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THE COMPRESSION SEMIGROUP OF A CONE IS CONNECTED *

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Abstract: Let $W \subset \mathbb{R}^n$ be a pointed and generating cone and denote by $S(W)$ the semigroup of matrices with positive determinant leaving W invariant. The purpose of this paper is to prove that $S(W)$ is path connected. This result has the following consequence: Semigroups with nonempty interior in the group $\mathrm{Sl}(n,\mathbb{R})$ are classified into types, each type being labelled by a flag manifold. The semigroups whose type is given by the projective space $\mathbb{P}^{n-1}$ form one of the classes. It is proved here that the semigroups in $\mathrm{Sl}(n,\mathbb{R})$ leaving invariant a pointed and generating cone are the only maximal connected in the class of $\mathbb{P}^{n-1}$.

1 – Introduction

Let W be a convex cone in $\mathbb{R}^n$ and form its compression semigroup of matrices

$$S(W) = \Big\{ g \in \mathrm{Gl}^+(n,\mathbb{R}) \colon \; gW \subset W \Big\} ~,$$

where $\mathrm{Gl}^+(n,\mathbb{R})$ stands for the group of real matrices having positive determinant. The purpose of this paper is to prove that $S(W)$ is connected if mild conditions on W are assumed. Precisely, recall that W is said to a be pointed cone in case $\pm v \in W$ implies $v = 0$. Also, W is generating if $\mathbb{R}^n = W + (-W)$, or equivalently, if $\operatorname{int} W \neq \emptyset$, where int stands for the interior of a set with respect to the standard topology of $\mathbb{R}^n$.

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Theorem 1. *If W is pointed and generating then $S(W)$ is path connected.*

Clearly, $S(W)$ is a closed subsemigroup of $\mathrm{Gl}^+(n,\mathbb{R})$. Moreover, it is known — and easy to prove — that in case W is pointed and generating, $S(W)$ has nonempty interior in $\mathrm{Gl}^+(n,\mathbb{R})$, taken with its standard topology (cf. Proposition 4 below).

Apart from being fruitful examples of semigroups in Lie groups the interest in the semigroups $S(W)$ stays in the fact that they form (in essence) a class of maximal semigroups in the special linear group $\mathrm{Sl}(n,\mathbb{R})$. In order to discuss this we note first that the identity matrix 1 as well as the scalar matrices $\lambda\cdot 1$, $\lambda>0$, are in $S(W)$. Analogously, a matrix $g\in S(W)$ if and only if $(\det g)^{1/n}g\in S(W)$. Therefore if we consider the compression semigroup

$$S_W \;=\; S(W)\cap \mathrm{Sl}(n,\mathbb{R}) \;=\; \Big\{g\in \mathrm{Sl}(n,\mathbb{R}):\; gW\subset W\Big\}\;,$$

it follows that $S(W)=\mathbb{R}^+\cdot S_W$ and S_W is the image of $S(W)$ under the continuous map $g\mapsto(\det g)^{1/n}g$. Hence if one of the semigroups $S(W)$ or S_W is connected, the same happens to the other. In what follows we take advantage of the theory of semigroups in semi-simple Lie groups and work within $\mathrm{Sl}(n,\mathbb{R})$. The proof of Theorem 1 will be accomplished by showing that S_W is connected.

To see the connection between S_W and maximal semigroups in $\mathrm{Sl}(n,\mathbb{R})$ let $[W]$ be the subset of the projective space $\mathbb{P}^{n-1}$ underlying W, that is, $[W]$ is the subset of lines in $\mathbb{R}^n$ contained in $W\cup -W$. Put

$$S[W] \;=\; \Big\{g\in \mathrm{Sl}(n,\mathbb{R}):\; g[W]\subset [W]\Big\}\;.$$

It was proved in [11], Theorem 6.12, that $S[W]$ is a maximal semigroup of $\mathrm{Sl}(n,\mathbb{R})$ (see also [9], for more details about maximal semigroups). Clearly $g\in S[W]$ if and only if $g\in S_W$ or $gW\subset -W$. It is rather easy to prove the existence of $g\in S[W]$ such that $gW\subset -W$ (see Lemma 11, below), so that S_W is not a maximal semigroup. However, by proceeding like in the proof that $S[W]$ is connected we get that S_W is a *maximal connected* semigroup in the sense that if $S_W\subset T$ with T a connected subsemigroup of $\mathrm{Sl}(n,\mathbb{R})$ then either $T=S_W$ or $T=\mathrm{Sl}(n,\mathbb{R})$.

