in the space of bounded mappings of the sequence $g_n(\cdot)$. Therefore there exists a bounded mapping $g(\cdot)\colon I \to H$ such that $\|g_n - g\|_\infty \to 0$.

Step4. *Existence of a solution.*

Since $(x_n \circ \theta_n)$ and $(v_n \circ \theta_n)$ converge uniformly on I to x and u respectively, then by the continuity of G on $I \times \alpha\mathbb{B} \times l\mathbb{B}$, the closedness of the set $G(t, x(t), u(t))$, and the fact that $g_n(t) \in G(\theta_n(t), x_n(\theta_n(t)), v_n(\theta_n(t)))$ a.e. on I (by (3.13)), we obtain $g(t) \in G(t, x(t), u(t))$ a.e. on I.

Recall that $v_n(\theta_n(t)) \in K(x_n(\theta_n(t)))$, for all $t \in I$ and all $n \geq n_0$. It follows then by the closedness and the continuity of K that $u(t) \in K(x(t))$, for all $t \in I$ and hence (3.1) holds.

By (3.13) one can assume without loss of generality that the sequence f_n converges weakly in $L^1(I, H)$ to some mapping f. Therefore, from (3.13) once again, we can classically (see Theorem V-14 in [11]) conclude that $f(t) \in F(t, x(t), u(t))$ a.e. on I, because by hypothesis F is scalarly u.s.c. with convex weakly compact values. We apply now Castaing techniques (see for example [8]). The weak convergence of $(\dot{v}_n - (f_n + g_n))$ to $\dot{u} - (f + g)$ in $L^1(I, H)$ (by what precedes and

Step 2) entails (Mazur's lemma) that for a.e. $t \in I$

$$\dot{u}(t) - f(t) - g(t) \in \bigcap_n \overline{co}\Big[\dot{v}_k(t) - f_k(t) - g_k(t),\ k \geq n\Big] .$$

Fix such t in I an any $\xi \in H$. Then the last relation gives

$$\Big\langle \dot{u}(t) - f(t) - g(t),\, \xi \Big\rangle \leq \inf_n \sup_{k\geq n} \Big\langle \dot{v}_n(t) - f_n(t) - g_n(t),\, \xi \Big\rangle .$$

Hence by (3.20), one obtains

$$\Big\langle \dot{u}(t) - f(t) - g(t),\, \xi \Big\rangle \leq \limsup_n\, \sigma\Big(-\delta\, \partial d_{K(x_n(\nu_n(t)))}(v_n(\nu_n(t)),\, \xi\Big) .$$

Since $|\nu_n(t) - t| \leq e_n$ on $[0,T[$, then $\nu_n(t) \to t$ uniformly on $[0,T[$. It follows then by Remark 2.1 and Theorem 2.1 that for a.e. $t \in I$ and any $\xi \in H$,

$$\Big\langle \dot{u}(t) - f(t) - g(t),\, \xi \Big\rangle \leq \sigma\Big(-\delta\, \partial d_{K(x(t))}(u(t)),\, \xi\Big) .$$

By Proposition 2.1 we have $\partial d_{K(x(t))}(u(t))$ is a convex closed set and so the last inequality entails

$$\dot{u}(t) - f(t) - g(t) \in -\delta\, \partial d_{K(x(t))}(u(t)) \subset -N\Big(K(x(t)); u(t)\Big) ,$$

because $u(t) \in K(x(t))$ (by (3.1)). Thus

$$\begin{aligned} \dot{u}(t) &\in -N\Big(K(x(t)); u(t)\Big) + f(t) + g(t) \\ &\subset -N\Big(K(x(t)); u(t)\Big) + F\Big(t, x(t), u(t)\Big) + G\Big(t, x(t), u(t)\Big) , \end{aligned}$$

and so (3.2) holds and the proof of the theorem is complete. ∎

It would be interesting in the infinite dimensional setting to ask whether the compactness assumption on K, i.e., $K(x) \subset \mathcal{K}_1 \subset l\,\mathbb{B}$, can be replaced by, $K(x) \subset l\,\mathbb{B}$, the boundness of the set-valued mapping K. Here we give a positive answer when K is anti-monotone, G satisfies the strong linear growth condition, that is,

$$G(t,x,u) \subset (1 + \|x\| + \|u\|)\kappa_2 \subset \rho_2(1 + \|x\| + \|u\|)\mathbb{B} ,$$

for all $(t,x,u) \in [0, \frac{\varsigma}{l}] \times \operatorname{gph} K$, where κ_2 is a convex compact subset in H and $\rho_2 \geq 0$, and F satisfies one of the two following assumptions:

1 – The monotony with respect to the third variable on $[0, \frac{\varsigma}{l}] \times \operatorname{gph} K$, that is, for any $(t_i, x_i, u_i) \in [0, \frac{\varsigma}{l}] \times \operatorname{gph} K$ and any $z_i \in F(t_i, x_i, u_i)$ $(i = 1,2)$ one has

$$\langle z_1 - z_2,\, u_1 - u_2\rangle \geq 0 ;$$

2 – The strong linear growth condition, that is,

$$F(t,x,u) \subset (1+\|x\|+\|u\|)\kappa_1 \subset \rho_1(1+\|x\|+\|u\|)\mathbb{B} ,$$

for all $(t,x,u) \in [0,\frac{\varsigma}{l}] \times \operatorname{gph} K$, where κ_1 is a convex compact subset in H and $\rho_1 \geq 0$.

We need to recall the definition of anti-monotone set-valued mappings. We will say that K is anti-monotone if the set-valued mapping $-K$ is monotone in the usual sense, that is, for any $(x_i,u_i) \in \operatorname{gph} K$ $(i=1,2)$ one has

$$\langle u_1-u_2,\, x_1-x_2\rangle \leq 0 .$$

In the following theorem we prove the first case when F is monotone with respect to the third variable.

Theorem 3.2. *Let $F,G\colon [0,+\infty[\times H\times H \rightrightarrows H$ be two set-valued mappings and $\varsigma>0$ such that $x_0+\varsigma\mathbb{B} \subset \mathcal{V}_0$. Assume that the following assumptions are satisfied:*

(i) *K is anti-monotone and for all $x \in \operatorname{cl}(\mathcal{V}_0)$, $K(x) \subset l\,\mathbb{B}$, for some $l>0$;*

(ii) *F is scalarly u.s.c. on $[0,\frac{\varsigma}{l}] \times \operatorname{gph} K$ with nonempty convex weakly compact values;*

(iii) *G satisfies the strong linear growth condition and it is uniformly continuous on $[0,\frac{\varsigma}{l}] \times \alpha\mathbb{B} \times l\mathbb{B}$ into nonempty compact subsets of H, for $\alpha := \|x_0\|+\varsigma$;*

(iv) *F satisfies the linear growth condition, that is,*

$$F(t,x,u) \subset \rho_1(1+\|x\|+\|u\|)\mathbb{B} ,$$

for all $(t,x,u) \in [0,\frac{\varsigma}{l}] \times \operatorname{gph} K$ for some $\rho_1 \geq 0$;

(v) *F is monotone with respect to the third variable on $[0,\frac{\varsigma}{l}] \times \operatorname{gph} K$.*

Then for every $T \in\,]0,\frac{\varsigma}{l}]$ there is a Lipschitz solution $x\colon I:=[0,T] \to \operatorname{cl}(\mathcal{V}_0)$ of (SSPMP) satisfying $\|\dot{x}(t)\| \leq l$ and $\|\ddot{x}(t)\| \leq l\lambda + 2(\rho_1+\rho_2)(1+\alpha+l)$ a.e. on I.

Proof: An inspection of the proof of Theorem 3.1 shows that the compactness assumption on K was used in Step 2 and Step 3 to get the uniform convergence of both sequences $x_n(\cdot)$ and $v_n(\cdot)$ and the relative strong compactness of $g_n(\cdot)$.

Then we have to prove Step 2 and Step 3. First, we need, for technical reasons, to fix n_0 satisfying $((4\sqrt{T}+3)l\lambda + 2(\zeta_1+\zeta_2))\sqrt{e_{n_0}} \leq \frac{r}{2}$.

