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OSCILLATION OF SOLUTIONS OF A PAIR OF COUPLED NONLINEAR DELAY DIFFERENTIAL EQUATIONS

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Abstract: New oscillation criteria of Kamenev-type and Philos-type are established for a pair of coupled nonlinear delay differential equations. Our results improve the results of Kowng and Wong and the recent results of Li and Cheng. The relevance of the results obtained is illustrated with a number of carefully selected examples.

1 – Introduction

In this paper, we are concerned with the oscillation of all solutions of a pair of coupled nonlinear delay differential equations of the form

$$\left.\begin{aligned} x'(t) &= a(t)\, f(y(\sigma(t))) \\ y'(t) &= -b(t)\, g(x(\sigma(t))) \end{aligned}\right\}, \qquad t \geq t_0 , \tag{1.1}$$

where

($\mathbf{H}_1$) $a, b \in C([t_0,\infty), \mathbb{R}^+)$ and $\sigma \in C^1([t_0,\infty), \mathbb{R}^+)$, $\sigma(t) \leq t$, $\sigma'(t) > 0$;

($\mathbf{H}_2$) $f \in C^1(\mathbb{R},\mathbb{R})$, $uf(u) > 0$, $f'(u) \geq k > 0$, and $g \in C(\mathbb{R},\mathbb{R})$, $ug(u) > 0$, $\frac{g(u)}{u} \geq k_1 > 0$ for $u \neq 0$.

We will restrict our attention to those solutions of the differential system (1.1) that exist on some ray $[T_0, \infty)$, where $T_0 \geq t_0$ may depend upon the particular solution involved. Note that under quite general conditions there will always exist

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solutions that are continuable to an interval of the form $[T_0, \infty)$, even though non-continuable solutions will also exist [2]. We make the standing hypothesis that (1.1) does possess such continuable solutions. As usual, a continuous real valued function defined on an interval $[T_0, \infty)$ is said to be oscillatory if it has arbitrarily large zeros; otherwise it will be called nonoscillatory. A solution $(x(t), y(t))$ of the system (1.1) will be called oscillatory if both $x(t)$ and $y(t)$ are oscillatory; otherwise it will be called nonoscillatory.

The system (1.1) is naturally classified into four cases according to whether

$$\int_{t_0}^{\infty} a(t)\,dt = \infty, \quad \int_{t_0}^{\infty} a(t)\,dt < \infty, \quad \int_{t_0}^{\infty} b(t)\,dt = \infty \quad \text{and} \quad \int_{t_0}^{\infty} b(t)\,dt < \infty \ .$$

However, by symmetry considerations, we will restrict our attention to the cases where

$$\int_{t_0}^{\infty} a(t)\,dt = \infty \ , \tag{1.2}$$

and

$$\int_{t_0}^{\infty} a(t)\,dt < \infty \ . \tag{1.3}$$

A particular case of (1.1) is the following system

$$\left.\begin{aligned} x'(t) &= a(t)\,f(y(t)) \\ y'(t) &= -b(t)\,g(x(t)) \end{aligned}\right\}, \qquad t \ge t_0 \ . \tag{1.4}$$

A few number of oscillation and nonoscillation criteria of solutions of (1.4) have already been derived, (see for example, Kordonis and Philos [1], Kwong and Wong [2], Mirzov [5–7] and the recent results of Li and Cheng [4]). It seems that nothing is known regarding the qualitative behavior of solutions of the system (1.1). Therefore our aim in this paper is to provide some new sufficient conditions for having (1.1) oscillatory, when (1.2) holds, using the techniques of Philos [8] and Li [3] regarding second order differential equations. The case when (1.3) holds will be treated in a separate paper. Our results improve the results of Kwong and Wong [2] and the recent results of Li and Cheng [4]. The relevance of our results becomes clear through some carefully selected examples.

In the sequel, when we write a functional inequality we will assume that it holds for all sufficient large values of t.

2 – Main results

In this section we will give some new sufficient conditions for having system (1.1) oscillatory. Before stating our main results we need the following lemma, the proof of which is similar to that of Lemma 1.1 in [4]. For the sake of completeness we will include the proof.

Lemma 2.1. *Assume that condition (H_1) and (H_2) hold. Suppose further that the function $a(t)$ is not identically zero on any interval of the form $[T_0, \infty)$, where $T_0 \geq t_0$. Then the component function $x(t)$ of a nonoscillatory solution $(x(t), y(t))$ of (1.1) is also nonoscillatory.*

Proof: Assume to the contrary that $x(t)$ is oscillatory but $y(t)$ and $y(\sigma(t))$, for some $T_0 \geq t_0$, are positive for every $t \geq T_0$. Therefore $x'(t) = a(t) f(y(\sigma(t))) \geq 0$ for every $t \in [T_0, +\infty)$ and $x'(t)$ cannot be identically zero on any interval $[T, +\infty)$, $(T \geq T_0)$. Then neither $x(t)$ can be identically zero on any interval $[T, +\infty)$, $(T \geq T_0)$ nor $x'(t)$ can be negative on $[T_0, +\infty)$. This contradicts the oscillatory property of $x(t)$. The case where $y(t)$ and $y(\sigma(t))$, for some T_0, are negative for $t \geq T_0$ is similarly proved.