Corollary 2. *S_W is maximal connected in* $\mathrm{Sl}(n,\mathbb{R})$.

There is a converse to this corollary, ensuring that a semigroup in a certain class of maximal connected subsemigroups of $\mathrm{Sl}(n,\mathbb{R})$ must be S_W for some pointed and generating cone W. This is the class of semigroups whose type is the projective space $\mathbb{P}^{n-1}$. We refer the reader to [8] and [9] for the definition of the type of a semigroup and in particular of the type of S_W (see also [7] for a discussion specific to semigroups in $\mathrm{Sl}(n,\mathbb{R})$). It was observed in [8], Example 4.10, that if a semigroup is connected and of type $\mathbb{P}^{n-1}$ then it is contained in S_W for some pointed generating cone $W \subset \mathbb{R}^n$. Therefore we get from the fact that S_W is connected the following characterization of the maximal connected semigroups of the projective space type:

Corollary 3. *Let $\mathcal{C}$ stand for the class of semigroups $S \subset \mathrm{Sl}(n,\mathbb{R})$, with $\operatorname{int} S \neq \emptyset$, which are maximal connected of type $\mathbb{P}^{n-1}$. Then*

$$\mathcal{C} = \Big\{S_W \colon\, W \subset \mathbb{R}^n \text{ is a pointed generating cone}\Big\} .$$

Finally we mention that the semigroup of all matrices leaving invariant a cone W — without any determinantal restriction — is trivially a convex cone in the space of matrices, and hence connected. Our results, however, refer to the semigroups $S(W)$ and S_W which are far from being convex cones. In fact, it was proved in [10] that the topology of these semigroups is rather rich, since they have the same homotopy groups as the orthogonal group $\mathrm{SO}(n-1)$.

2 – S_W is connected

In this section we prove the main result of this paper, namely Theorem 1. From now on we let $W \subset \mathbb{R}^n$ stand for a pointed generating convex cone. As before denote by S_W the semigroup of matrices in $\mathrm{Sl}(n,\mathbb{R})$ leaving W invariant.

We refer the reader to Hilgert, Hofmann and Lawson [2] for the general theory of semigroups. In particular, the concept of Lie wedge $\mathcal{L}(S)$ of a semigroup $S \subset \mathrm{Sl}(n,\mathbb{R})$ is defined by

$$\mathcal{L}(S) = \Big\{X \in \mathfrak{sl}(n,\mathbb{R}) \colon\, \exp(tX) \in \operatorname{cl} S \text{ for all } t \geq 0\Big\}$$

where $\mathfrak{sl}(n,\mathbb{R})$ is the Lie algebra of trace zero $n{\times}n$-matrices. In what follows we denote by S_{inf} the semigroup generated by $\mathcal{L}(S_W)$, namely

$$S_{\mathrm{inf}} = \langle \exp(\mathcal{L}(S_W)) \rangle .$$

Since S_W is closed, it follows that S_{inf} is a subsemigroup of S_W. Furthermore, being generated by one-parameter semigroups S_{inf} contains the identity and is path connected. It is a consequence of the next statement that $\mathcal{L}(S)$ is a generating cone in $\mathfrak{sl}(n,\mathbb{R})$, implying that $S_{\inf}$ has nonempty interior in $\mathrm{Sl}(n,\mathbb{R})$ and that the interior of S_W is dense in S_W, i.e., $S_W = \mathrm{cl}(\mathrm{int}\, S_W)$.

Proposition 4. *Suppose that $V \subset \mathbb{R}^n$ is a codimension one subspace with $V \cap W = \{0\}$. Take a basis*

$$\beta = \{f_1, \ldots, f_n\}$$

of $\mathbb{R}^n$ such that $f_1 \in W$ and $\{f_2, \ldots, f_n\} \subset V$. Let $H \in \mathfrak{sl}(n,\mathbb{R})$ be such that its matrix with respect to β is

$$H = \mathrm{diag}\{n-1, -1, \ldots, -1\} \ .$$

Then $H \in \mathcal{L}(S_W)$. Moreover, if $f_1 \in \mathrm{int}\, W$ then $H \in \mathrm{int}\, \mathcal{L}(S_W)$ so that $\exp(tH) \subset \mathrm{int}\, S_W$ for all $t > 0$.