Observe by (3.13) and the strong linear growth of G that for every $t \in I$ and every $n \geq n_0$

$$g_n(t) \in G\Big(\theta_n(t), x_n(\theta_n(t)), v_n(\theta_n(t))\Big) \subset (1+\alpha+l)\,\kappa_2 \ .$$

Then the set $\{g_n(t) \colon n \geq n_0\}$ is relatively strongly compact in H for all $t \in I$. On the other hand as $g_n(\cdot)$ is equioscillating by the same arguments in Step 3 in the proof of Theorem 3.1, then we get the relative strong compactness of $g_n(\cdot)$ for the uniform convergence in the space of bounded mappings. Consequently, we may assume without loss of generality that $g_n(\cdot)$ converges uniformly to a bounded mapping g, i.e.,

$$\|g_n - g\|_\infty \to 0 \quad \text{as} \ \ n \to +\infty \ . \tag{3.29}$$

Now we prove the uniform convergence of $x_n(\cdot)$. Put for all positive integers m and $n \geq n_0$

$$w_{m,n}(t) := \frac{1}{2}\,\|x_n(t) - x_m(t)\|^2 \ .$$

Then

$$\frac{d^+ w_{m,n}}{dt}(t) = \Big\langle x_m(t) - x_n(t),\, u_m(t) - u_n(t)\Big\rangle, \qquad \text{for all} \ \ t \in [0,T[\ ,$$

since u_n is the right-derivative of x_n. Observe that for any $t \in [0,T[$, there exist positive integers i and j such that $t \in I_{n,i} \cap I_{m,j}$. Then $u_m(t) = u_{m,j}$ belongs to $K(x_m(t_{m,j}))$ and $u_n(t) = u_{n,i}$ belongs to $K(x_n(t_{n,i}))$. It follows by the anti-monotony of K that

$$\Big\langle x_m(t_{m,j}) - x_n(t_{n,i}),\, u_m(t) - u_n(t)\Big\rangle \leq 0 \ ,$$

and thus

$$\frac{d^+ w_{m,n}}{dt}(t) \leq \Big\langle x_m(t) - x_m(t_{m,j}),\, u_m(t) - u_n(t)\Big\rangle + \Big\langle x_n(t_{n,i}) - x_n(t),\, u_m(t) - u_n(t)\Big\rangle .$$

Since (3.10) holds and since all the mappings x_n have the same Lipschitz constant l, we have

$$\frac{d^+ w_{m,n}}{dt}(t) \ \leq \ 2\,l^2|t - t_{m,j}| + 2\,l^2|t - t_{n,i}| \ \leq \ 2\,l^2(e_m + e_n) \ .$$

Moreover, $w_{m,n}(0)=0$. Hence $w_{m,n}(t)\le 2\,l^2(e_m+e_n)\,t$, and so

$$\|x_m-x_n\|_\infty \;\le\; 2\,l\,\sqrt{T}\,(\sqrt{e}_m+\sqrt{e}_n)\;, \tag{3.30}$$

which ensures that $x_n(\cdot)$ is a Cauchy sequence for the uniform convergence, hence it converges uniformly to a Lipschitz mapping $x(\cdot)$ with ratio l. So, we have

$$\|x_n-x\|_\infty\to 0 \quad \text{as } \; n\to+\infty\;.$$

Now, we proceed to prove the Cauchy property of the sequence $v_n(\cdot)$ for the uniform convergence in the space of continuous mappings $\mathcal{C}(I,H)$. We will follow the idea used in [6].

Fix $m,n\ge n_0$ and fix also $t\in I$ with $t\ne t_{m,j}$ for $j=0,\dots,\mu_m-1$ and $t\ne t_{n,i}$ for $i=0,\dots,\mu_n-1$. Observe by the Lipschitz property of K and the relations (3.16), (3.19), and (3.30) that

$$\begin{aligned}
d_{K(x_n(\nu_n(t)))}(v_m(t)) \;&\le\; \mathcal{H}\Big(K(x_n(\nu_n(t))),K(x_m(\nu_m(t)))\Big)+\|v_m(\nu_m(t))-v_m(t)\|\\
&\le\; \lambda\|x_n(\nu_n(t))-x_m(\nu_m(t))\|+\|v_m(\nu_m(t))-v_m(t)\|\\
&\le\; \lambda\Big[\|x_n(\nu_n(t))-x_m(\nu_n(t))\|+\|x_m(\nu_m(t))-x_m(\nu_n(t))\|\Big]\\
&\quad\; +\Big(l\lambda+2(\zeta_1+\zeta_2)\Big)\,|\nu_m(t)-t|\\
&\le\; \lambda\Big[2\,l\sqrt{T}(\sqrt{e}_n+\sqrt{e}_m)+l|\nu_m(t)-\nu_n(t)|\Big]+\Big(l\lambda+2(\zeta_1+\zeta_2)\Big)e_m\\
&\le\; (2\sqrt{T}+1)\,l\lambda\sqrt{e}_n+\Big(2(\sqrt{T}+1)\,l\lambda+2(\zeta_1+\zeta_2)\Big)\,\sqrt{e}_m\\
&\le\; \Big[(4\sqrt{T}+3)\,l\lambda+2(\zeta_1+\zeta_2)\Big]\,\sqrt{e}_{n_0}\;\le\;\frac{r}{2}\;<\;r\;.
\end{aligned}$$

Put $\alpha_1:=2(\sqrt{T}+1)$ and $\widetilde{e}_n:=\max\{\sqrt{e}_n,\|g_n-g\|_\infty\}$, for all $n\ge n_0$. Then, by (3.20) and (P_r) in Proposition 2.2 entail

$$\begin{aligned}
&\Big\langle \dot v_n(t)-(f_n(t)+g_n(t)),\,v_n(\nu_n(t))-v_m(t)\Big\rangle\;\le\\
&\qquad\le\;\frac{2\,\delta}{r}\,\|v_n(\nu_n(t))-v_m(t)\|^2+\delta\,d_{K(x_n(\nu_n(t)))}(v_m(t))\\
&\qquad\le\;\frac{2\,\delta}{r}\,\Big[\|v_n(\nu_n(t))-v_n(t)\|+\|v_n(t)-v_m(t)\|\Big]^2\\
&\qquad\quad\;+\delta\Big[\alpha_1\,l\,\lambda\,\widetilde{e}_n+\Big(\alpha_1\,l\,\lambda+2(\zeta_1+\zeta_2)\Big)\,\widetilde{e}_m\Big]\\
&\qquad\le\;\frac{2\,\delta}{r}\Big[\delta\,\widetilde{e}_n+\|v_n(t)-v_m(t)\|\Big]^2\;+\;\delta\,\Big(\alpha_1\,l\,\lambda+2(\zeta_1+\zeta_2)\Big)\,(\widetilde{e}_n+\widetilde{e}_m)\;.
\end{aligned}$$

This last inequality and (3.19) yield

$$\begin{aligned}
&\Big\langle \dot v_n(t),\, v_n(t) - v_m(t))\Big\rangle \;\le \\
&\qquad \le \Big\langle f_n(t) + g_n(t),\, v_m(t) - v_n(\nu_n(t))\Big\rangle + \Big\langle \dot v_n(t),\, v_n(t) - v_n(\nu_n(t))\Big\rangle \\
&\qquad\quad + \frac{2\,\delta}{r}\Big[\delta\,\widetilde e_n + \|v_n(t) - v_m(t)\|\Big]^2 + \delta\Big(\alpha_1\, l\,\lambda + 2(\zeta_1+\zeta_2)\Big)\,(\widetilde e_n + \widetilde e_m) \\
&\qquad \le \Big\langle f_n(t) + g_n(t),\, v_m(t) - v_n(\nu_n(t))\Big\rangle + \frac{2\,\delta}{r}\Big[\delta\,\widetilde e_n + \|v_n(t) - v_m(t)\|\Big]^2 \\
&\qquad\quad + \delta\Big(\alpha_1\, l\,\lambda + 2(\zeta_1+\zeta_2)\Big)\,(\widetilde e_n + \widetilde e_m) \;+\; \delta^2\,\widetilde e_n\;.
\end{aligned}$$