If $b(t)$ is not identically zero on any interval of the form $[T_0, \infty)$, where $T_0 \geq t_0$, then the component function $y(t)$ of a nonoscillatory solution $(x(t), y(t))$ of (1.1) is also nonoscillatory. Therefore, under the additional condition

($\mathbf{H}_3$) $a(t)$ and $b(t)$ are not identically zero on any interval of the form $[T_0, \infty)$, where $T_0 \geq t_0$,

each component function of a nonosillatory solution $(x(t), y(t))$ is eventually of one sign.

It is remarkable that for any solution (x, y) of the differential system (1.1), in the case where the coefficients a and b are assumed to be not identically zero on any interval of the form $[T_1, \infty)$, $T_1 \geq t_0$, from the first equation of (1.1) it follows easily that the oscillation of x implies that y is also oscillatory. So if (x, y) is a nonoscillatory solution of (1.1) then x is always nonoscillatory. ∎

Now, we present some new oscillation results for system (1.1), using Kamenev-type integral average conditions [3].

Theorem 2.1. *Assume that* (H_1)–(H_3) *hold. Let* $r(t) = \frac{1}{a(t)}$ *and* $\rho \in C^1[[t_0,\infty),\mathbb{R}^+)$ *be such that*

$$\limsup_{t\to\infty} \frac{1}{t^n} \int_{t_0}^{t} (t-s)^n \left(\rho(s)\,q(s) - \frac{(\rho'(s))^2\; r(\sigma(\sigma(s)))}{4\rho(s)\,\sigma'(s)\,\sigma'(\sigma(s))}\right) ds \;=\; \infty \;, \tag{2.1}$$

for some nonnegative integer n, *where*

$$q(t) = k\,k_1\, b(\sigma(t))\,\sigma'(t) \;. \tag{2.2}$$

Then every solution of (1.1) oscillates.

Proof: Assume that the differential system (1.1) admits a nonoscillatory solution $(x(t),y(t))$ on an interval $[T_0,\infty)$, where $T_0 \geq t_0$. From (H_3) it follows that the coefficients a and b are not identically zero on any interval of the form $[T_0,\infty)$, $T_0 \geq t_0$. So, as pointed out in Lemma 2.1, $x(t)$ is always nonoscillatory. Without loss of generality we shall assume that $x(t) \neq 0$ for $t \geq T_0$. Furthermore, we observe that the substitution $u=-x$ and $v=-y$ transforms the system (1.1) into a system of the same form subject to the same assumptions of the theorem. Thus we restrict our discussion only to the case where $x(t)$ and $x(\sigma(t)) > 0$ are positive on $[T_0,\infty)$.

From (H_3), as $a(t)$ is positive and not identically zero on any interval $[T_0,\infty)$ the differential system (1.1) reduces to the second order nonlinear delay differential equation

$$(r(t)\,x'(t))' + b(\sigma(t))\,\sigma'(t)\,f'(y(\sigma(t)))\,g(x(\sigma(\sigma(t)))) = 0\,, \qquad t \geq T_0 \;. \tag{2.3}$$

From (1.2) we have

$$\int_{t_0}^{\infty} \frac{1}{r(t)}\,dt \;=\; \infty \;. \tag{2.4}$$

From (H_2) and (2.3) it follows that

$$(r(t)\,x'(t))' + k\,k_1 b(\sigma(t))\,\sigma'(t)\,x(\sigma(\sigma(t))) \leq 0\,, \qquad t \geq T_0 \;, \tag{2.5}$$

which implies that

$$(r(t)\,x'(t))' < 0 \quad \text{ for } \; t \geq T_0 \;. \tag{2.6}$$

Therefore $r(t)x'(t)$ is a decreasing function. We claim that

$$x'(t) \geq 0, \quad \text{ for } \; t \geq T_0 \;. \tag{2.7}$$

If not, there is a $T_1 > T_0$ such that $x'(T_1) < 0$. It follows from (2.6) that

$$x(t) \le x(T_1) + r(T_1)\, x'(T_1) \int_{T_1}^{t} \left(\frac{1}{r(s)}\right) ds \; . \tag{2.8}$$

Hence, by (2.4) we have $\lim_{t\to\infty} x(t) = -\infty$, which contradicts the fact that $x(t) > 0$ for $t \ge T_0$.