Proof: Take $x \in W$, $x \neq 0$. Since $V \cap W = \{0\}$, it follows that $\mathbf{B} = (f_1 + V) \cap W$ is a cone basis of W in the affine subspace $f_1 + V$. Hence up to multiplication by a positive scalar we have

$$x \ = \ f_1 + a_2 f_2 + \cdots + a_n f_n \ .$$

Therefore,

$$\begin{aligned} Hx \ &= \ (n-1) f_1 - (a_2 f_2 + \cdots + a_n f_n) \\ &= \ n\, f_1 - x \ , \end{aligned}$$

that is, $x + Hx = n f_1 \in W$. By the invariance theorem for cones (see [2], Theorem I.5.27), it follows that $\exp tH \in S_W$ for all $t \geq 0$, which means that $H \in \mathcal{L}(S_W)$.

Now, assume that $f_1 \in \mathrm{int}\, W$. Note that the cone basis $\mathbf{B}$ is compact since W is a pointed cone. Also, the map

$$(A, x) \in \mathfrak{sl}(n,\mathbb{R}) \times \mathbb{R}^n \ \longmapsto \ x + Ax \in \mathbb{R}^n$$

is continuous. Hence, given $x \in \mathbf{B}$ and a neighborhood U of $n f_1$ in W, there are neighborhoods O_x of H and C_x of x such that for $A \in O_x$ and $y \in C_x$, it holds $y + Ay \in U \subset W$. By compactness of $\mathbf{B}$ there exists a neighborhood O of H such that $x + Ax \in U$ for all $x \in \mathbf{B}$ and $A \in O$. It follows that the open set O is contained in $\mathcal{L}(S_W)$, implying that $H \in \mathrm{int}(\mathcal{L}(S_W))$. Clearly, this implies that $tH \in \mathrm{int}(\mathcal{L}(S_W))$ for all $t > 0$. Hence, using the fact that the exponential mapping is a diffeomorphism around the identity we conclude that $\exp(tH) \in \mathrm{int}\, S_W$ for small values of $t > 0$. Therefore the formula $\exp(tH) = \exp((t/n)H)^n$ implies that $\exp(tH) \in \operatorname{int} S_W$ for all $t > 0$. ∎

Taking H as in this proposition with $f_1 \in \operatorname{int} W$ we have that $\exp(tH) \in \operatorname{int} S_W$ if $t > 0$, hence for all $g \in S_W$, $\exp(tH)g$ and $g\exp(tH)$ belong to $\operatorname{int} S_W$ if $t > 0$. Therefore, any $g \in S_W$ can be linked to $\operatorname{int} S_W$ by a continuous path inside S_W. Since this fact is used in the proof that S_W is connected we emphasize it.

Corollary 5. *Let H be as in the previous proposition with $f_1 \in \operatorname{int} W$. Take $g \in S_W$. Then $\exp(tH)g$ and $g\exp(tH)$ belong to $\operatorname{int} S_W$ if $t > 0$.*

Before proceeding we note the following simple, but useful, fact about matrices in $\operatorname{int} S_W$:

Lemma 6. *If $g \in \operatorname{int} S_W$ then $gW \subset \operatorname{int} W \cup \{0\}$.*

Proof: If $x \neq 0$, the assignment $h \in \mathrm{Sl}(n, \mathbb{R}) \mapsto hx \in \mathbb{R}^n$ is an open mapping because $\mathrm{Sl}(n, \mathbb{R})$ acts transitively on $\mathbb{R}^n \setminus \{0\}$. Hence $(\operatorname{int} S_W)x = \{hx : h \in \operatorname{int} S_W\}$ is open if $x \neq 0$. Since $(\operatorname{int} S_W)\, x \subset W$, it follows that $gx \in \operatorname{int} W$ for all $x \in W$, $x \neq 0$. ∎

The following statement is central in the proof that S_W is connected, it concerns the Jordan decomposition of the matrices in $\operatorname{int} S_W$.

Lemma 7. *Let $g \in \operatorname{int} S_W$ be given. Then there exists a basis $\beta = \{f_1, \ldots, f_n\}$ of $\mathbb{R}^n$ with $f_1 \in \operatorname{int} W$ and*

$$\operatorname{span}\{f_2, \ldots, f_n\} \cap W = 0 \ ,$$

such that the matrix of g with respect to β is written in blocks as

$$g = \begin{pmatrix} \lambda & 0 \\ 0 & h \end{pmatrix}$$

where $\lambda > 0$ and h is an $(n-1) \times (n-1)$-matrix with $\det h > 0$. Furthermore λ is a principal eigenvalue, i.e., $|\mu| < \lambda$ if μ is an eigenvalue of h.

