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ON MODULI OF REGULAR SURFACES WITH $K^2 = 8$ AND $p_g = 4$

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Abstract: Let S be a surface of general type with not birational bicanonical map and that does not contain a pencil of genus 2 curves. If $K_S^2 = 8$, $p_g(S) = 4$ and $q(S) = 0$ then S can be given as double cover of a quadric surface. We show that its moduli space is generically smooth of dimension 38, and single out an open subset. Note that for these surfaces $h^2(S, T_S)$ is not zero.

1 – Introduction

It is known that if X is a surface of general type with a pencil of genus 2 curves then its bicanonical map is non birational (see [1]). On the other hand, there are also surfaces with non birational bicanonical map, which have no pencil of curves of genus 2. According with [2], these surfaces are said to be *special*. Under the assumption that $p_g \geq 4$, the classification of all special surfaces has been completed in [2]. There are three main types of such surfaces, while others of them can be obtained by specialization. Two of these types are classically known (see [1] and [3]), and also their moduli space has been studied (see [4]). Here we concern with the third type, discovered in [2]. These surfaces have the following invariants: $K^2 = 8$, $p_g = 4$ and $q = 0$.

We recall theorem (3.1) in [2].

Theorem 1.1. *If S is a minimal regular surface of general type with $K^2 = 8$ and $p_g = 4$ which is special, then S is one of the following types:*

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(**i**) *The canonical system K has four distinct simple base points p_1, p_2, q_1, q_2. The canonical map ϕ_{K_S} is of degree 2 onto a smooth quadric Q of $\mathbb{P}^3$. If $p\colon \widetilde{S} \to S$ is the blow-up of the points p_1, p_2, q_1, q_2, then there exists a morphism $\varphi\colon \widetilde{S} \to Q \subset \mathbb{P}^3$ such that $\varphi = \phi_{K_S} \circ p$. The morphism φ is generically finite of degree 2, with branch curve B on Q of type $B = \eta_1 + \eta_2 + \eta_1' + \eta_2' + B'$, where η_1, η_1' are two distinct lines of the same ruling of Q, η_2, η_2' are two distinct lines of the other ruling, B' is a curve of type $(8,8)$ not containing η_i, η_i', having 4-uple points at the intersection of the four lines, and no further essential singularity.*

(**ii**) *The canonical system $|K_S|$ has a fixed component which is an irreducible (-2)-curve Z. The linear system $|K_S - Z|$ has no fixed component but has two distinct simple base points. The canonical map ϕ_{K_S} has degree 2 onto a smooth quadric Q of $\mathbb{P}^3$. If $p\colon \widetilde{S} \to S$ is the blow-up of the base points of $|K_S - Z|$, then there exists a morphism $\varphi\colon \widetilde{S} \to Q$ such that $\varphi = \phi_{K_S} \circ p$. The morphism φ is generically finite of degree 2, with branch curve B on Q of type $B = \eta + \eta' + B'$, where η, η' are two distinct lines of the same ruling of Q, B' is a curve of type $(8,8)$ not containing η, η', having two $[4,4]$-points at the intersection of the η, η' with a line of the other ruling, and tangent lines η, η', and no further essential singularity.*

Remark 1.2. Here, the essential singularities are the ones that affect the invariants of S.

The surfaces in theorem 1.1(ii) are specialization of the surfaces in theorem 1.1(i). We call *general* the latter surfaces and *particular* the former ones (see remark (3.10) in [2]). □

For the tangent bundle one has

$$\chi(T_S) = -10\,\chi(\mathcal{O}_S) + 2\,K_S^2 = -34\ . \tag{1}$$

We will prove the following:

Theorem 1.3. *The family $\mathcal{F}$ of regular surfaces with $K^2 = 8$, $p_g = 4$ with non trivial torsion and without a pencil of genus 2 curves described in theorem 1.1(i) corresponds to an open subset of its moduli space, which is irreducible, smooth of dimension* 38.

The prove is based on the geometric description of S by means of the double map on the quadric surface Q.