Define now the function

$$w(t) = \rho(t)\, \frac{r(t)\, x'(t)}{x(\sigma(\sigma(t)))}, \qquad \text{for} \;\; t \ge T_0 \; . \tag{2.9}$$

Differentiating (2.9) and using (2.5), we have

$$w'(t) \le -\rho(t)\, q(t) + \frac{\rho'(t)}{\rho(t)}\, w(t) - \sigma'(t)\, \sigma'(\sigma(t))\, \rho(t)\, r(t)\, \frac{x'(t)\, x'(\sigma(\sigma(t)))}{x(\sigma(\sigma(t)))} \; . \tag{2.10}$$

Since, the function $r(t)x'(t)$ is nonincreasing, this leads to

$$r(\sigma(\sigma(t)))\, x'(\sigma(\sigma(t))) \ge r(t)\, x'(t), \qquad \text{for} \;\; t \ge T_0 \; . \tag{2.11}$$

In order to simplify the notations we introduce

$$\gamma_1(s) = \frac{\rho'(s)}{\rho(s)}, \qquad W_1(s) = \frac{\sigma'(s)\, \sigma'(\sigma(s))}{\rho(s)\, r(\sigma(\sigma(s)))} \; . \tag{2.12}$$

Using (2.10) and (2.11) we find that $w(t) > 0$ and satisfies

$$\begin{aligned} w'(t) &\le -\rho(t)\, q(t) + \gamma_1(t)\, w(t) - 2\, W_1(t)\, w^2(t) \\ &< -\rho(t)\, q(t) + \gamma_1(t)\, w(t) - W_1(t)\, w^2(t) \; , \end{aligned} \tag{2.13}$$

which implies

$$w'(t) < -\rho(t)\, q(t) + \frac{(\gamma_1(t))^2}{4\, W_1(t)} - \left[\sqrt{W_1(t)}\, w(t) - \frac{\gamma_1(t)}{2\sqrt{W_1(t)}}\right]^2 . \tag{2.14}$$

Thus

$$w'(t) < -\left[\rho(t)\, q(t) - \frac{(\gamma_1(t))^2}{4\, W_1(t)}\right], \qquad \text{for} \;\; t \ge T_0 \; .$$

Multiplying the last inequality by $(t-s)^n$ and integrating it from T_0 to t we have

$$\int_{T_0}^{t} (t-s)^n \left[\rho(s)\, q(s) - \frac{(\gamma_1(s))^2}{4\, W_1(s)}\right] ds < -\int_{T_0}^{t} (t-s)^n\, w'(s)\, ds \; . \tag{2.15}$$

Since

$$(2.16)\qquad \int_{T_0}^{t} (t-s)^n\, w'(s)\, ds \;=\; n\int_{T_0}^{t} (t-s)^{n-1}\, w(s)\, ds \;-\; w(T_0)\,(t-T_0)^n$$

we obtain

$$(2.17)\qquad \frac{1}{t^n}\int_{T_0}^{t} (t-s)^n\, Q(s)\, ds \;\le\; w(T_0)\left(\frac{t-T_0}{t}\right)^n - \frac{n}{t^n}\int_{T_0}^{t} (t-s)^{n-1}\, w(s)\, ds$$

where

$$Q(s) \;=\; \rho(s)\, q(s) - \frac{(\gamma_1(s))^2}{4\, W_1(s)}\ .$$

Hence

$$(2.18)\qquad \frac{1}{t^n}\int_{T_0}^{t} (t-s)^n\, Q(s)\, ds \;\le\; w(T_0)\left(\frac{t-T_0}{t}\right)^n,$$

since $w(t) > 0$. Then

$$(2.19)\qquad \lim_{t\to\infty}\sup \frac{1}{t^n}\int_{T_0}^{t} (t-s)^n\, Q(s)\, ds \;\to\; w(T_0) < \infty$$

which contradicts the condition (2.1). Therefore every solution of (1.1) oscillates and the proof is complete. ∎

From Theorem 2.1 we have the following result.

Theorem 2.2. *Assume that all the assumptions of Theorem 2.1 hold, except the condition (2.1) which is replaced by*

$$(2.20)\qquad \lim_{t\to\infty}\sup \int_{t_0}^{t} \left(\rho(s)\, q(s) - \frac{(\gamma_1(s))^2}{4\, W_1(s)}\right) ds \;=\; \infty\ .$$

Then every solution of (1.1) oscillates.

The following examples illustrate this theorem.

Example 2.1. Consider the pair of coupled nonlinear delay differential equations

$$\left.\begin{aligned} x'(t) &= \frac{1}{1+\cos^2 t}\, y\,(t-2\pi)\,\left[1+y^2\,(t-2\pi)\right] \\ y'(t) &= -\frac{1}{1+\sin^2 t}\, x\,(t-2\pi)\,\left[1+x^2\,(t-2\pi)\right] \end{aligned}\right\}, \qquad t \ge 4\pi \ . \tag{2.21}$$

Here

$$a(t) = \frac{1}{1+\cos^2 t}\,, \qquad b(t) = \frac{1}{1+\sin^2 t}\,, \qquad \sigma(t) = t - 2\pi \ ,$$
$$f(y) = y(1+y^2) \quad \text{and} \quad g(x) = x(1+x^2) \ .$$

Then

$$\sigma(\sigma(t)) = t - 4\pi\,, \qquad b(\sigma(t)) = \frac{1}{1+\sin^2 t} \quad \text{and} \quad r(\sigma(\sigma(t))) = 1 + \cos^2 t \ ,$$
$$f'(y) = 1 + 3y^2 \ge 1 = k \quad \text{and} \quad \frac{g(x)}{x} = 1 + x^2 \ge 1 = k_1 \ .$$