On the other hand by (3.13) and (3.19) we have

$$\begin{aligned}
&\Big\langle f_n(t) + g_n(t),\, v_m(t) - v_n(\nu_n(t))\Big\rangle \;= \\
&\qquad\qquad = \Big\langle f_n(t) + g_n(t),\, v_m(\theta_m(t)) - v_n(\theta_n(t))\Big\rangle \\
&\qquad\qquad\quad + \Big\langle f_n(t) + g_n(t),\, v_m(t) - v_m(\theta_m(t))\Big\rangle \\
&\qquad\qquad\quad + \Big\langle f_n(t) + g_n(t),\, v_n(\theta_n(t)) - v_n(\nu_n(t))\Big\rangle \\
&\qquad\qquad \le \Big\langle f_n(t) + g_n(t),\, v_m(\theta_m(t)) - v_n(\theta_n(t))\Big\rangle \;+\; \delta^2(\widetilde e_m + \widetilde e_n)
\end{aligned}$$

Therefore, we get for some positive constant α_2 independent of m, n, and t

$$\begin{aligned}
\Big\langle \dot v_n(t),\, v_n(t) - v_m(t))\Big\rangle \;\le\; &\Big\langle f_n(t) + g_n(t),\, v_m(\theta_m(t)) - v_n(\theta_n(t))\Big\rangle \;+\; \alpha_2(\widetilde e_m + \widetilde e_n) \\
&+ \frac{2\,\delta}{r}\,\Big[\delta\,\widetilde e_n + \|v_n(t) - v_m(t)\|\Big]^2\;.
\end{aligned}$$

In the same way, we also have

$$\begin{aligned}
\Big\langle \dot v_m(t),\, v_m(t) - v_n(t)\Big\rangle \;\le\; &\Big\langle f_m(t) + g_m(t),\, v_n(\theta_n(t)) - v_m(\theta_m(t))\Big\rangle + \alpha_2(\widetilde e_m + \widetilde e_n) \\
&+ \frac{2\,\delta}{r}\,\Big[\delta\,\widetilde e_m + \|v_n(t) - v_m(t)\|\Big]^2\;.
\end{aligned}$$

It then follows from both last inequalities (note that $\|v_n(t)\| \le \|u_0\| + \delta\,T$) that we have for some positive constant β_1 independent of m, n, and t

$$\begin{aligned}
\Big\langle \dot v_m(t) - \dot v_n(t),\, v_m(t) - v_n(t)\Big\rangle \;\le\; &\Big\langle f_m(t) - f_n(t),\, v_n(\theta_n(t)) - v_m(\theta_m(t))\Big\rangle \\
&+ \Big\langle g_m(t) - g_n(t),\, v_n(\theta_n(t)) - v_m(\theta_m(t))\Big\rangle \\
&+ \frac{2\,\delta}{r}\,\|v_n(t) - v_m(t)\|^2 + \frac{\beta_1}{2}(\widetilde e_m + \widetilde e_n)\;.
\end{aligned}$$

By (3.13) one has for all $t \in I$

$$\Big(x_n(\theta_n(t)), v_n(\theta_n(t))\Big) \in \operatorname{gph} K \quad \text{and} \quad f_n(t) \in F\Big(\theta_n(t), x_n(\theta_n(t)), v_n(\theta_n(t))\Big)$$

and hence by the monotony of F with respect to the third variable on $I \times \operatorname{gph} K$ we get

$$\Big\langle f_m(t) - f_n(t),\, v_n(\theta_n(t)) - v_m(\theta_m(t))\Big\rangle \le 0 .$$

On the other hand, one has for some $\beta_2 > 0$ (because $\|v_n(t)\| \le \|u_0\| + \delta T$)

$$\Big\langle g_m(t) - g_n(t),\, v_n(\theta_n(t)) - v_m(\theta_m(t))\Big\rangle \le \frac{\beta_2}{2} \|g_m - g_n\|_\infty \le \frac{\beta_2}{2} (\widetilde{e}_n + \widetilde{e}_m) .$$

Thus we obtain

$$\frac{d}{dt}\Big(\|v_m(t) - v_n(t)\|^2\Big) \le (\beta_1 + \beta_2)\,(\widetilde{e}_m + \widetilde{e}_n) + \frac{4\delta}{r} \|v_m(t) - v_n(t)\|^2 .$$

As $\|v_m(0) - v_n(0)\|^2 = 0$, Gronwall's inequality yields for all $t \in I$

$$\|v_m(t) - v_n(t)\|^2 \le \int_0^t \Big[(\beta_1 + \beta_2)\,(\widetilde{e}_m + \widetilde{e}_n)\, \exp\int_s^t \Big(\frac{4\,\delta}{r}\, d\tau\Big)\Big]\, ds$$

and hence for some positive constant β independent of m, n, and t we have

$$\|v_m(t) - v_n(t)\|^2 \le \beta(\widetilde{e}_m + \widetilde{e}_n) .$$

The Cauchy property in $\mathcal{C}(I, H)$ of the sequence $(v_n)_n$ is thus established and hence this sequence converges uniformly to some Lipschitz mapping u with ratio $l\lambda + 2(\zeta_1 + \zeta_2)$.

Thus the proof of the theorem is complete. ∎

Now, we prove the case when F satisfies the strong linear growth.

Theorem 3.3. *Let $F, G \colon [0, +\infty[\times H \times H \rightrightarrows H$ be two set-valued mappings and $\varsigma > 0$ such that $x_0 + \varsigma\mathbb{B} \subset \mathcal{V}_0$. Assume that the following assumptions are satisfied:*

(i) *K is anti-monotone and for all $x \in \operatorname{cl}(\mathcal{V}_0)$, $K(x) \subset l\,\mathbb{B}$, for some $l > 0$;*

(ii) *F is scalarly u.s.c. on $[0, \frac{\varsigma}{l}] \times \operatorname{gph} K$ with nonempty convex weakly compact values;*

(iii) *G is uniformly continuous on $[0, \frac{\varsigma}{l}] \times \alpha\mathbb{B} \times l\,\mathbb{B}$ into nonempty compact subsets of H, for $\alpha := \|x_0\| + \varsigma$;*

(**iv**) *F and G satisfy the strong linear growth condition.*

Then for every $T \in \,]0, \frac{\zeta}{l}]$ there is a Lipschitz solution $x\colon\, I := [0,T] \to \mathrm{cl}(\mathcal{V}_0)$ of (SSPMP) satisfying $\|\dot{x}(t)\| \le l$ and $\|\ddot{x}(t)\| \le l\lambda + 2(\rho_1 + \rho_2)(1+\alpha+l)$ a.e. on I.

Proof: As in the proof of Theorem 3.2 we have to prove Step 2 and Step 3 in Theorem 3.1, i.e., the uniform convergence of both sequences $x_n(\cdot)$ and $v_n(\cdot)$ and the relative strong compactness of $g_n(\cdot)$. Using the anti-monotony of K we can show as in the proof of Theorem 3.2 the uniform convergence of $x_n(\cdot)$ and so we may assume that (3.30) holds. Also, the relative strong compactness of $g_n(\cdot)$ can be proved as in the proof of Theorem 3.2 by using the strong linear growth condition of G. So we may assume that (3.29) holds. Thus it remains only to prove the uniform convergence of $v_n(\cdot)$. To do that we need, for technical reasons, to fix n_0 as in the proof of Theorem 3.2, i.e., satisfying $((4\sqrt{T}+3)\, l\lambda + 2(\zeta_1+\zeta_2))\, \widetilde{e}_{n_0} \le \frac{r}{2}$.

Put $h_n(t) := \displaystyle\int_0^t f_n(s)\, ds$ and $w_n(t) := v_n(t) - h_n(t)$ for all $t \in I$. By the strong linear growth condition of F and our construction in Theorem 3.1 we have

$$f_n(t) \in (1+\alpha+l)\, \kappa_1 \quad \text{and} \quad h_n(t) \in T(1+\alpha+l)\kappa_1 \quad \text{for all } \ t \in I\ . \tag{3.31}$$

Then Arzelà–Ascoli's theorem ensures that we may extract a subsequence of h_n that converges uniformly to a mapping h with $h(t) = \displaystyle\int_0^t f(s)\, ds$ and f is the weak limit of a subsequence of f_n in $L^1(I,H)$. Put for all $n \ge n_0$

$$\widehat{e}_n := \max\Big\{\delta^{-1}\|h_n - h\|_\infty,\, \widetilde{e}_n\Big\}, \qquad \text{for all } \ n \ge n_0\ . \tag{3.32}$$

Now, we proceed to prove the Cauchy property of the sequence $v_n(\cdot)$ for the uniform convergence in the space of continuous mappings $\mathcal{C}(I,H)$.

