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NEW LIMITING DISTRIBUTIONS OF MAXIMA OF INDEPENDENT RANDOM VARIABLES

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Abstract: This paper deals with the limiting distribution of the maximum, under linear normalization, of k_n independent real random variables, where $\{k_n\}$ is a non decreasing positive integer sequence satisfying $\lim_{n\to+\infty} k_n = +\infty$.

It is proven that, if the sequence of random variables verifies a new Uniformity Assumption of Maxima depending on the behaviour of the sequence $\{k_n\}$, which is a suitable extension of the Galambos assumption (Galambos, 1978), a new class of limiting distribution of maxima arises in the theory of extremes. This class contains the Mejzler's class of log-concave distributions (Mejzler, 1956) and also the class of max-semistable distributions introduced in Grinevich (1992).

1 – Introduction

Let $\{X_n\}$ be a sequence of independent real random variables with distribution functions (d.f.'s) sequence $\{F_n\}$, and $\{k_n\}$ an integer sequence verifying

(1) $$k_{n+1} \geq k_n \geq 1 \quad \text{and} \quad \lim_{n\to+\infty} k_n = +\infty \ .$$

In this paper we characterize the class of all non degenerate limits in

(2) $$\lim_{n\to+\infty} P(M_{k_n} \leq x/a_n + b_n) \,=\, G(x) \ ,$$

where $M_{k_n} = \max\{X_1, X_2, ..., X_{k_n}\}$ and $\{a_n\}$, $a_n > 0$, and $\{b_n\}$ are real sequences. The convergence (2) holds for all continuity points of the non degenerate distribution function (d.f.) G.

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In the particular case $k_n = n$, Mejzler (1956) introduced the class of all possible limiting distributions of the maxima, linearly normalized, which are usually called *log-concaves* distributions. This class of Mejzler, denoted by class M, coincides with the class of max-selfdecomposable real d.f.'s under linear normalization.

In what follows $w_G = \sup\{x\colon\ G(x) < 1\}$ and $\alpha_G = \inf\{x\colon\ G(x) > 0\}$.

Theorem 1. *(Mejzler, 1956) Let G be a non degenerate d.f.. Suppose that there exist two real sequences $\{a_n\}$, $a_n > 0$, and $\{b_n\}$ and a real sequence of independent random variables such that*

$$\lim_{n\to+\infty} P(M_n \le x/a_n + b_n) \;=\; G(x) \;, \tag{3}$$

for all continuity points of the non degenerate d.f. G. Suppose further that $\lim\limits_{n\to+\infty} F_{q_n}(x/a_n + b_n) = 1$, for any x with $G(x) > 0$ and for any sequence $\{q_n\}$ of integers with $1 \le q_n \le n$. Then

- **i)** $w_G = +\infty$ *and* $\log G(x)$, $x > \alpha_G$, *is a concave function or*
- **ii)** $w_G < +\infty$ *and* $\log G(w_G - e^{-x})$, $x \in \mathbb{R}$, *is a concave function or*
- **iii)** $\alpha_G > -\infty$ *and* $\log G(\alpha_G + e^{x})$, $x \in \mathbb{R}$, *is a concave function.*

Conversely any d.f. satisfying i), ii) or iii) can occur as a limit in the given set-up.

In Galambos (1978) are obtained the conclusions of Mejzler's Theorem changing its assumption by the Uniformity Assumption of Maxima. This assumption is stronger than the one of Mejzler but it is easier to use in practice.

It is well known that if $\{X_n\}$ is a sequence of independent and identically distributed (i.i.d.) random variables, the limit in (3) is a max-stable distribution. Thus a max-stable distribution is log-concave.