Let $\rho(t) = 1$. A straightforward computation yields that all the assumptions of Theorem 2.2 are satisfied. Then every solution of (2.21) oscillates. In fact, one such solution is $(x(t), y(t)) = (\sin t, \cos t)$. □

Example 2.2. Consider the pair of coupled nonlinear delay differential equations

$$\left.\begin{aligned} x'(t) &= \frac{1}{1+\cos^2 t}\, y\,(t-2\pi)\,\left[1+y^2\,(t-2\pi)\right] \\ y'(t) &= -\frac{9(1+\cos^2 t)}{(10+\cos^2 t)}\, x\,(t-2\pi)\left[\frac{1}{9} + \frac{1}{1+x^2\,(t-2\pi)}\right] \end{aligned}\right\}, \qquad t \ge 4\pi \ . \tag{2.22}$$

Here

$$a(t) = \frac{1}{1+\cos^2 t}\,, \qquad b(t) = \frac{9(1+\cos^2 t)}{(10+\cos^2 t)}\ , \tag{2.23}$$

$$\sigma(t) = t - 2\pi\,, \qquad \sigma(\sigma(t)) = t - 4\pi\,, \qquad r(\sigma(\sigma(t))) = 1 + \cos^2 t \ ,$$

$$f(y) = y(1+y^2)\,, \qquad f'(y) = 1 + 3y^2 \ge 1 = k \ , \tag{2.24}$$

$$\frac{g(x)}{x} = \left[\frac{1}{9} + \frac{1}{1+x^2}\right] \ge k_1 = \frac{1}{9} \ . \tag{2.25}$$

One can easily show that all the assumptions of the Theorem 2.2 are satisfied if we choose $\rho(t) = 1$. Hence every solution of (2.22) oscillates. Again, $(x(t), y(t)) = (\sin t, \cos t)$ is an oscillatory solution of (2.22). □

Example 2.3. Consider the pair of coupled nonlinear differential equations

$$\left.\begin{aligned} x'(t) &= t\,y(t) \\ y'(t) &= -\frac{2}{t^3}\,x(t) \end{aligned}\right\}, \qquad t \geq 1\ . \tag{2.26}$$

Here

$$a(t) = t\,, \qquad b(t) = \frac{2}{t^3}\,, \qquad \sigma(t) = t\ . \tag{2.27}$$

$$f(y) = y\,, \qquad f'(y) = 1 = k\,, \qquad \frac{g(x)}{x} = 1 = k_1\ .$$

Let $\rho(t) = t^2$. Then condition (2.20) is satisfied and from Theorem 2.2 every solution of (2.26) oscillates. In fact, one such solution is

$$(x(t), y(t)) \;=\; \Big(t\sin(\ln t),\, \frac{1}{t}\Big(\sin(\ln t) + \cos(\ln t)\Big)\Big)\ . \ \square$$

Remark 2.1. Note that the results of Kwong and Wong [2] cannot be applied to (2.26) since the assumption (2.5) of Theorem 1 in [2] does not hold. Therefore our results in Theorems 2.1 and 2.2 improve the results of Kwong and Wong [2]. □

Remark 2.2. For $f(y) = y\left[\frac{1}{9} + \frac{1}{1+y^2}\right]$, we note that

$$f'(y) = \frac{(y^2-2)\,(y^2-9)}{9\,(1+y^2)^2} \tag{2.28}$$

changes sign on $\mathbb{R}$ four times. Therefore, the condition (H_2) in this case is not satisfied and consequently Theorem 2.1 cannot be applied. It seems interesting to find other oscillation criteria for the case where $f(y)$ is not monotonic. □

Next, we present some new oscillation results for (1.1), using integral average conditions of Philos-type. Following Philos [8], we introduce a class of functions $\Re$, defined as follows. Let

$$\mathbf{D}_0 = \Big\{(t,s)\colon\ t > s \geq t_0\Big\} \quad \text{and} \quad \mathbf{D} = \Big\{(t,s)\colon\ t \geq s \geq t_0\Big\}\ . \tag{2.29}$$

A function $H \in C(\mathbf{D}, \mathbb{R})$ is said to belong to the class $\Re$ if

(I) $H(t,t) = 0$ for $t \geq t_0$, $H(t,s) > 0$ for $t > s \geq t_0$;

(II) H has a continuous and nonpositive partial derivative on $\mathbf{D}_0$ with respect to the second variable.

Theorem 2.3. *Assume that (H_1)–(H_3) hold. Let $r(t) = \frac{1}{a(t)}$, $\rho \in C^1([t_0, \infty), \mathbb{R}^+)$, $H \in \Re$ and $h \in C(\mathbf{D}, \mathbb{R})$ be such that*

$$-\frac{\partial H(t,s)}{\partial s} = h(t,s)\sqrt{H(t,s)} \quad \text{for all } (t,s) \in \mathbf{D}_0 \,, \tag{2.30}$$

and

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)\,\rho(s)\,q(s) - \frac{\rho(s)\,r(\sigma(\sigma(s)))\,Q^2(t,s)}{4\,\sigma'(s)\,\sigma'(\sigma(s))} \right] ds \,=\, \infty\,, \tag{2.31}$$

where

$$Q(t,s) \,=\, h(t,s) - \frac{\rho'(s)}{\rho(s)}\sqrt{H(t,s)}\;. \tag{2.32}$$

Then every solution of (1.1) oscillates.