Fix $m,n \ge n_0$ and fix also $t \in I$ with $t \ne t_{m,j}$ for $j = 0, \dots, \mu_m - 1$ and $t \ne t_{n,i}$ for $i = 0, \dots, \mu_n - 1$. As in the proof of Theorem 3.2 we get for almost every $t \in I$

$$\begin{aligned}\Big\langle \dot{w}_n(t) - g_n(t),\, v_n(\nu_n(t)) - v_m(t)\Big\rangle \;\le\; & \frac{2\,\delta}{r}\Big[\delta\, \widetilde{e}_n + \|v_n(t) - v_m(t)\|\Big]^2 \\ & + \delta\Big(\alpha_1\, l\, \lambda + 2(\zeta_1+\zeta_2)\Big)\, (\widetilde{e}_n + \widetilde{e}_m)\ .\end{aligned}$$

Then by (3.19), (3.13), (3.31), and (3.32) one gets

$$\begin{aligned}
&\Big\langle \dot w_n(t),\, w_n(t)-w_m(t)\Big\rangle \;\le\\
&\qquad\le\; \Big\langle \dot w_n(t),\, w_n(t)-w_n(\nu_n(t))\Big\rangle + \Big\langle g_n(t),\, v_n(\nu_n(t))-v_m(t)\Big\rangle\\
&\qquad\quad+\; \delta\Big(\alpha_1\, l\,\lambda + 2(\zeta_1+\zeta_2)\Big)\,(\widetilde e_n+\widetilde e_m) + \Big\langle \dot w_n(t),\, h_m(t)-h_n(\nu_n(t))\Big\rangle\\
&\qquad\quad+\; \frac{2\,\delta}{r}\Big[\delta\,\widetilde e_n + \|w_n(t)-w_m(t)\| + \|h_n(t)-h_m(t)\|\Big]^2\\
&\qquad\le\; \delta^2 e_n + \zeta_2\,\delta\, e_n + \Big\langle g_n(t),\, v_n(t)-v_m(t)\Big\rangle\\
&\qquad\quad+\; \delta\Big(\alpha_1\, l\,\lambda + 2(\zeta_1+\zeta_2)\Big)\,(\widetilde e_n+\widetilde e_m) + \delta(\delta\,\widehat e_n + \delta\,\widehat e_m + \zeta_1\, e_n)\\
&\qquad\quad+\; \frac{2\,\delta}{r}\Big[2\,\delta(\widehat e_n+\widehat e_m) + \|w_n(t)-w_m(t)\|\Big]^2 .
\end{aligned}$$

Therefore, we get for some $\beta_1>0$ (independent of m,n, and t)

$$\begin{aligned}
\Big\langle \dot w_n(t),\, w_n(t)-w_m(t)\Big\rangle \;\le\; &\Big\langle g_n(t),\, v_n(t)-v_m(t)\Big\rangle + \frac{\beta_1}{2}\,(\widehat e_n+\widehat e_m)\\
&+\frac{2\,\delta}{r}\Big[2\,\delta(\widehat e_n+\widehat e_m) + \|w_n(t)-w_m(t)\|\Big]^2 .
\end{aligned}$$

In the same way, we also have

$$\begin{aligned}
\Big\langle \dot w_m(t),\, w_m(t)-w_n(t))\Big\rangle \;\le\; &\Big\langle g_m(t),\, v_m(t)-v_n(t)\Big\rangle + \frac{\beta_1}{2}\,(\widehat e_n+\widehat e_m)\\
&+\frac{2\,\delta}{r}\Big[2\,\delta(\widehat e_n+\widehat e_m) + \|w_n(t)-w_m(t)\|\Big]^2 .
\end{aligned}$$

It then follows from both last inequalities, the relation (3.32), the definition of $\widetilde e_n$ and the equiboundedness of v_n and w_n, that for some $\beta_2>0$ independent of m,n, and t one has

$$\Big\langle \dot w_m(t)-\dot w_n(t),\, w_m(t)-w_n(t)\Big\rangle \;\le\; \frac{2\delta}{r}\,\|w_n(t)-w_m(t)\|^2 + \frac{\beta_2}{2}(\widehat e_m+\widehat e_n) .$$

Thus we obtain

$$\frac{d}{dt}\Big(\|w_m(t)-w_n(t)\|^2\Big) \;\le\; \beta_2(\widehat e_m+\widehat e_n) + \frac{4\,\delta}{r}\,\|w_m(t)-w_n(t)\|^2 .$$

As $\|w_m(0)-w_n(0)\|^2=0$, Gronwall's inequality yields for some $\beta>0$ independent of m,n, and t

$$\|w_m(t)-w_n(t)\|^2 \;\le\; \beta^2(\widehat e_m+\widehat e_n) ,$$

for all $t \in I$. Finally, by (3.32) one obtains

$$\|v_m(t) - v_n(t)\| \leq \beta(\widehat{e}_m + \widehat{e}_n)^{1/2} + \delta(\widehat{e}_m + \widehat{e}_n) \leq (\beta + \delta)\,(\widehat{e}_m + \widehat{e}_n)^{1/2} \,.$$

The Cauchy property in $\mathcal{C}(I, H)$ of the sequence $(v_n)_n$ is thus established and hence this sequence converges uniformly to some Lipschitz mapping u with ratio $l\lambda + 2(\zeta_1 + \zeta_2)$. Thus the proof of the theorem is complete. ∎

Remark 3.1. Observe that in the proof of Theorems 3.1, 3.2, and 3.3, the constant of Lipschitz of $\dot{x}$ (the derivative of the solution x) as well as the construction of the sequences and their convergences depend upon the initial point x_0, the neighbourhood $\mathcal{V}_0$, and the constant T. Nevertheless, an inspection of the proof of Theorem 3.1 shows that if we take $\mathcal{V}_0 = H$ and if we replace the linear growth condition of F and G by the following bounded-linear growth condition (bounded in x and linear growth in u)

$$\text{(BLGC)} \qquad F(t, x, u) \subset \rho_1(1 + \|u\|)\,\mathbb{B} \quad \text{and} \quad G(t, x, u) \subset \rho_2(1 + \|u\|)\,\mathbb{B}$$

for all $(t, x, u) \in [0, +\infty[\times \operatorname{gph} K$ for some $\rho_1, \rho_2 \geq 0$, then for every $T > 0$ there exists a solution $x\colon [0, T] \to H$ independently upon the constant T. Consequently, by extending in the evident way the solution x to $[0, +\infty[$ by considering the interval $[0, 1]$ and next the interval $[1, 2]$, etc, we obtain the following global existence result:

Theorem 3.4. *Let $x_0 \in H$, $u_0 \in K(x_0)$, and $G, F\colon [0, +\infty[\times H \times H \rightrightarrows H$ be two set-valued mappings. Assume that the following assumptions are satisfied:*

- **(i)** *For all $x \in H$, $K(x) \subset \mathcal{K}_1 \subset l\,\mathbb{B}$, for some convex compact set $\mathcal{K}_1$ in H and some $l > 0$;*
- **(ii)** *F is scalarly u.s.c. on $[0, +\infty[\times \operatorname{gph} K$ with nonempty convex weakly compact values;*
- **(iii)** *For any $\alpha > 0$, G is uniformly continuous on $[0, +\infty[\times \alpha\mathbb{B} \times l\,\mathbb{B}$ into nonempty compact subsets of H;*
- **(iv)** *F and G satisfy the bounded-linear growth condition (LGC).*