This lemma is well known in the theory of matrices (see Berman and Plemmons [1]). Below we offer another proof of it, having a Lie theoretic flavor.

2.1. Proof of Theorem 1

In view of Corollary 5, in order to prove that S_W is path connected it is enough to show that $\operatorname{int} S_W$ is path connected. We prove this by exhibiting, for any $g \in \operatorname{int} S_W$, a path in S_W joining it to S_{inf}. Since S_{inf} is path connected, this implies that $\operatorname{int} S_W$ is path connected as well.

Fix $g \in \operatorname{int} S_W$, and let $\beta = \{f_1, \ldots, f_n\}$ be a basis given by Lemma 7, providing a block decomposition of g.

Let $P \subset \mathrm{Sl}(n, \mathbb{R})$ be the subgroup of those linear maps whose matrices with respect to β have the same block structure as g:

$$P = \left\{ \begin{pmatrix} \mu & 0 \\ 0 & Q \end{pmatrix} : \mu > 0,\ Q \in \mathrm{Gl}^+(n-1, \mathbb{R}),\ \mu \det Q = 1 \right\}.$$

Clearly, P is a closed and connected subgroup of $\mathrm{Sl}(n, \mathbb{R})$. By construction, $g \in (\operatorname{int} S_W) \cap P$. Let $H \in \mathfrak{sl}(n, \mathbb{R})$ be such that its matrix with respect to β is

$$H = \operatorname{diag}\{n-1, -1, \ldots, -1\}\ . \tag{1}$$

By Proposition 4, $H \in \operatorname{int} \mathcal{L}(S_W)$ and $\exp(tH) \in (\operatorname{int} S_{\text{inf}}) \cap P$ for all $t > 0$. Put

$$\Gamma = (\operatorname{int} S_{\text{inf}}) \cap P\ .$$

Then Γ is a semigroup with nonempty interior in P (with respect to the topology of P).

Define the map $\phi : P \to \mathrm{Sl}(n-1, \mathbb{R})$ by

$$\phi \begin{pmatrix} \mu & 0 \\ 0 & Q \end{pmatrix} = (\det Q)^{-1/n-1} Q = \mu^{1/n-1} Q\ .$$

It is checked immediately that ϕ is a surjective homomorphism. Hence it is an open mapping, so that $\phi(\Gamma)$ is a semigroup with nonempty interior in $\mathrm{Sl}(n-1, \mathbb{R})$.

Now, $\exp(tH) \in \Gamma$, for all $t > 0$. Since

$$\exp(tH) = \operatorname{diag}\{e^{t(n-1)}, e^{-t}, \ldots, e^{-t}\}\ ,$$

it follows that $\phi(\exp(tH)) = 1$. Therefore, $1 \in \phi(\Gamma)$ implying that $\phi(\Gamma) = \mathrm{Sl}(n-1, \mathbb{R})$ because $\mathrm{Sl}(n-1, \mathbb{R})$ is connected. Combining this fact together with the definition of ϕ we get the

Lemma 8. *For all $h' \in \mathrm{Sl}(n-1, \mathbb{R})$ there exists $a > 0$ such that*

$$g' = \begin{pmatrix} a & 0 \\ 0 & a^{-1/n-1} h' \end{pmatrix} \in \Gamma = (\operatorname{int} S_{\text{inf}}) \cap P \subset \operatorname{int} S_{\text{inf}}\ . \tag{2}$$

Let us show now that there is a path linking the given $g \in \operatorname{int} S_W$ to $S_{\inf}$. We can write

$$g = \begin{pmatrix} \lambda & 0 \\ 0 & h \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & (\det h)^{1/n-1} h' \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1/n-1} h' \end{pmatrix}$$

where $h' = (\det h)^{-1/n-1} h \in \mathrm{Sl}(n-1, \mathbb{R})$. For this h', the above lemma ensures the existence of $a > 0$ such that the corresponding g' as in (2) belongs to Γ. There are the following possibilities:

1. $\lambda \leq a$. Then $e^{(n-1)T}\lambda = a$ for some $T \geq 0$. Hence if H is given by (1) then
$$\exp(TH)g = \begin{pmatrix} e^{(n-1)T}\lambda & 0 \\ 0 & (e^{(n-1)T}\lambda)^{-1/n-1} h' \end{pmatrix}.$$
Substituting in this equality $e^{(n-1)T}\lambda = a$ we get from (2) that
$$\exp(TH)g = g' \in S_{\inf} .$$
Since $\exp(tH)g \in \operatorname{int} S_W$ for all $t \geq 0$, the path $t \mapsto \exp(tH)g$, $t \in [0, T]$, joins g to $g' \in S_{\inf}$, without leaving $\operatorname{int} S_W$.
2. $\lambda > a$. In this case we reverse the roles of g and g' to get $T > 0$ such that $\exp(TH)g' = g$, providing the path $t \mapsto \exp(tH)g'$, $t \in [0, T]$, linking g' to g inside $\operatorname{int} S_W$.

Therefore for arbitrary $g \in \operatorname{int} S_W$ there exists a path inside $\operatorname{int} S_W$ joining g to $S_{\inf}$ concluding the proof of Theorem 1. ∎

2.2. Proof of Lemma 7

We start with the following lemma which holds for an arbitrary semigroup S contained in $\mathrm{Sl}(n, \mathbb{R})$ and having nonempty interior.

Lemma 9. *Given $h \in \operatorname{int} S$ let $V \subset \mathbb{R}^n$ be an h-invariant subspace with $\dim V \geq 2$ and such that $|\mu|$ is constant as μ runs through the eigenvalues of the restriction $\bar{h}$ of h to V. Then S is transitive on the rays of V. Precisely, let P_V be the subgroup*

$$P_V = \left\{ h \in \mathrm{Sl}(n, \mathbb{R}) : \; hV = V \right\} .$$

Then $\Gamma = S \cap P_V$ is a semigroup with nonempty interior in P_V and for two rays r_1 and r_2 in V, starting at the origin, there exists $h' \in \Gamma$ such that $h' r_1 = r_2$.

Proof: The first step in the proof consists in projecting Γ to the group $\mathrm{Sl}(V)$, of unimodular linear maps of V. This need to be done only if V is a proper subspace. In this case the restriction of P_V to V is the whole linear group $\mathrm{Gl}(V)$, which has two connected components, say $\mathrm{Gl}^{\pm}(V)$, with $1 \in \mathrm{Gl}^{+}(V)$. Clearly $h \in \Gamma$ so that Γ is a semigroup with nonempty interior in P_V. Denote also by Γ its restriction to V. It follows that $\Gamma^{+} = \Gamma \cap \mathrm{Gl}^{+}(V)$ also has nonempty interior, because $Q^2 \in \mathrm{Gl}^{+}(V)$ if $Q \in \mathrm{Gl}(V)$.

Consider the onto homomorphism $\psi\colon \mathrm{Gl}^{+}(V) \to \mathrm{Sl}(V)$ given by

$$Q \in \mathrm{Gl}^{+}(V) \longmapsto (\det Q)^{1/k} Q\,, \qquad k = \dim V\ .$$

The image $\Gamma_1 = \psi(\Gamma^{+})$ is a semigroup with nonempty interior in $\mathrm{Sl}(V)$.

Now, the restriction $\bar{h}$ of h to V belongs to $\operatorname{int}\Gamma$. By assumption the eigenvalues of $\bar{h}$ are of the form

$$e^{a}(\cos\theta_1 + i\,\sin\,\theta_1),\ \ldots,\ e^{a}(\cos\theta_s + i\,\sin\,\theta_s)\ ,$$

with fixed a. So that $\bar{h}$ decomposes in Jordan blocks of the types

$$e^{a}\begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \qquad e^{a}\begin{pmatrix} \begin{matrix} \cos\theta_j & -\sin\theta_j \\ \sin\theta_j & \cos\theta_j \end{matrix} & & * \\ & \ddots & \\ 0 & & \begin{matrix} \cos\theta_j & -\sin\theta_j \\ \sin\theta_j & \cos\theta_j \end{matrix} \end{pmatrix}.$$

In case $\theta_j = 2\pi\, q_j$, $j = 1, \ldots, s$, with q_j rational, a quick glance at these blocks show that some power of $\bar{h}$ has real eigenvalues so that there exists $h_1 \in \operatorname{int}\Gamma$ whose restriction $\bar{h}_1$ to V has the form

$$\bar{h}_1 = \lambda \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \tag{3}$$

with $\lambda > 0$. The existence of such $\bar{h}_1$, coming from the semigroup, can be ensured without the restrictive assumption that the eigenvalues of $\bar{h}$ are rational multiples of π. In fact, since $h \in \operatorname{int}\Gamma$, there exists $h_2 \in \operatorname{int}\Gamma$ having the same block structure as h and such that the arguments of the eigenvalues of the restriction of h_2 to V are rational multiples of 2π. Thus we can argue with h_2 in place of h to get the desired element $\bar{h}_1$ like in (3).