Proof: Assume that the differential system (1.1) admits a nonoscillatory solution $(x(t), y(t))$ on an interval $[T_0, \infty)$, where $T_0 \geq t_0$. Now as in the proof of the Theorem 2.1 we consider the function w defined by (2.9). Therefore by similar arguments we have that $w(t) > 0$, and then for all $t > T \geq T_0$ the inequality (2.13) can be obtained.

Again to simplify the notation we denote

$$\gamma_1(s) = \frac{\rho'(s)}{\rho(s)}\,, \qquad W_1(s) = \frac{\sigma'(s)\,\sigma'(\sigma(s))}{\rho(s)\,r(\sigma(\sigma(s)))}\;.$$

Then from (2.13) for all $t > T \geq T_0$, we have

$$\begin{aligned}
\int_T^t H(t,s)\,\rho(s)\,q(s)\,ds \;\leq\; & \\
&\leq \int_T^t H(t,s)\,\gamma_1(s)\,w(s)\,ds - \int_T^t H(t,s)\,w'(s)\,ds - \int_T^t H(t,s)\,W_1(s)\,w^2(s)\,ds \\
&= -H(t,s)\,w(s)\Big|_T^t \\
&\quad - \int_T^t \left[-\frac{\partial H(t,s)}{\partial s}\,w(s) - H(t,s)\,\gamma_1(s)\,w(s) + H(t,s)\,W_1(s)\,w^2(s)\right] ds
\end{aligned}$$

$$
\begin{aligned}
&= H(t,T)\,w(T) \\
&\quad - \int_T^t \left[\sqrt{H(t,s)}\left(h(t,s) - \sqrt{H(t,s)}\,\gamma_1(s)\right) w(s) + H(t,s)\,W_1(s)\,w^2(s)\right] ds \\
&= H(t,T)\,w(T) - \int_T^t \left[\sqrt{H(t,s)\,W_1(s)}\,w(s) + \frac{1}{2}\,\frac{Q(t,s)}{\sqrt{W_1(s)}}\right]^2 + \int_T^t \frac{Q^2(t,s)}{4\,W_1(s)}\,ds\,.
\end{aligned}
$$

Therefore, we conclude that

$$
\begin{aligned}
&\int_T^t \left[H(t,s)\,\rho(s)\,q(s) - \frac{Q^2(t,s)}{4\,W_1(s)}\right] ds \;\le \\
&\qquad \le\; H(t,T)\,w(T) - \int_T^t \left[\sqrt{H(t,s)\,W_1(s)}\,w(s) + \frac{1}{2}\,\frac{Q(t,s)}{\sqrt{W_1(s)}}\right]^2 ds\,.
\end{aligned}
\tag{2.33}
$$

By virtue of (2.33) and (II) we obtain for $t > T \ge T_0$,

$$
\int_T^t \left[H(t,s)\,\rho(s)\,q(s) - \frac{Q^2(t,s)}{4\,W_1(s)}\right] ds \;\le\; H(t,T)\,w(T)\,. \tag{2.34}
$$

Then by (2.34) and (II), we have

$$
\frac{1}{H(t,t_0)} \int_{t_0}^t \left[H(t,s)\,\rho(s)\,q(s) - \frac{Q^2(t,s)}{4\,W_1(s)}\right] ds \;\le\; \int_{t_0}^{T_0} \rho(s)\,q(s)\,ds + w(T_0)\,. \tag{2.35}
$$

Inequality (2.35) yields

$$
\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^t \left[H(t,s)\,\rho(s)\,q(s) - \frac{Q^2(t,s)}{4\,W_1(s)}\right] ds \;\le \\
&\qquad\qquad \le \int_{t_0}^{T_0} \rho(s)\,q(s)\,ds \,+\, w(T_0) \;<\; \infty\,,
\end{aligned}
\tag{2.36}
$$

and assumption (2.31) is contradicted. Therefore every solution of (1.1) oscillates. The proof is complete. ∎

The following theorem follows directly from Theorem 2.3.

Theorem 2.4. *Assume that all the assumptions of Theorem 2.3 hold except the condition (2.31) which is replaced by*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s)\rho(s)\, q(s)\, ds \;=\; \infty \,, \tag{2.37}$$

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \frac{\rho(s)\, r(\sigma(\sigma(s)))\, Q^2(t,s)}{\sigma'(s)\, \sigma'(\sigma(s))}\, ds \;<\; \infty \,. \tag{2.38}$$

Then every solution of (1.1) oscillates.

The following two oscillation criteria are useful when condition (2.31) cannot be easily verified.