Then there is a Lipschitz solution $x\colon [0, +\infty[\to H$ to

$$\begin{cases} \ddot{x}(t) \in -N\Big(K(x(t)); \dot{x}(t)\Big) + F\Big(t, x(t), \dot{x}(t)\Big) + G\Big(t, x(t), \dot{x}(t)\Big), & \text{a.e. } [0, +\infty[\,; \\ \dot{x}(t) \in K(x(t)), \quad \textit{for all } \ t \in [0, +\infty[\,; \\ x(0) = x_0 \quad \textit{and} \quad \dot{x}(0) = u_0 \,. \blacksquare \end{cases}$$

Remark 3.2. As in Remark 3.1, global existence results can be obtained in Theorems 3.2 and 3.3 when we take $\mathcal{V}_0 = H$ and we replace in Theorems 3.2 (resp. Theorems 3.3) the linear growth for F and the strong linear growth for G (resp. the strong linear growth for both F and G) by the bounded-linear growth (BLGC) for F and the strong bounded-linear growth for G (resp. the strong bounded-linear growth for both F and G, i.e.,

$$F(t,x,u) \subset (1+\|u\|)\,\kappa_1 \quad \text{and} \quad G(t,x,u) \subset (1+\|u\|)\,\kappa_2 \ ,$$

for all $(t,x,u) \in [0,\infty[\times \operatorname{gph} K$, where κ_1 an κ_2 are two convex compact sets in H.) □

4 – Existence results when F is globally measurable and scalarly u.s.c. w.r.t. (x,u)

In the previous section we have proved many existence results for the problem (SSPMP) when the perturbation F is assumed to be globally scalarly u.s.c. Our aim in the present section is to prove that for the problem (SSPCP) (the Second order Sweeping Process with a Convex Perturbation F, i.e., the case when $G = \{0\}$), the global scalarly upper semicontinuity of F on $[0,\frac{\varsigma}{l}] \times \operatorname{gph} K$ can be replaced by the following weaker assumptions:

- (A_1) For any $t \in [0,\frac{\varsigma}{l}]$, the set-valued mapping $F(t;\cdot,\cdot)$ is scalarly u.s.c. on *gph* K;
- (A_2) F is scalarly measurable with respect to the σ-field of $[0,\frac{\varsigma}{l}] \times \operatorname{gph} K$ generated by the Lebesgue sets in $[0,\frac{\varsigma}{l}]$ and the Borel sets in the space H.

Our proof here is based on an approximation method. The idea is to approximate a set-valued mapping F that satisfies (A_1) and (A_2) by a sequence of globally scalarly u.s.c. set-valued mappings F_n and study the convergence of the solutions x_n of $(\text{SSPCP})_n$ associated with each F_n (the existence of such solutions is ensured by our results in Theorems 3.1–3.3). We will use a special approximation F_n of F defined by

$$F_n(t,x,u) \;:=\; \frac{1}{\eta_n}\int_{I_{t,\eta_n}} F(s,x,u)\,ds$$

for all $(t,x,u) \in I\times H\times H$, where I is some compact interval, η_n is a sequence of strictly positive numbers converging to zero and $I_{t,\eta_n} := I \cap [t,t+\eta_n]$. For more details concerning this approximation we refer the reader to [22, 10] and the references therein. We need the two following lemmas. For their proofs we refer to [22, 10].

Lemma 4.1. *Let $T > 0$, S be a Suslin metrizable space, and $F\colon [0,T] \times S$ be a set-valued mapping with nonempty convex weakly compact values. Assume that F satisfies the following assumptions:*

(**a**) *For any $t \in [0,T]$, $F(t;\cdot)$ is scalarly u.s.c. on S;*

(**b**) *F is scalarly measurable w.r.t. the σ-field of $[0,T] \times S$ generated by the Lebesgue sets in $[0,T]$ and the Borel sets in the topological space S;*

(**c**) *$F(t,y) \subset \rho(1 + \|y\|)\,\mathbb{B}$, for all $(t,y) \in [0,T] \times S$ and for some $\rho > 0$. Then F_n is a globally scalarly u.s.c. set-valued mapping on $[0,T]\times S$ with nonempty convex compact values satisfying*

$$F_n(t,y) \subset \rho T\,(1 + \|y\|)\,\mathbb{B}\ ,$$

for all $(t,y) \in [0,T] \times S$ and all n.

Lemma 4.2. *Let $T>0$, S be a Suslin metrizable space and $F\colon [0,T]\times S \rightrightarrows H$ be a set-valued mapping with nonempty convex weakly compact values. Assume that F is bounded on $[0,T] \times S$ and that satisfies the hypothesis (a), (b) and (c) in Lemma 4.1. Then for any sequence y_n of Lebesgue measurable mappings from $[0,T]$ to S which converges pointwisely to a Lebesgue measurable mapping y, any sequence z_n in $L^1([0,T],H)$ weakly converging to z in $L^1([0,T],H)$ and satisfying $z_n(t) \in F_n(t,y_n(t))$ a.e. on I one has*

$$z(t) \in F(t,y(t)) \qquad \text{a.e. on} \ \ [0,T]\ .$$

Now we are able to prove our first result in this section.

Theorem 4.1. *Let $F\colon [0,+\infty[\, \times H \times H \rightrightarrows H$ be a set-valued mapping and $\varsigma > 0$ such that $x_0+\varsigma\,\mathbb{B} \subset \mathcal{V}_0$. Assume that the hypothesis (i), (iv) in Theorem 3.1 are satisfied and assume that F satisfies (A_1) and (A_2). Then for every $T \in\,]0,\frac{\varsigma}{l}]$ there exists a Lipschitz solution $x\colon [0,T]\to \mathrm{cl}(\mathcal{V}_0)$ of (SSPCP) satisfying $\|\dot{x}(t)\|\le l$ and $\|\ddot{x}(t)\| \le l\lambda + 2\,T\rho_1(1 + \alpha + l)$ a.e. on $[0,T]$.*

Proof: Let $T \in\,]0,\frac{\varsigma}{l}]$ and put $I := [0,T]$ and $S := \alpha\mathbb{B} \times l\,\mathbb{B}$. Clearly S is a Suslin metrizable space. Let η_n be a sequence of strictly positive numbers that converges to zero. For each $n \geq 1$ we put

$$F_n(t,x,u) := \frac{1}{\eta_n} \int_{I_{t,\eta_n}} F(s,x,u)\, ds$$

for all $(t,x,u) \in I \times H \times H$. By Lemma 4.1 the set-valued mappings F_n are scalarly u.s.c. on $I \times S$ with nonempty convex compact values and satisfies

$$F_n(t,x,u) \subset T\rho_1(1+\|x\|+\|u\|)\mathbb{B} \subset T\rho_1(1+\alpha+l)\mathbb{B} =: T\,\zeta_1\,\mathbb{B}\ ,$$

for any $(t,x,u) \in I \times S$ and all $n \geq 1$. So that we can apply the result of Theorem 3.1. For each $n \geq 1$, there exists a Lipschitz mapping $x_n\colon I \to \mathrm{cl}(\mathcal{V}_0)$ satisfying

$$(\mathrm{SSPCP})_n \quad \begin{cases} \ddot{x}_n(t) \in -N\Big(K(x_n(t)); \dot{x}_n(t)\Big) + F_n\Big(t, x_n(t), \dot{x}_n(t)\Big), & \text{a.e. on } I\ ; \\ \dot{x}_n(t) \in K(x_n(t)), \quad \text{for all } t \in I\ ; \\ x_n(0) = x_0 \quad \text{and} \quad \dot{x}_n(0) = u_0\ , \end{cases}$$

with $\|\dot{x}_n(t)\| \leq l$ and $\|\ddot{x}_n(t)\| \leq l\lambda + 2\,T\,\zeta_1$ a.e. on I and for all $n \geq 1$.