1.1. Notations and set up

We recall the notations used in [2]: we consider $n = \eta_1 \cap \eta_2$, $n' = \eta_1' \cap \eta_2'$, $m = \eta_2 \cap \eta_1'$, $m' = \eta_1 \cap \eta_2'$, points on the quadric Q. We denote by E_i, E_i' for $i = 1, 2$ the exceptional curves in $\widetilde{S}$ corresponding to the points p_1, p_2, q_1, q_2 of S by the blow up p.

We introduce further notations. We denote by Γ_1 and Γ_2 the two pencils of lines on Q to which η_1 and η_2 belong respectively. Let $bl : Y \to Q$ be the blow up of Q on n, n', m, m', and denote by E_n, $E_{n'}$,E_m, $E_{m'}$ the exceptional curves corresponding to the points n, n', m, m'. We write

$$E \;=\; E_n + E_{n'} + E_m + E_{m'} \ .$$

We mark with a bar the strict transforms of the divisors of Q on Y.

We have the following commutative diagram (cf. the proof of theorem (3.1)(i) in [2]):

$$\begin{array}{ccc} \widetilde{S} & \xrightarrow{p} & S \\ \downarrow\psi & & \downarrow\phi_K \\ Y & \xrightarrow{bl} & Q \end{array} \tag{2}$$

The curves E_1, E_2, E_1', E_2' are sent to the lines $\eta_1, \eta_2, \eta_1', \eta_2'$ respectively, by $\phi_K \circ p$.

Note that $\psi\colon \widetilde{S} \to Y$ is a $2\colon1$ morphism branched along a divisor B_Y of Y. In fact, there are curves on $\widetilde{S}$, denoted by $\bar{N}, \bar{N}', \bar{M}, \bar{M}'$ in [2], which are sent on E_n, $E_{n'}$,E_m, $E_{m'}$ respectively. By theorem 1.1 B_Y belongs to the linear system

$$|10\,\bar\Gamma_1 + 10\,\bar\Gamma_2 - 6\,E| \;=\; \bar\eta_1 + \bar\eta_2 + \bar\eta_1' + \bar\eta_2' + |B_Y'| \ ,$$

where

$$B_Y' \;\in\; |8\,\bar\Gamma_1 + 8\,\bar\Gamma_2 - 4\,E| \ .$$

Since Y and $\widetilde{S}$ are smooth and ψ is finite, the branch locus B_Y is smooth.

2 – The number of moduli of S

It is possible to compute the number of moduli of the surface $\widetilde{S}$ (and therefore of S) by applying the projection formula to the tangent sheaf:

$$h^i(\widetilde{S}, T_{\widetilde{S}}) \;=\; h^i\Big(Y, T_Y(-\log B_Y)\Big) + h^i\Big(Y, T_Y(-D)\Big)\,, \qquad i = 0, 1, 2\ , \tag{3}$$

where $2D \sim B_Y$ (cf. [6]). Note that

$$D \in |5\,\bar{\Gamma}_1 + 5\,\bar{\Gamma}_2 - 3\,E| \ .$$

Proposition 2.1.

$$h^2\Big(Y, T_Y(-\log B_Y)\Big) \ = \ 0 \ .$$

Proof: Consider the exact sequence

$$0 \ \to \ T_Y(-\log B_Y) \ \to \ T_Y \ \longrightarrow \ \mathcal{O}_{B_Y}(B_Y) \ \to \ 0 \ . \tag{4}$$

The curve B_Y is the disjoint union of 5 components: there are 4 rational curves composing E, plus the curve B'_Y, of genus 43, which can be easily computed by adjunction formula. By Serre duality, $H^1(B_Y, \mathcal{O}_{B_Y}(B_Y)) = 0$. Moreover $H^2(Y, T_Y) = H^2(Q, T_Q) = 0$. Hence, the long exact sequence of cohomology coming from (4) implies that $H^2(Y, T_Y(-\log B_Y)) \cong H^2(Y, T_Y) = 0$. ∎

Lemma 2.2.

$$H^k\Big(Y, bl^* T_Q(-D)\Big) = 0, \qquad \text{for} \ \ k = 0, 2 \ ,$$
$$H^1\Big(Y, bl^* T_Q(-D)\Big) \cong \mathbb{C}^8 \ .$$