On the other hand, if the random variables of $\{X_n\}$ are i.i.d., with d.f. F, but $\{k_n\}$ satisfies

$$k_{n+1} \ge k_n \ge 1 \quad \text{and} \quad \lim_{n\to+\infty} \frac{k_{n+1}}{k_n} = r, \quad r \in [1, +\infty[\;, \tag{4}$$

a new class of limiting distribution of M_{k_n}, linearly normalized, appears in the theory of extremes (Grinevich 1992, 1993). This is the class MSS of *max-semistable* distributions. Following Grinevich (1992) we shall say that a real non degenerate d.f. G is max-semistable if there are reals $r > 1$, $a > 0$ and b such that

$$G(x) = G^r(x/a + b)\,, \qquad x \in \mathbb{R}\;, \tag{5}$$

or equivalently, if there exist a sequence of i.i.d. random variables with d.f. F and two real sequences $\{a_n\}$, $a_n>0$, and $\{b_n\}$ such that (2) holds for each continuity point of G. Canto e Castro *et al.* (1999) gives a characterization of max-semistable distributions, simpler than the one given in Grinevich (1993), and gives necessary and sufficient conditions on F such that (2) holds.

We further remark that the Geometric d.f., the Binomial Negative d.f. and the von Misès d.f. $F(x) = 1-\exp(-x-\frac{1}{2}\sin x),\ x>0$, do not belong to any max-stable domain of attraction. Nevertheless these d.f.'s belong to a max-semistable domain of attraction.

In section 2, we will consider sequences of independent and, in general, non identically distributed random variables and an integer sequence $\{k_n\}$ satisfying (1). We prove that if a new *Uniformity Assumption of Maxima* holds and (2) occur, then G is a *log-semiconcave* distribution. This new class of d.f.'s contains the *log-concave* class of Mejzler and also contains the Grinevich's class.

2 – Results

In what follows we only consider the case where the sequence $\{k_n\}$ verifies (1) and $\dfrac{k_{n+1}}{k_n} \not\to 1,\ n\to+\infty$. In fact, considering $\lim\limits_{n\to+\infty} \dfrac{k_{n+1}}{k_n} = 1$, we obtain a natural extension to the case considered by Mejzler and so, *mutatis mutandis*, we easily prove that the class of limit distributions in (2) is the Mejzler's class.

Definition 1. Let $\{k_n\}$ be an integer sequence satisfying the assumptions (1) and $\dfrac{k_{n+1}}{k_n} \not\to 1,\ n\to+\infty$. The sequence $\{X_n\}$ satisfies the *Uniformity Assumption of Maxima* on $\{k_n\}$ if there exist two real sequences $\{a_n\}$, $a_n>0$, and $\{b_n\}$ such that

$$\overline{F}_{k_n,max} := \max_{i=1,\dots,k_n}\Big(1-F_i(x/a_n+b_n)\Big) \to 0\,, \qquad n\to+\infty\,, \tag{1}$$

holds and, for all $m\geq 0$,

$$\lim_{n\to+\infty} \sum_{j=1}^{k_{n-m}} \Big(1-F_j(x/a_n+b_n)\Big) = w_*(m,x) \tag{2}$$

where $w_*(m,\cdot)$ verifies $0<w_*(m,x)<w_*(m,y)<+\infty$, for some $y<x$. □

For the proof of Theorem 2 we need the following lemma.

Lemma 1. *If $a_{n,j}$, for $j = 1, ...n$ and $n \geq 1$, are real numbers in $[0,1]$ such that*

$$\lim_{n\to+\infty} \max\Big\{a_{n,j},\ j = 1,...,n\Big\} = 0$$

and $\prod_{j=1}^{n}(1-a_{n,j}) > 0$, $n > n_0$, for some n_0, then

$$\log \prod_{j=1}^{n}(1-a_{n,j}) = \Big(1+o(1)\Big)\Big(-\sum_{j=1}^{n} a_{n,j}\Big) .$$

Theorem 1. *Let $\{k_n\}$ be an integer sequence verifying (1) and suppose that $\frac{k_{n+1}}{k_n} \not\to 1$, $n \to +\infty$.*