Now,

$$\psi(\bar{h}_1) = \begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{pmatrix}.$$

This implies that $\Gamma_1 = \mathrm{Sl}(V)$. In fact, $\psi(\bar{h}_1) \in \operatorname{int}\Gamma_1$ and $\psi(\bar{h}_1)$ can be approximated by a matrix of the form $\exp(X)$ with X having purely imaginary eigenvalues. This permits to show that $1 \in \operatorname{int}\Gamma_1$ concluding that $\Gamma_1 = \mathrm{Sl}(V)$ (see [6], Lemma 4.1, for details).

From $\Gamma_1 = \mathrm{Sl}(V)$ and $\dim V \geq 2$ it follows at once that Γ_1 is transitive on the rays of V. The lemma is then a direct consequence of the definition of ψ. ∎

An application of the above lemma to g yields the

Corollary 10. *Fix $g \in \operatorname{int} S_W$. Let $V \subset \mathbb{R}^n$ be a g-invariant subspace such that $|\mu|$ is constant as μ runs through the eigenvalues of the restriction of g to V. Then $\dim V = 1$ if $V \cap W \neq 0$.*

Proof: If $V \cap W \neq 0$ there exists a ray of V contained in W. On the other hand the lemma implies that S_W is transitive on the rays of V if $\dim V \geq 2$. Hence $V \subset W$ if $\dim V \geq 2$ contradicting the assumption that W is a pointed cone. ∎

In order to continue we put

$$\rho = \max\Big\{|\lambda| : \lambda \text{ is an eigenvalue of } g\Big\}$$

for a fixed $g \in \operatorname{int} S_W$. Let V^+ be the direct sum of the generalized eigenspaces $V_\lambda = \ker(g-\lambda)^n$, with $|\lambda| = \rho$. Also, let V^- be the sum of the remaining generalized eigenspaces of g. We claim that $V^+ \cap W \neq \{0\}$. To see this write for $u \in \mathbb{R}^n$, $u = u^+ + u^-$ with $u^\pm \in V^\pm$. Then as $k \to +\infty$, $(1/\rho)^k g^k u^-$ converges to zero. Furthermore, the fact that the eigenvalues of g in V^+ have constant modulus ρ, implies that there exists a subsequence k_l such that $\lim (1/\rho)^{k_l} g^{k_l} u^+ = v$, as $l \to +\infty$. This limit is not zero if $u^+ \neq 0$. Thus when $u^+ \neq 0$, $(1/\rho)^{k_l} g^{k_l} u$ converges to $v \in V^+$. In particular take $u \in W$ such that $u^+ \neq 0$. The existence of such u follows from the assumption that W is generating. Then $0 \neq v \in V^+ \cap W$ because $(1/\rho)^{k_l} g^{k_l} u \in W$ and W is closed, showing the claim.

By Corollary 10 we conclude that $\dim V^+ = 1$. Hence there exists just one eigenvalue, say $\lambda_{\max}$, with $|\lambda_{\max}| = \rho$, which is by force real. Furthermore the eigenspace V^+ is contained in $W \cup (-W)$ and since $gW \subset W$, it follows that $\lambda_{\max} > 0$.

Take an eigenvector $f_1 \in V^+ \cap W$. Then $\lambda_{\max} f_1 = g f_1 \in \operatorname{int} W$ by Lemma 6. Hence $f_1 \in \operatorname{int} W$. Therefore, the proof of Lemma 7 follows as soon as we show that $V^- \cap W = \{0\}$.