Theorem 2.5. *Assume that* (H_1)*–*(H_3) *hold. Let* $r(t) = \frac{1}{a(t)}$, $\rho \in C^1([t_0,\infty),\mathbb{R}^+)$, $H \in \Re$ *and* $h \in C(\mathbf{D},\mathbb{R})$ *be such that (2.30) holds. Furthermore suppose that*

$$0 \;<\; \inf_{s\ge t_0} \left[\liminf_{t\to\infty} \frac{H(t,s)}{H(t,t_0)}\right] \;\le\; \infty \,, \tag{2.39}$$

and

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \frac{Q^2(t,s)}{W_1(s)}\, ds \;<\; \infty \,, \tag{2.40}$$

where $Q(t,s)$ *and* $W_1(s)$ *are given by (2.32) and (2.11), respectively. Let* $\psi \in C([t_0,\infty),\mathbb{R})$ *be such that*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \psi_+^2(s)\, W_1(s)\, ds \;=\; \infty \tag{2.41}$$

and

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left(H(t,s)\, \rho(s)\, q(s) - \frac{Q^2(t,s)}{4\, W_1(s)} \right) ds \;\ge\; \psi(T) \,, \tag{2.42}$$

where $\psi_+(t) = \max\{\psi(t), 0\}$. *Then every solution of (1.1) oscillates.*

Proof: As in the proof of the Theorem 2.3, assume that (1.1) has a nonoscillatory solution. Defining again $w(t)$ by (2.9), by similar arguments, we obtain the inequality (2.33). Therefore for $t > T \geq T_0$ we have

$$\frac{1}{H(t,T)} \int_T^t \left[H(t,s)\,\rho(s)\,q(s) - \frac{Q^2(t,s)}{4\,W_1(s)}\right] ds \;\leq$$
$$\leq\; w(T) - \frac{1}{H(t,T)} \int_T^t \left[\sqrt{H(t,s)\,W_1(s)}\,w(s) + \frac{Q(t,s)}{2\sqrt{W_1(s)}}\right]^2 ds$$

and consequently

$$\lim_{t\to\infty}\sup \frac{1}{H(t,T)} \int_T^t \left[H(t,s)\,\rho(s)\,q(s) - \frac{Q^2(t,s)}{4\,W_1(s)}\right] ds \;\leq$$
$$\leq\; w(T) - \lim_{t\to\infty}\inf \frac{1}{H(t,T)} \int_T^t \left[\sqrt{H(t,s)\,W_1(s)}\,w(s) + \frac{Q(t,s)}{2\sqrt{W_1(s)}}\right]^2 ds\;.$$

On the other hand inequality (2.42) implies that

$$w(T) \;\geq\; \psi(T) + \lim_{t\to\infty}\inf \frac{1}{H(t,T)} \int_T^t \left[\sqrt{H(t,s)\,W_1(s)}\,w(s) + \frac{Q(t,s)}{2\sqrt{W_1(s)}}\right]^2 ds\,, \tag{2.43}$$

and so, for every $T \geq T_0$ one has

$$w(T) \;\geq\; \psi(T) \tag{2.44}$$

and

$$\lim_{t\to\infty}\inf \frac{1}{H(t,T_0)} \int_{T_0}^t \left[\sqrt{H(t,s)\,W_1(s)}\,w(s) + \frac{Q(t,s)}{2\sqrt{W_1(s)}}\right]^2 ds \;\leq\; w(T_0) - \psi(T_0)$$
$$=\; M \;<\; \infty\;.$$

Therefore, for $t \geq T_0$, we have

$$\begin{aligned}
\infty \;&>\; \lim_{t\to\infty}\inf \frac{1}{H(t,T_0)} \int_{T_0}^t \left[\sqrt{H(t,s)\,W_1(s)}\,w(s) + \frac{Q(t,s)}{2\sqrt{W_1(s)}}\right]^2 ds \\
&\geq\; \lim_{t\to\infty}\inf \frac{1}{H(t,T_0)} \int_{T_0}^t \left[H(t,s)\,W_1(s)\,w^2(s) + \sqrt{H(t,s)}\,Q(t,s)\,w(s)\right] ds\;.
\end{aligned} \tag{2.45}$$

Defining the functions $\alpha(t)$ and $\beta(t)$ as

$$\alpha(t) = \frac{1}{H(t,T_0)} \int_{T_0}^{t} H(t,s)\, W_1(s)\, w^2(s)\, ds \ ,$$

$$\beta(t) = \frac{1}{H(t,T_0)} \int_{T_0}^{t} \sqrt{H(t,s)}\, Q(t,s)\, w(s)\, ds \ ,$$

(2.45) can be written as

$$\lim_{t\to\infty} [\alpha(t) + \beta(t)] < \infty \ . \tag{2.46}$$

Now we claim that

$$\int_{T_0}^{\infty} W_1(s)\, w^2(s)\, ds \ < \ \infty \ . \tag{2.47}$$

Suppose on the contrary that

$$\int_{T_0}^{\infty} W_1(s)\, w^2(s)\, ds \ = \ \infty \ . \tag{2.48}$$

By (2.39), there is a positive constant ζ satisfying

$$\inf_{s\geq t_0} \left[\lim_{t\to\infty} \inf \frac{H(t,s)}{H(t,t_0)}\right] \ > \zeta \ > \ 0 \ , \tag{2.49}$$

and from (2.48) it follows that for every positive number, μ, there exists a $T_1 \geq T_0$ such that

$$\int_{T_0}^{t} W_1(s)\, w^2(s)\, ds \ \geq \ \frac{\mu}{\zeta} \qquad \text{for} \ \ t \geq T_1 \ .$$