1. *If there exist two real sequences $\{a_n\}$, $a_n > 0$, and $\{b_n\}$ and a real sequence of independent random variables verifying the Uniformity Assumption of Maximum on $\{k_n\}$, then (2) holds, for all continuity points of G, and G verifies at least one of the following conditions:*

   i) *$w_G = +\infty$ and there exists a positive real α such that $H_1(x) = \frac{G(x)}{G(x+\alpha)}$, $x > \alpha_G$, is a non decreasing and right continuous function,*

   ii) *$w_G < +\infty$ and there exists a positive real $\alpha < 1$ such that*

$$H_2(x) = \begin{cases} \dfrac{G(x)}{G\Big(\alpha(x-w_G)+w_G\Big)}, & x < w_G, \\ 1, & x \geq w_G\ , \end{cases}$$

   is a non decreasing and right continuous function or

   iii) *$\alpha_G > -\infty$ and there exist a real $\alpha > 1$ such that*

$$H_3(x) = \begin{cases} 0, & x < \alpha_G, \\ \dfrac{G(x)}{G\Big(\alpha(x-\alpha_G)+\alpha_G\Big)}, & x \geq \alpha_G\ , \end{cases}$$

   is a non-degenerate d.f..

2. *Conversely if a non degenerate d.f. G verifies one of the three conditions i), ii) or iii), then there exist real sequences $\{a_n\}$, $a_n > 0$, and $\{b_n\}$ and a sequence of independent random variables satisfying the Uniformity Assumption of Maxima on $\{k_n\}$. Consequently (2) holds for all continuity point of G.*

Proof: Consider $u_n = x/a_n + b_n$.
Using Lemma 1 we obtain

$$\log P(M_{k_{n-m}} \le u_n) = \log \prod_{j=1}^{k_{n-m}} F_j(u_n) \tag{3}$$
$$= \Big(1 + o(1)\Big) \Big(- \sum_{j=1}^{k_{n-m}} \Big(1 - F_j(u_n)\Big)\Big)$$

or equivalently

$$\lim_{n\to+\infty} P(M_{k_{n-m}} \le u_n) = \exp(-w_*(m,x)) \ .$$

Attending to (3), the Khintchine's Theorem gives us

$$\lim_{n\to+\infty} \frac{a_{n-m}}{a_n} = A_m \ , \qquad \lim_{n\to+\infty} a_{n-m}(b_n - b_{n-m}) = B_m \tag{4}$$

and $G(A_m x + B_m) = \exp(-w_*(m,x))$, for all $m \ge 1$. Then, for x such that $\alpha_G < A_m x + B_m \le w_G$, we have

$$\lim_{n\to+\infty} \prod_{j=k_{n-m}+1}^{k_n} F_j(u_n) = \frac{G(x)}{G(A_m x + B_m)} \ . \tag{5}$$

It is clear that the function on the right hand side of (5) is non decreasing in x and belongs to $[0,1]$.

In order to specify the possible forms of A_m and B_m we observe that, using (4) it is easy to establish the following functional equations, stated for all positive integers m and p

$$A_{m+p} = A_m A_p$$

and

$$B_{m+p} = A_p B_m + B_p = A_m B_p + B_m \ .$$

Thus $A_m = 1$ and $B_m = m\,\alpha$ for some $\alpha > 0$, or $A_m = \alpha^m$ for some real $\alpha \ne 1$ and $B_m = \beta(\alpha^m - 1)$ for some real β.

In the first case, the right hand side of (5) becomes $\frac{G(x)}{G(x+m\alpha)}$ with $\alpha > 0$. Moreover $\frac{G(x)}{G(x+m\alpha)}$ is non decreasing in x if and only if the same holds with $H_1(x)$.

In the second case, the right hand side of (5) becomes $\frac{G(x)}{G(\alpha^m(x+\beta)-\beta)}$.

Since $\alpha^m(x+\beta) - \beta \ge x$ is equivalent to $\alpha < 1$ and $x \le -\beta$ or $\alpha > 1$ and $x \ge -\beta$ it is obvious that $\alpha < 1$ and $-\beta = w_G$ or $\alpha > 1$ and $-\beta = \alpha_G$. Hence H_2 and H_3 are non decreasing and right continuous functions.

We prove now the part 2 of the theorem.