Since $\dot{x}_n(t) \in K(x_n(t)) \subset \mathcal{K}_1$ for all $n \geq 1$ and all $t \in I$, then we get the relative strong compactness of the set $\{\dot{x}_n(t)\colon n \geq 1\}$ in H for all $t \in I$. Therefore, by Arzelà–Ascoli's theorem we may extract from $\dot{x}_n$ a subsequence that converges uniformly to some Lipschitz mapping $\dot{x}$. By integrating, we get the uniform convergence of the sequence x_n to x because they have the same initial value $x_n(0) = x_0$, for all $n \geq 1$. Now, by $(\mathrm{SSPCP})_n$ there is for any $n \geq 1$ a Lebesgue measurable mapping $f_n\colon I \to H$ such that

$$(4.1) \qquad f_n(t) \in F_n\Big(t, x_n(t), \dot{x}_n(t)\Big) \subset T\rho_1\Big(1 + \|x_n(t)\| + \|\dot{x}_n(t)\|\Big)\mathbb{B} \subset T\,\zeta_1\,\mathbb{B}$$

and

$$(4.2) \qquad f_n(t) - \ddot{x}_n(t) \in N\Big(K(x_n(t)); \dot{x}_n(t)\Big) \cap \delta\,\mathbb{B} = \delta\,\partial d_{K(x_n(t))}(\dot{x}_n(t))\ ,$$

for a.e. $t \in I$, where $\delta := l\lambda + 3\,T\zeta_1$. Observe by (4.1) and $(\mathrm{SSPCP})_n$ that f_n and $\ddot{x}_n(\cdot)$ are equibounded in $L^1(I,H)$ and so subsequences may be extracted that converge in the weak topology of $L^1(I,H)$. Without loss of generality, we may suppose that these subsequences are f_n and $(\ddot{x}_n)_n$ respectively. Denote by f and w their weak limits respectively. Then, for each $t \in I$

$$u_0 + \int_0^t \ddot{x}(s)\, ds = \dot{x}(t) = \lim_{n\to\infty} \dot{x}_n(t) = u_0 + \lim_{n\to\infty} \int_0^t \ddot{x}_n(s)\, ds = u_0 + \int_0^t w(s)\, ds\ ,$$

which gives the equality $\ddot{x}(t) = w(t)$ for almost all $t \in I$, that is, $(\ddot{x}_n)_n$ converges weakly in $L^1(I,H)$ to $\ddot{x}$.

It follows then from $(\mathrm{SSPCP})_n$ and the Lipschitz property of K that

$$d_{K(x(t))}(\dot{x}_n(t)) \leq \mathcal{H}\Big(K(x(t)), K(x_n(t)\Big) \leq \|x_n(t) - x(t)\| \to 0 ,$$

and hence one obtains $\dot{x}(t) \in K(x(t))$, because the set $K(x(t))$ is closed.

Now, we apply Castaing techniques (see for example [8]). The weak convergence in $L^1(I,H)$ of $(\ddot{x}_n)_n$ and $(f_n)_n$ to $\ddot{x}$ and f respectively entail for almost all $t \in I$ (by Mazur's lemma)

$$f(t) - \ddot{x}(t) \in \bigcap_n \overline{co}\Big\{f_k(t) - \ddot{x}_k(t)\colon\, k \geq n\Big\} .$$

Fix any such $t \in I$ and consider any $\xi \in H$. The last relation ensures

$$\Big\langle \xi,\, f(t) - \ddot{x}(t)\Big\rangle \leq \inf_n \sup_{k\geq n} \Big\langle \xi,\, f_k(t) - \ddot{x}_k(t)\Big\rangle ,$$

and hence according to (4.2) and Theorem 2.1 we get

$$\Big\langle \xi,\, f(t) - \ddot{x}(t)\Big\rangle \leq \limsup_n \sigma\Big(\delta\, \partial d_{K(x_n(t))}(\dot{x}_n(t)),\, \xi\Big) \leq \sigma\Big(\delta\, \partial d_{K(x(t))}(\dot{x}(t)),\, \xi\Big) .$$

As the set $\partial d_{K(t)}(u(t))$ is closed and convex (see Proposition 2.1), we obtain

$$f(t) - \ddot{x}(t) \in \delta\, \partial d_{K(x(t))}(\dot{x}(t)) \subset N\Big(K(x(t)); \dot{x}(t)\Big) , \tag{4.3}$$

because $\dot{x}(t) \in K(x(t))$. Now we check that $f(t) \in F(t, x(t), \dot{x}(t))$ a.e. on I. Since F is bounded on $I \times S$, f_n converges weakly to f in $L^1(I,H)$, and $(x_n, \dot{x}_n)$ is a sequence of Lebesgue measurable mappings from I to S (because $\dot{x}_n(t) \in K(x_n(t)) \subset l\,\mathbb{B}$ and $\|x_n(t)\| \leq \alpha$ for all $t \in I$) converging uniformly to $(x, \dot{x})$, it follows then from Lemma 4.2 that $f(t) \in F(t, x(t), \dot{x}(t))$ for a.e. on I. Consequently, we obtain by (4.3)

$$\ddot{x}(t) \in -N\Big(K(x(t)); \dot{x}(t)\Big) + F\Big(t, x(t), \dot{x}(t)\Big) .$$

Thus completing the proof of the theorem. ∎

Now we prove our second main result in this section.

Theorem 4.2. *Let $F\colon [0, +\infty[\times H \times H \rightrightarrows H$ be a set-valued mapping and $\varsigma > 0$ such that $x_0 + \varsigma\,\mathbb{B} \subset \mathcal{V}_0$. Assume that the hypothesis (i) and (iv) in Theorem 3.3 and (A_1) and (A_2) are satisfied. Then for every $T \in\,]0, \frac{\varsigma}{l}]$ there exists a Lipschitz solution $x : [0,T] \to \mathrm{cl}(\mathcal{V}_0)$ of (SSPCP) satisfying $\|\dot{x}(t)\| \leq l$ and $\|\ddot{x}(t)\| \leq l\lambda + 2T\rho_1(1 + \alpha + l)$ a.e. on $[0,T]$.*

Proof: We do as in the proof of Theorem 4.1 to get, for all $n \geq 1$, a Lipschitz solution x_n of $(\mathrm{SSPCP})_n$ with the estimates with $\|\dot{x}_n(t)\| \leq l$ and $\|\ddot{x}_n(t)\| \leq l\lambda + 2T\zeta_1$ a.e. on I. Then, we prove the uniform convergence of the sequences $x_n(\cdot)$ and $\dot{x}_n(\cdot)$. For this end, we denote $w_{m,n}(t) := \frac{1}{2}\|x_n(t) - x_m(t)\|^2$, for all $t \in I$ and for every $m, n \geq 1$. Then

$$\frac{d^+ w_{m,n}}{dt}(t) = \Big\langle \dot{x}_m(t) - \dot{x}_n(t),\, x_m(t) - x_n(t) \Big\rangle, \qquad \text{for all } \ t \in [0, T[\ .$$

Therefore by $(\mathrm{SSPP})_n$ and the anti-monotony of K we get

$$\frac{d^+ w_{m,n}}{dt}(t) \leq 0\ ,$$

for all $t \in [0, T[$. Moreover, by $(\mathrm{SSPP})_n$ one has $w_{m,n}(0) = \frac{1}{2}\|x_n(0) - x_m(0)\|^2 = 0$. Hence $w_{m,n}(t) = 0$ for all $t \in I$ and then $x_n(\cdot)$ is a constant sequence. Let x be its limit. Then $(\dot{x}_n)$ and $(\ddot{x}_n)$ converge uniformly to $\dot{x}$ and $\ddot{x}$ respectively.