Proof: One has

$$H^k\Big(Y, bl^* T_Q(-D)\Big) \ =$$
$$= \ H^k\Big(Y, \mathcal{O}_Y(-3\,\bar{\Gamma}_1 - 5\,\bar{\Gamma}_2 - 3\,E)\Big) \oplus H^k\Big(Y, \mathcal{O}_Y(-5\,\bar{\Gamma}_1 - 3\,\bar{\Gamma}_2 - 3\,E)\Big) \ .$$

In fact $T_Q = \mathcal{O}_Q(2\Gamma_1) \oplus \mathcal{O}_Q(2\Gamma_2)$. The rest follows from Riemann–Roch formula. ∎

Proposition 2.3. $h^1(S, T_S) \leq 38$.

Proof: Since $h^0(S, T_S) = 0$, being S of general type, then $h^1(S, T_S) = h^2(S, T_S) - \chi(T_S) = 34 + h^2(S, T_S)$, by (1). The proposition follows once we prove that $h^2(S, T_S) \leq 4$. It is sufficient to verify that $h^2(\tilde{S}, T_{\tilde{S}}) \leq 4$. In fact, it is $h^2(S, T_S) = h^2(\widetilde{S}, T_{\widetilde{S}})$, see for instance [5].

Consider now the exact sequence

$$0 \ \to \ T_Y(-D) \ \to \ bl^* T_Q(-D) \ \to \ N^*_{E/Y}(-D) \ \to \ 0 \ . \tag{5}$$

Since $N^*_{E/Y}(-D) = \mathcal{O}_{\mathbb{P}^1}(-2)^{\oplus 4}$, we get

$$H^0\Big(Y, N^*_{E/Y}(-D)\Big) = 0 \quad \text{and} \quad H^1\Big(Y, N^*_{E/Y}(-D)\Big) = \mathbb{C}^4 \ .$$

From (5) and lemma 2.2 one has the following exact sequence:

$$0 \to H^1\Big(Y, T_Y(-D)\Big) \to \mathbb{C}^8 \to \mathbb{C}^4 \to H^2\Big(Y, T_Y(-D)\Big) \to 0 \ .$$

In particular, $h^2(Y, T_Y(-D)) \leq 4$. From (3) and proposition 2.1 one finally has:

$$h^2(\tilde{S}, T_{\tilde{S}}) \ = \ h^2\Big(Y, T_Y(-\log B_Y)\Big) + h^2\Big(Y, T_Y(-D)\Big) \ \leq \ 4 \ . \ \blacksquare$$

2.1. Proof of Theorem 1.3

The irreducibility has been proved in [2].

Consider the family $\mathcal{F}$ of surfaces as in theorem 1.1. It is sufficient to show that $\dim \mathcal{F} = h^1(S, T_S) = 38$. Since the general surface S of $\mathcal{F}$ is the double cover of a nonsingular quadric Q of $\mathbb{P}^3$ branched on a divisor B, we can compute the dimension $\dim \mathcal{F}$ by computing the dimension of the linear system $\Sigma(B)$ of the divisors B. We recall that $B = \eta_1 + \eta_2 + \eta'_1 + \eta'_2 + B'$, where $\eta_1 + \eta'_1$ and $\eta_2 + \eta'_2$ are lines of the same pencil on the quadric, B' belongs to the sublinear system $\Sigma(B')$ cut on Q by the surfaces of degree 8, having quadruple points at the 4 intersection points of the 4 lines. Thus $\dim \Sigma(B) = 4 + (\dim \Sigma(B'))$. Since each of the quadruple points gives 10 conditions then

$$\dim \Sigma(B') \ = \ h^0(Q, \mathcal{O}_Q(8)) - 40 - 1 \ = \ 40 \ .$$

Hence

$$\dim \mathcal{F} \ = \ \dim \Sigma(B) - \dim \mathrm{Aut}\,(Q) \ = \ 40 + 4 - 6 \ = \ 38 \ .$$

Therefore $h^1(S, T_S) \geq \dim \mathcal{F} = 38$. By proposition 2.3, the equality holds. $\blacksquare$

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