We observe first that H_i, $i=1,2$, could not verify $\lim\limits_{x\to-\infty} H_i(x)=0$ but the sequences of d.f.'s which we are going to define must have a left tail verifying $\lim\limits_{x\to-\infty} F_j(x)=0,\ j\geq 1$.

i) Let $\{F_j\}$ be a sequence of d.f.'s, which for $x>x_0$, for a certain real x_0, are defined by

$$F_{k_j}(x) \;=\; \frac{G(x-\alpha j)}{G(x-\alpha j+\alpha)}\,, \qquad j\geq 1\ ,$$

and

$$F_j(x) \;=\; \frac{G(x+\alpha j)}{G(x+\alpha j+\alpha)}\,, \qquad j\in\mathbb{N}\backslash\{k_n,\, n\geq 1\}\ .$$

Therefore with $u_n=x+\alpha n$ we have, as $n\to+\infty$,

$$\sum_{j=1}^{k_{n-m}}\Big(1-F_j(u_n)\Big) \;\sim\; \log\prod_{j=1}^{n-m}\frac{G(u_n-\alpha j+\alpha)}{G(u_n-\alpha j)} \;+\; \log\prod_{j\in B_m}\frac{G(u_n+\alpha j+\alpha)}{G(u_n+\alpha j)}\ ,$$

where $B_m=\{1,2,...,k_{n-m}\}\setminus\{k_1,k_2,...,k_{n-m-1},k_{n-m}\}$.

Attending to the fact that

$$\log\prod_{j=1}^{n-m}\frac{G(u_n-\alpha j+\alpha)}{G(u_n-\alpha j)} \;=\; o(1)-\log G(x+\alpha m)$$

and

$$\begin{aligned}\log\prod_{j\in B_m}\frac{G(u_n+\alpha j+\alpha)}{G(u_n+\alpha j)} \;&\leq\; \log\prod_{j=1}^{k_{n-m}}\frac{G(u_n+\alpha j+\alpha)}{G(u_n+\alpha j)}\\ &=\; \log G\Big(x+\alpha(n+1+k_{n-m})\Big) \;-\; \log G\Big(x+\alpha(n+1)\Big)\\ &=\; o(1)\,, \qquad n\to+\infty,\end{aligned}$$

we get

$$\lim_{n\to+\infty}\sum_{j=1}^{k_{n-m}}\Big(1-F_j(u_n)\Big) \;=\; -\log G(x+\alpha m)\ ,$$

for all $m\geq 0$.

Hence, since (1) is immediate, we have a sequence of independent variables of d.f.'s $\{F_n\}$ verifying the Uniformity Assumption of Maxima on $\{k_n\}$ and

$$\lim_{n\to+\infty} P(M_{k_n}\leq x+\alpha n) \;=\; G(x)\ .$$

In the second case we define, for $x_0 < x < w_G$,

$$F_{k_j}(x) \;=\; \frac{G\Big(\alpha^{-j}(x-w_G)+w_G\Big)}{G\Big(\alpha^{-j+1}(x-w_G)+w_G\Big)}\,, \qquad j \ge 1\ ,$$

$$F_j(x) \;=\; \frac{G\Big(\alpha^{j}(x-w_G)+w_G\Big)}{G\Big(\alpha^{j+1}(x-w_G)+w_G\Big)}\,, \qquad j \in \mathbb{N}\backslash\{k_n,\, n\ge 1\}\ ,$$

and $F_j(x)=1$ for $x > w_G$. With $u_n = \alpha^n x + w_G(1-\alpha^n)$ we obtain the desired result. In this case we also have proved that

$$\lim_{n\to+\infty} P\Big(M_{k_n} \le \alpha^n x + w_G(1-\alpha^n)\Big) \;=\; G(x)\ .$$

Finally, using H_3 we define $F_j(x)=0$ for $x<\alpha_G$ and, for $x\ge\alpha_G$,

$$F_{k_j}(x) \;=\; \frac{G\Big(\alpha^{-j}(x-\alpha_G)+\alpha_G\Big)}{G\Big(\alpha^{-j+1}(x-\alpha_G)+\alpha_G\Big)}\,, \qquad j \ge 1\ ,$$

and

$$F_j(x) \;=\; \frac{G\Big(\alpha^{j}(x-\alpha_G)+\alpha_G\Big)}{G\Big(\alpha^{j+1}(x-\alpha_G)+\alpha_G\Big)}\,, \qquad j \in \mathbb{N}\backslash\{k_n,\, n\ge 1\}\ .$$