To check that $V^- \cap W = \{0\}$ we note first that $V^- \cap \operatorname{int} W = \{0\}$, since otherwise W would meet both half-spaces determined by the codimension one subspace V^-. But this would contradict the fact that W is a pointed cone. In fact, for v_1 and v_2 in different sides of V^- the ray defined by $g^k v_1$ approaches, say, the ray spanned by f_1, as $k \to +\infty$, whereas the ray $g^k v_2$ approaches the ray spanned by $-f_1$. Since $g^k v$, $v \in W$, does not leave W, we would have $\pm f_1 \in W$. Finally,

$$g(V^- \cap W) \;=\; gV^- \cap gW \;\subset\; V^- \cap (\operatorname{int} W \cup \{0\})$$

because $gW \subset \operatorname{int} W \cup \{0\}$ by Lemma 6. Hence $g(V^- \cap W) = \{0\}$ so that $V^- \cap W = \{0\}$, concluding the proof of Lemma 7. ∎

3 – Complements

This section is devoted to the proof of some facts related to the main result. We start with the

Proof of Corollary 2: Let T be a connected semigroup with nonempty interior containing S_W properly. Note first that T is not contained in $S[W]$. To see this suppose to the contrary that $T \subset S[W]$. Then $Tx \subset W \cup (-W)$ for all $x \in W$. However, T is connected so that if $0 \neq x \in W$ then Tx is contained in a connected component of $(W \cup (-W)) \backslash \{0\}$, which is by force W because Tx is connected and contains x, as $1 \in T$. Therefore, $T \subset S_W$ contradicting the assumption on T. Now, the proof that $T = \mathrm{Sl}(n, \mathbb{R})$ follows the same steps as the proof that $S[W]$ is maximal (see [11], Theorem 6.12). We sketch it: By Proposition 4, any line outside $[W]$ is spanned by an eigenvector of some $h \in \operatorname{int} S_W$. This implies that $[W]$ and $\mathbb{P}^{n-1} \backslash [W]$ are the two control sets of S_W in $\mathbb{P}^{n-1}$. Therefore S_W is transitive in $\operatorname{int}[W]$ as well as in $\mathbb{P}^{n-1} \backslash [W]$. Since T is not contained in $S[W]$, there exists $g \in T$ such that $gx \in \mathbb{P}^{n-1} \backslash [W]$ for some $x \in \operatorname{int}[W]$. Also for any $y \in \mathbb{P}^{n-1}$ there exists $g_1 \in S_W$ with $g_1 y \in \operatorname{int}[W]$ (because $[W]$ is the invariant control set of S_W in $\mathbb{P}^{n-1}$). It follows that T acts transitively in $\mathbb{P}^{n-1}$. Thus $T = \mathrm{Sl}(n, \mathbb{R})$, by [11], Theorem 6.2. ∎

Now, we discuss the relation between S_W and $S[W]$. By definition $S[W] = S_W \cup S_W^{||}$, where

$$S_W^{||} = \left\{ g \in \mathrm{Sl}(n,\mathbb{R}) : \ gW \subset -W \right\} .$$

The following lemma shows that $S_W^{||}$ is not empty.

Lemma 11. *S_W is properly contained in $S[W]$.*

Proof: We must show that there exists $g \in \mathrm{Sl}(n,\mathbb{R})$ such that $gW \subset -W$. For this purpose take $H = \mathrm{diag}\{n-1, -1, \dots, -1\}$ with respect to a basis $\beta = \{f_1, \dots, f_n\}$ satisfying the requirements of Proposition 4, namely $f_1 \in \operatorname{int} W$ and $\mathrm{span}\{f_2, \dots, f_n\} \cap W = \{0\}$. Since $\mathrm{Sl}(n,\mathbb{R})$ acts transitively on $\mathbb{R}^n$, there exists $g_1 \in \mathrm{Sl}(n,\mathbb{R})$ such that $g_1 f_1 = -f_1$. By continuity $U = g_1^{-1}(\mathrm{int}(-W))$ is a neighborhood of f_1. Now, by construction of H there exists a large enough $t > 0$ such that if $h = \exp tH$ then $hW \subset U$. Hence $g_1 hW \subset -W$ so that $g = g_1 h$ belongs to $S[W]$ but not to S_W. ∎

Clearly, there are the inclusions $S_W S_W^{||} \subset S_W^{||}$ and $(S_W^{||})^2 \subset S_W$. The former shows in particular that $S_W^{||}$ has nonempty interior. In case n is even, $-1 \in \mathrm{Sl}(n,\mathbb{R})$, hence $-1 \in S_W^{||}$ for any W. Actually, -1 maps W exactly onto $-W$ hence the following statement implies that in even dimensions, $S_W^{||} = -S_W$.