Therefore, for every $t \geq T_1$, we have

$$\begin{aligned} \alpha(t) &= \frac{1}{H(t,T_0)} \int_{T_0}^{t} H(t,s)\, d\left[\int_{T_0}^{s} W_1(u)\, w^2(u)\, du\right] \\ &= \frac{1}{H(t,T_0)} \int_{T_0}^{t} -\frac{\partial H(t,s)}{\partial s} \left[\int_{T_0}^{s} W_1(u)\, w^2(u)\, du\right] ds \end{aligned}$$

$$\geq \frac{1}{H(t,T_0)} \int_{T_1}^{t} -\frac{\partial H(t,s)}{\partial s} \left[\int_{T_1}^{s} W_1(u)\, w^2(u)\, du \right] ds$$

$$\geq \frac{\mu}{\zeta} \frac{1}{H(t,T_0)} \int_{T_1}^{t} -\frac{\partial H(t,s)}{\partial s}\, ds \;=\; \frac{\mu}{\zeta} \frac{H(t,T_1)}{H(t,T_0)} \; .$$

But by (2.49), there exists a $T_2 \geq T_1$ such that

$$\frac{H(t,T_1)}{H(t,T_0)} \geq \zeta \quad \text{for all} \;\; t \geq T_2 \; ,$$

which implies that $\alpha(t) \geq \mu_1$ for all $t \geq T_2$ and since μ is arbitrary, we conclude

(2.50) $$\lim_{t\to\infty} \alpha(t) = \infty \; .$$

Next, consider a sequence $t_n \to \infty$ satisfying

$$\lim_{n\to\infty} [\alpha(t_n) + \beta(t_n)] = \lim_{t\to\infty} [\alpha(t) + \beta(t)] \; .$$

In view of (2.46), there exists a constant μ_2 such that

(2.51) $$\alpha(t_n) + \beta(t_n) \leq \mu_2 \, , \qquad n = 1, 2, \ldots \; .$$

But from (2.50) one has

(2.52) $$\lim_{n\to\infty} \alpha(t_n) = \infty \; ,$$

and (2.51) implies

(2.53) $$\lim_{n\to\infty} \beta(t_n) = -\infty \; .$$

Then, by (2.51) and (2.53), one has for n large enough

$$1 + \frac{\beta(t_n)}{\alpha(t_n)} \;\leq\; \frac{M}{\alpha(t_n)} \;<\; \frac{1}{2} \; .$$

and consequently

$$\frac{\beta(t_n)}{\alpha(t_n)} \;\leq\; -\frac{1}{2} \; .$$

which implies that

(2.54) $$\lim_{n\to\infty} \frac{\beta(t_n)}{\alpha(t_n)}\, \beta(t_n) \;=\; \infty \; .$$

On the other hand by Schwarz's inequality, we have for every positive integer n

$$\begin{aligned}\beta^2(t_n) &= \left[\frac{1}{H(t_n,T_0)}\int_{T_0}^{t_n}\sqrt{H(t_n,s)}\,Q(t_n,s)\,w(s)\,ds\right]^2\\ &\le \left\{\frac{1}{H(t_n,T_0)}\int_{T_0}^{t_n}\frac{Q^2(t_n,s)}{W_1(s)}\,ds\right\}\left\{\frac{1}{H(t_n,T_0)}\int_{T_0}^{t_n}H(t_n,s)\,W_1(s)\,w^2(s)\,ds\right\}\\ &\le \alpha(t_n)\left\{\frac{1}{H(t_n,T_0)}\int_{T_0}^{t_n}\frac{Q^2(t_n,s)}{W_1(s)}\,ds\right\}\ ,\end{aligned}$$

But (2.49) guarantees that for n large enough

$$\frac{H(t_n,T_0)}{H(t_n,t_0)} > \zeta\ ,$$

and consequently

$$\frac{\beta^2(t_n)}{\alpha(t_n)} \le \frac{1}{\zeta H(t_n,t_0)}\int_{T_0}^{t_n}\frac{Q^2(t_n,s)}{W_1(s)}\,ds\ .$$