Now, by $(\mathrm{SSPP})_n$ there is for any $n \geq 1$ a Lebesgue measurable mapping $f_n \colon I \to H$ such that

$$f_n(t) \in F_n\Big(t, x(t), \dot{x}(t)\Big) \subset T\Big(1 + \|x(t)\| + \|\dot{x}(t)\|\Big)\mathcal{K} \subset T\,\zeta_1 \mathbb{B} \tag{4.4}$$

and

$$f_n(t) - \ddot{x}(t) \in N\Big(K(x(t)); \dot{x}(t)\Big) \cap \delta\,\mathbb{B} = \delta\,\partial d_{K(x(t))}(\dot{x}(t))\ , \tag{4.5}$$

for a.e. $t \in I$, where $\delta := l\lambda + 3T\zeta_1$. Observe by (4.4) that f_n is equibounded in $L^1(I, H)$ and so a subsequence may be extracted that converges in the weak topology of $L^1(I, H)$. Without loss of generality, we may suppose that this subsequence is f_n. Denote by f its weak limit. Then, by using Mazur's lemma and the properties of the subdifferential of the distance function in Proposition 2.1, it is easy to conclude that for almost every $t \in I$

$$f(t) \in \delta\,\partial d_{K(x(t))}(\dot{x}(t)) + \ddot{x}(t) \subset N\Big(K(x(t)); \dot{x}(t)\Big) + \ddot{x}(t)\ . \tag{4.6}$$

Finally, with the same arguments, as in the proof of Theorem 4.1, we can check that $f(t) \in F(t, x(t), \dot{x}(t))$ a.e. on I and so we obtain by (4.6)

$$\ddot{x}(t) \in -N\Big(K(x(t)); \dot{x}(t)\Big) + F\Big(t, x(t), \dot{x}(t)\Big)\ .$$

Thus completing the proof. ∎

Remark 4.1. The generalization of Theorem 3.2, in the same way as like in Theorems 4.1–4.2, to the case of set-valued mappings F satisfying the assumptions (A_1) and (A_2), depends on the monotony of the approximation F_n which is the key of Theorem 3.2. Since one cannot be sure that the monotony of F whether implies or not the monotony of F_n, then it is not clear for us the generalization of Theorem 3.2. Thus, the question will be what are the assumptions on F implying the monotony of F_n? Under such assumptions the both proofs in Theorems 4.1–4.2 still work to obtain a generalization of Theorem 3.2. □

5 – Solution sets

Throughout this section, let $r \in]0,+\infty]$, Ω be an open subset in H, $F\colon [0,+\infty[\times H \times H \rightrightarrows H$ be a set-valued mapping, and $K\colon \mathrm{cl}(\Omega) \rightrightarrows H$ be a Lipschitz set-valued mapping with ratio $\lambda > 0$ taking nonempty closed uniformly r-prox-regular values in H. In this section we are interested by some topological properties of the solution set of the problem (SSPCP). Let $x_0 \in \Omega$, $u_0 \in K(x_0)$, and $T > 0$ such that $x_0 + T\,l\,\mathbb{B} \subset \Omega$. We denote by $\mathcal{S}_F(x_0,u_0)$ the set of all continuous mappings $(x,u)\colon [0,T] \to \mathrm{cl}(\Omega)\times H$ such that

$$\text{(SSPCP)} \qquad \begin{cases} u(0) = u_0 \ ; \\ x(t) = x_0 + \displaystyle\int_0^t u(s)\,ds \ , \qquad \text{for all} \ \ t \in [0,T] \ ; \\ u(t) \in K(x(t)), \qquad \text{for all} \ \ t \in [0,T] \ ; \\ \dot u(t) \in -N\Big(K(x(t)); u(t)\Big) + F\Big(t, x(t), u(t)\Big), \qquad \text{a.e. on} \ \ [0,T] \ . \end{cases}$$

Proposition 5.1. *Assume that the hypothesis of one of the Theorems 3.1, 3.2 and 3.3 are satisfied and that* $\mathrm{gph}\,K$ *is strongly compact in* $\mathrm{cl}(\Omega) \times l\,\mathbb{B}$. *Then the set* $\mathcal{S}_F(x_0,u_0)$ *is relatively strongly compact in* $\mathcal{C}([0,T], H\times H)$.

Proof: By Theorem 3.1, 3.2, and 3.3 the set of solution (x,u) of (SSPCP) are equi-Lipschitz and for any $t \in [0,T]$ one has $\{(x(t),u(t))\colon (x,u) \in \mathcal{S}_F(x_0,u_0)\}$ is relatively strongly compact in $H\times H$ because it is contained in the strong compact set $\mathrm{gph}\,K$. Then Arzelà–Ascoli's theorem gives the relative strong compactness of the set $\mathcal{S}_F(x_0,u_0)$ in $\mathcal{C}(I, H\times H)$. ∎

Remark 5.1. Assume that $\Omega = H$ and let T be any strictly positive number.

Put

$$\mathcal{S}_F(\operatorname{gph} K) \ := \bigcup_{(x_0,u_0)\,\in\,\operatorname{gph} K} \mathcal{S}_F(x_0,u_0)\ .$$

With the same arguments, as in the proof of Proposition 5.1, we can show that under the same hypothesis in Proposition 5.1 the set $\mathcal{S}_F(\operatorname{gph} K)$ is relatively strongly compact in $\mathcal{C}([0,T],H{\times}H)$. ◻

Now we wish to prove the closedness of the set-valued mapping $\mathcal{S}_F$.

Proposition 5.2. *Assume that the hypothesis of one of the Theorems 3.1, 3.2, and 3.3 are satisfied. Then the set-valued mapping $\mathcal{S}_F$ has a closed graph in $\Omega \times K(\Omega) \times \mathcal{C}([0,T],H{\times}H)$.*

Proof: Let $((x_0^n,u_0^n))_n \in \Omega{\times}K(\Omega)$ and $((x^n,u^n))_n \in \mathcal{C}([0,T],H{\times}H)$ with $(x^n,u^n) \in \mathcal{S}_F((x_0^n,u_0^n))$ such that $(x_0^n,u_0^n) \to (x_0,u_0) \in \Omega{\times}K(\Omega)$ uniformly, and $(x^n,u^n) \to (x,u) \in \mathcal{C}([0,T],H{\times}H)$ uniformly. We have to show that $(x,u) \in \mathcal{S}_F(x_0,u_0)$. First observe that for n sufficiently large $x_0^n \in x_0 + l\,T\,\mathbb{B}$. Now, it is not difficult to check that the closedness of gph K and the uniform convergence of both sequences $((x_0^n,u_0^n))_n$ and $((x^n,u^n))_n$ imply that $(x(0),u(0)) = (x_0,u_0)$ and that $u(t) \in K(x(t))$ for all $t \in [0,T]$. On the other hand one has for all $t \in [0,T]$

$$x(t) \;=\; \lim_n x^n(t) \;=\; x_0 + \lim_n \int_0^t u^n(s)\,ds \;=\; x_0 + \int_0^t u(s)\,ds\ .$$

It remains then to show that

$$\dot u(t) \,\in\, -N\Big(K(x(t));u(t)\Big) + F\Big(t,x(t),u(t)\Big), \qquad \text{a.e. on}\ \ [0,T]\ .$$

For every n, one has

$$\dot u^n(t) \,\in\, -N\Big(K(x^n(t));u^n(t)\Big) + F\Big(t,x^n(t),u^n(t)\Big), \qquad \text{a.e. on}\ \ [0,T]\ .$$

Then for every n there exists a measurable selection f^n such that

$$\text{(5.1)}\qquad f^n(t) \in F\Big(t,x^n(t),u^n(t)\Big) \ \ \text{and} \ \ -\dot u^n(t) + f^n(t) \in N\Big(K(x^n(t));u^n(t)\Big)\ ,$$

for a.e. $t \in [0,T]$. By Theorems 3.1, 3.2, and 3.3 one has for n sufficiently large

$$\text{(5.2)}\quad \|\dot u^n(t)\| \,\le\, l\lambda + 2\,\rho_1(1+\|x_0^n\| + Tl + l) \,\le\, l\lambda + 2\,\rho_1(1+\|x_0\| + 2\,Tl + l)\ .$$

By (iv) in Theorems 3.1, 3.2, and 3.3 and the fact that $u^n(t) \in K(x^n(t))$ one gets

$$\text{(5.3)}\qquad\qquad \|f^n(t)\| \,\le\, \rho_1(1+\|x_0\| + Tl + l)\ .$$

Therefore, we may suppose without loss of generality that $\dot{u}^n \to \dot{u}$ and $f^n \to f$ weakly in $L^1([0,T],H)$. Since $F(t,\cdot,\cdot)$ is scalarly upper semicontinuous with convex compact values, then we get easily that $f(t) \in F(t,x(t),u(t))$ a.e. $t \in [0,T]$. Now by (5.1), (5.2), (5.3) and Theorem 4.1 in [6] we have for $\delta := l\lambda + 3\,\rho_1(1+\|x_0\|+2\,Tl+l)$

$$-\dot{u}^n(t) + f^n(t) \,\in\, \delta\,\partial d_{K(x^n(t))}(u^n(t)) \qquad \text{a.e. } \ t \in [0,T] \ .$$