Proposition 12. *Suppose that there exists $k \in \mathrm{Sl}(n,\mathbb{R})$ satisfying $kW = -W$. Then $S_W^{||} = kS_W = S_W k$.*

Proof: Clearly, kS_W and $S_W k$ are contained in $S_W^{||}$. For the reverse inclusions note that $k^{-1}W = -W$. Pick $g \in S_W^{||}$. Then $gW \subset -W$, so that $gk^{-1}W \subset W$ and $k^{-1}gW \subset W$, that is, gk^{-1} and $k^{-1}g$ are in S_W. ∎

Under the assumption of this proposition it follows at once that $S_W^{||}$ is connected. Since the existence of k mapping W onto $-W$ depends on the geometry of the specific W, we prove next that in general

Proposition 13. *$S_W^{||}$ is connected. Hence S_W and $S_W^{||}$ are the connected components of $S[W]$.*

Proof: Take $g, h \in S_W^{||}$. Both gW and hW are pointed generating cones contained in $-W$. Take H and β like in Proposition 4 with the first element f_1

of β contained in $\mathrm{int}(hW)$. Like in that proposition $H \in \mathcal{L}(S_W)$ and for large enough t_0, $\exp(t_0 H)(-W) \subset hW$. In particular,

$$\exp(t_0 H)(gW) \subset hW \ .$$

Hence $h^{-1}\exp(t_0 H)g \in S_W$, that is, $\exp(t_0 H)g \in hS_W$. Since S_W is path connected, this implies the existence of a path in $S[W]$ linking $\exp(t_0 H)g$ to h. However, $H \in \mathcal{L}(S_W)$, so that $\exp(t_0 H)g$ and g are in the same path component of $S_W^{||}$, concluding the proof of that $S_W^{||}$ is connected. ∎

In general $S_{\mathrm{inf}} = \langle \exp(\mathcal{L}(S_W)) \rangle$ is a proper subsemigroup of S_W. As observed by K.-H. Neeb (personal communication) the inclusion $S_{\mathrm{inf}} \subset S_W$ is proper for the semigroup $\mathrm{Sl}^+(n,\mathbb{R}) = S_W$ of positive matrices, where W is the orthant

$$W = \mathcal{O}^+(n) = \Big\{(x_1,\ldots,x_n) \in \mathbb{R}^n : x_i \geq 0\Big\} \ .$$

To see this note that in case $n \geq 3$ the unit group of S_W, $H(S_W) = S_W \cap S_W^{-1}$ is not connected. In fact, it easy to check that g must permutes the basic vectors if $g \in H(S_W)$ so that $H(S_W) = \Pi \times A$ where A is the group of diagonal matrices with positive entries and Π is the group of permutation matrices with $\det = 1$. In case $n \geq 3$, Π — and hence $H(S_W)$ — is not connected. On the other hand, it is a general fact that the unit group of an infinitesimally generated semigroup like S_{inf} must be connected (see [2], Theorem V.2.8).

Finally, we observe that Corollary 3 completely determines the maximal connected semigroups of $\mathrm{Sl}(n,\mathbb{R})$ for $n = 2, 3$. In fact, for $n = 2$, any semigroup is of the projective type so that any maximal connected semigroup is S_W for some pointed and generating cone $W \subset \mathbb{R}^2$. It should be remarked here that for any such cone W there exists $g \in \mathrm{Sl}(2,\mathbb{R})$ such that $W = g\mathcal{O}^+(2)$. Since $S_{gW} = g\,S_W\,g^{-1}$, it follows that up to conjugation $\mathrm{Sl}^+(2,\mathbb{R})$ is the only maximal connected semigroup of $\mathrm{Sl}(2,\mathbb{R})$. For $n = 3$, there are two types of maximal semigroups, namely a semigroup is of type $\mathbb{P}^2$ or $\mathrm{Gr}_2(3)$, the Grassmannian of two-dimensional subspaces of $\mathbb{R}^3$. However, if a semigroup is of type $\mathrm{Gr}_2(3)$ then its inverse S^{-1} is of projective type (see [9], Proposition 6.3). Therefore there is the following characterization of the maximal connected semigroups in $\mathrm{Sl}(3,\mathbb{R})$:

Proposition 14. *A semigroup $S \subset \mathrm{Sl}(3,\mathbb{R})$, with $\mathrm{int}\, S \neq \emptyset$, is maximal connected if and only if there exists a pointed and generating cone $W \subset \mathbb{R}^3$ such that either $S = S_W$ or $S = S_W^{-1}$.*

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