Thus by (2.54) we have

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{T_0}^{t}\frac{Q^2(t,s)}{W_1(s)}\,ds = \infty\ , \tag{2.55}$$

which contradicts (2.40). Hence (2.47) holds and from (2.44) one obtains

$$\int_{T_0}^{\infty}\psi_+^2(s)\,W_1(s)\,ds \le \int_{T_0}^{\infty}w^2(s)\,W_1(s)\,ds < \infty\ ,$$

which contradicts (2.41). Therefore, every solution of (1.1) oscillates. ∎

Theorem 2.6. *Assume that* (H_1)*–*(H_3) *hold. Let* $r(t)=\frac{1}{a(t)}$, $\rho\in C^1([t_0,\infty),\mathbb{R}^+)$, $H\in\Re$ *and* $h\in C(\mathbf{D},\mathbb{R})$ *satisfying (2.30) and (2.39). Suppose there exists a function* $\psi\in C([t_0,\infty),\mathbb{R})$ *such that (2.41) holds,*

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}H(t,s)\,\rho(s)\,q(s)(s)\,ds < \infty\ ,$$

and

$$\lim_{t\to\infty}\sup \frac{1}{H(t,t_0)}\int_{t_0}^{t}\left(H(t,s)\,\rho(s)\,q(s)-\frac{Q^2(t,s)}{4\,W_1(s)}\right)ds\ \ge\ \psi(T)\ . \tag{2.56}$$

where $Q(t,s)$, $W_1(s)$ and $\psi_+(t)$ are as in Theorem 2.5. Then every solution of (1.1) oscillates.

Proof: Assuming as before that (1.1) has a nonoscillatory solution and defining $w(t)$ by (2.9), the inequality (2.33) can again be obtained. Therefore, for $t > T \ge T_0$ we have

$$\begin{aligned}\lim_{t\to\infty}\inf \frac{1}{H(t,T)}\int_{T}^{t}\left[H(t,s)\,\rho(s)\,q(s)-\frac{Q^2(t,s)}{4\,W_1(s)}\right]ds\ \le\\ \le\ w(T)-\lim_{t\to\infty}\sup\frac{1}{H(t,T)}\int_T^t\left[\sqrt{H(t,s)\,W_1(s)}\,w(s)+\frac{Q(t,s)}{2\,\sqrt{W_1(s)}}\right]^2 ds\ .\end{aligned}$$

It follows by (2.56) that for $T \ge T_0$

$$w(T)\ \ge\ \psi(T)+\lim_{t\to\infty}\sup\frac{1}{H(t,T)}\int_T^t\left[\sqrt{H(t,s)W_1(s)}\,w(s)+\frac{Q(t,s)}{2\sqrt{W_1(s)}}\right]^2 ds\,. \tag{2.57}$$

Hence, (2.44) holds for all $T \ge T_0$, and

$$\lim_{t\to\infty}\sup\frac{1}{H(t,T_0)}\int_T^t\left[\sqrt{H(t,s)W_1(s)}\,w(s)+\frac{Q(t,s)}{2\sqrt{W_1(s)}}\right]^2 ds\ \le\ w(T_0)-\phi(T_0)\ <\ \infty\,.$$

For $\alpha(t)$ and $\beta(t)$ defined as in the proof of Theorem 2.5, this implies that

$$\begin{aligned}&\lim_{t\to\infty}\sup[\alpha(t)+\beta(t)]\ \le\\ &\quad\le\ \lim_{t\to\infty}\sup\frac{1}{H(t,T_0)}\int_T^t\left[\sqrt{H(t,s)\,W_1(s)}\,w(s)+\frac{Q(t,s)}{2\,\sqrt{W_1(s)}}\right]^2 ds\ ,\end{aligned} \tag{2.58}$$

The remainder of the proof is similar to the proof of Theorem 2.5 and hence omitted. ∎

Under appropriate choices of the functions H and h, it is possible to derive from Theorems 2.3–2.6 other oscillation criteria for (1.1).

Taking, for example, for a nonnegative integer n, the function $H(t,s)$ given by

$$H(t,s) = (t-s)^n\,, \qquad (t,s) \in D\,. \tag{2.59}$$

we can easily check that $H \in \Re$. Furthermore the function

$$h(t,s) = n(t-s)^{(n-2)/2}\,, \qquad (t,s) \in D \tag{2.60}$$

is continuous and satisfies condition (II).

Other possibilities arise if we choose the functions H and h as follows:

$$H(t,s) = (e^t - e^s)^n, \quad h(t,s) = n\,e^s (e^t - e^s)^{(n-2)/2}\,, \qquad t \ge s \ge t_0\,,$$

or

$$H(t,s) = \left(\ln\frac{t}{s}\right)^n, \quad h(t,s) = \frac{n}{s}\left(\ln\frac{t}{s}\right)^{n/2-1}, \qquad t \ge s \ge t_0\,,$$

or more generally:

$$H(t,s) = \left(\int_s^t \frac{du}{\theta(u)}\right)^n, \quad h(t,s) = \frac{n}{\theta(s)}\left(\int_s^t \frac{du}{\theta(u)}\right)^{\frac{n}{2}-1}, \qquad t \ge s \ge t_0\,,$$

where $n > 1$ is an integer, and $\theta\colon\ [t_0,\infty) \to \mathbb{R}^+$ is a continuous function satisfying the condition

$$\lim_{t\to\infty} \int_{t_0}^t \frac{du}{\theta(u)} = \infty\,.$$

It is a simple matter to check that in all these cases the assumptions (I) and (II) are verified.

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