Then by using Mazur's lemma and Theorem 2.1, it is easy to conclude that for a.e. $t \in [0,T]$

$$f(t) - \dot{u}(t) \,\in\, \delta\,\partial d_{K(x(t))}(u(t)) \,\subset\, N\Big(K(x(t)); u(t)\Big) \ .$$

Thus we get for a.e. $t \in [0,T]$

$$\dot{u}(t) \,\in\, -N\Big(K(x(t)); u(t)\Big) + F\Big(t, x(t), u(t)\Big) \ ,$$

which completes the proof of the proposition. ∎

Remark 5.2. The proof of Proposition 5.1 shows that the the solution set $\mathcal{S}_F(x_0,u_0)$ associated to the problem (SSPMP) is relatively strongly compact in $\mathcal{C}([0,T],H{\times}H)$ whenever the graph $\operatorname{gph} K$ is strongly compact in H. Contrarily, our proof in Proposition 5.2 cannot provide the closedness of the graph of the set-valued mapping $\mathcal{S}_F$ associated to the problem (SSPMP). The difficulty that prevents to conclude is the absence of the convexity of G. □

6 – Particular case

In this section let H be a finite dimensional space and let us focus our attention to the special case when F is defined by

$$F(t,x,u) = -\partial^C f_t(x) + \gamma\, u \ ,$$

where $\gamma \in \mathbb{R}$, $f_t := f(t,\cdot)$, $f\colon\ [0,T] \times \operatorname{cl}(\mathcal{V}_0) \to \mathbb{R}$ is a globally measurable function and β-equi-Lispchitz w.r.t. the second variable, $\mathcal{V}_0$ is an open neighbourhood of x_0, and $T>0$ satisfies $x_0 + T\,l\,\mathbb{B} \subset \mathcal{V}_0$. Here $\partial^C f_t(x)$ denotes the Clarke subdifferential of f_t at x given by

$$\partial^C f_t(x) \,=\, \Big\{\xi \in H\colon\ \langle \xi, h\rangle \le f_t^0(x;h), \ \text{ for all } h \in H\Big\} \ ,$$

where $f_t^0(x;h)$ is the Clarke directional derivative of f_t at x in the direction h, that is,

$$f_t^0(x;h) := \limsup_{\substack{\delta\downarrow 0\\ x'\to x}} \delta^{-1}\Big[f(t,x'+\delta h) - f(t,x')\Big] .$$

It is not difficult to see that the set-valued mapping F satisfies the hypothesis (A_1), (A_2), and (iv) in Theorem 4.1. Indeed, for the hypothesis (A_1), (A_2) it suffices to observe that the support function associated with F is given by

$$\sigma\Big(F(t,x,u),\, h\Big) \;=\; \sigma\Big(-\partial^C f_t(x),\, h\Big) + \gamma\langle u,h\rangle \;=\; (-f_t)^0(x;h) + \gamma\langle u,h\rangle \ ,$$

for all $h \in H$. Then the measurability and the scalar u.s.c. of F follow easily from the hypothesis on f and the properties of the Clarke directional derivative. Since f_t is β-equi-Lispchitz w.r.t. the second variable we get

$$F(t,x,u) = -\partial^C f_t(x) + \gamma\, u \;\subset\; \beta\,\mathbb{B} + \gamma u \;\subset\; \rho_1(1+\|u\|)\,\mathbb{B} \ ,$$

with $\rho_1 := \max\{\beta, |\gamma|\}$ and so the hypothesis (iv) is satisfied. Now applying Theorem 4.1 we get the following result.

Theorem 6.1. *For every $u_0 \in K(x_0)$ there is a Lipschitz solution $x\colon [0,T]\to \mathrm{cl}(\mathcal{V}_0)$ to the Cauchy problem for the second order differential inclusion:*

$$\begin{cases} \ddot{x}(t) \in -N\Big(K(x(t)); \dot{x}(t)\Big) - \partial^C f_t(x(t)) + \gamma\,\dot{x}(t), & \textit{a.e. on } [0,T] \ ;\\ \dot{x}(t) \in K(x(t)), \quad \textit{for all } t\in[0,T] \ ; \\ x(0) = x_0 \quad \textit{and} \quad \dot{x}(0) = u_0 \ , \end{cases}$$

with $\|\dot{x}(t)\| \le l$ and $\|\ddot{x}(t)\| \le l\lambda + 2T\rho_1(1+\alpha+l)$. ∎

It would be interesting to ask whether the result in Theorem 6.1 remains true if we take f_t is not necessarily Lipschitz? Such problem is till now open and in our opinion is so hard to attacked it in a direct manner. Nevertheless in what follows we give a positive answer for a special case when f_t is the indicator function associated to some set-valued mapping C. To this aim we use the result stated in Theorem 6.1 for the distance function which satisfies all the hypothesis of that theorem and then we prove the viability of the solution x, that is, $x(t) \in C(t)$ for all $t \in I$. So applying Theorem 6.1 for $f_t = d_{C(t)}$ we get a Lipschitz mapping $x\colon [0,T]\to \mathrm{cl}(\mathcal{V}_0)$ such that

$$\begin{cases} \ddot{x}(t) \in -N\Big(K(x(t)); \dot{x}(t)\Big) - \partial^C d_{C(t)}(x(t)) + \gamma\,\dot{x}(t), & \text{a.e. on } [0,T] \ ;\\ \dot{x}(t) \in K(x(t)), \quad \text{for all } t\in[0,T] \ ; \\ x(0) = x_0 \quad \text{and} \quad \dot{x}(0) = u_0 \ , \end{cases}$$

with $\|\dot{x}(t)\| \leq l$ and $\|\ddot{x}(t)\| \leq l\lambda + 2T\rho_1(1+\alpha+l)$, where $\rho_1 := \max\{|\gamma|, 1\}$. Now we come back to our construction in Theorem 3.1. Observe that the solution x is always bounded by $\|x_0\| + lT$. Thus if we assume that for all $t \in [0,T]$ the set $C(t)$ contains the ball $M\mathbb{B}$ where $M := \|x_0\| + lT$, then we get $x(t) \in C(t)$ and consequently the solution would satisfy $\partial^C d_{C(t)}(x(t)) \subset N^C(C(t); x(t)) = \partial^C \psi_{C(t)}(x(t))$ and so

$$\ddot{x}(t) \in -N\Big(K(x(t)); \dot{x}(t)\Big) - N\Big(C(t); x(t)\Big) + \gamma\,\dot{x}(t) \;.$$

Therefore we obtain the following result.

Theorem 6.2. *Let $C\colon [0,T] \rightrightarrows H$ be any set-valued mapping such that its associated distance function to images $(t,x) \mapsto d_{C(t)}(x)$ is globally measurable. Assume that l, T, and x_0 satisfy $(\|x_0\| + lT)\,\mathbb{B} \subset C(t)$ for all $t \in [0,T]$. Then, for every $u_0 \in K(x_0)$ there is a Lipschitz mapping $x\colon [0,T] \to \mathrm{cl}(\mathcal{V}_0)$ satisfying*

$$\begin{cases} \ddot{x}(t) \in -N\Big(K(x(t)); \dot{x}(t)\Big) - N\Big(C(t); x(t)\Big) + \gamma\,\dot{x}(t), & \text{a.e. on } [0,T]\;; \\ \dot{x}(t) \in K(x(t)), \quad \text{for all } t \in [0,T]\;; \\ x(0) = x_0 \quad \text{and} \quad \dot{x}(0) = u_0\;, \end{cases}$$

with $\|\dot{x}(t)\| \leq l$ and $\|\ddot{x}(t)\| \leq l\lambda + 2\rho_1(1+\alpha+l)$, where $\rho_1 := \max\{|\gamma|, 1\}$. ∎

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Messaoud Bounkhel,
Department of Mathematics, College of Science,
King Saud University, P.O. Box 2455, Riyadh 11451 – SAUDI ARABIA
E-mail: bounkhel@ksu.edu.sa