Choosing $u_n = \alpha^n(x-\alpha_G)+\alpha_G$ we again obtain the desired results. ∎

The class of all non degenerate limiting d.f.'s G which arise in the last theorem will be denoted by $\mathcal{L}$.

We should remark that this new class contains the class of Mejzler. Indeed, if for instance $w_G = +\infty$ and $\log G$ is a concave function then, for all positive real α and $x > \alpha_G$, $\frac{G(x)}{G(x+\alpha)}$ is a non decreasing and continuous function and thus it verifies condition i) in Theorem 2. By applying similar arguments we can establish the other two inclusions.

Example 1. The Poisson d.f. $\mathcal{P}(\lambda)$ belongs to class $\mathcal{L}$, for all λ. □

Example 2. The Binomial d.f. $B(m,p)$ belongs to class $\mathcal{L}$, for all m and for all p. □

In Proposition 1 we prove that $\mathcal{L}$ contains the Grinevich's class. Before we do that, we present one illustrative example where it is shown that this inclusion is strict.

Example 3. The d.f.

$$G(x) = \begin{cases} \exp\Big(-e^{-2x}(4+\cos 2\pi x)\Big), & x<1, \\ \exp\Big(-e^{-1-x}(2-\cos \pi x)\Big), & 1\le x<2, \\ \exp\Big(-e^{-8-x}(8+\cos 2\pi x)\Big), & x\ge 2\ , \end{cases}$$

belongs to $\mathcal{L}$ but it is not log-concave neither is max-semistable. □

Proposition 1. *Any max-semistable d.f. belongs to $\mathcal{L}$.*

Proof: If G is non degenerate and satisfies (5) with $a=1$, that is $G(x) = G^r(x+b)$ for all x in $\mathbb{R}$, then $\frac{G(x)}{G(x+b)} = G^{r-1}(x+b)$. On the other hand, if $a>1$ in (5) then $b = w_G(1-a^{-1})$ and $G(x) = G^r(x/a + w_G(1-a^{-1}))$ which implies $\frac{G(x)}{G(w_G - a^{-1}(w_G-x))} = G^{r-1}(w_G - a^{-1}(w_G-x))$.

The case $0<a<1$ is similar. ∎

We present now a relation between $\mathcal{L}$ and concavity.

Proposition 2. *Suppose that G is a non degenerate d.f.. If G belongs to $\mathcal{L}$ then G verifies at least one of the following conditions:*

- **i)** *$w_G = +\infty$ and there exist a positive real α such that for all $x>\alpha_G$, $\{\log G(x+\alpha m)\}_m$ is a concave sequence,*
- **ii)** *$w_G < +\infty$ and there exist a real α in $]0,1[$ such that, for all $x>\alpha_G$, $\{\log G(w_G - \alpha^m(w_G-x))\}_m$ is a concave sequence or*
- **iii)** *$\alpha_G > -\infty$ and there exist a real $\alpha>1$ such that, for all $x>\alpha_G$, $\{\log G(\alpha_G + \alpha^m(x-\alpha_G))\}_m$ is a concave sequence.*

Proof: Suppose that $w_G = +\infty$ and for some $\alpha>0$ the function $\frac{G(x)}{G(x+\alpha)}$ is non decreasing for $x>\alpha_G$. Thus, for each real $x>\alpha_G$ and each $m\ge 1$, since $x+\alpha(m-1) < x+\alpha m$, we have

$$\log \frac{G\Big(x+\alpha(m-1)\Big)}{G(x+\alpha m)} \le \log \frac{G(x+\alpha m)}{G\Big(x+\alpha(m+1)\Big)}\ .$$

With $a_m(x) := \log G(x + \alpha m)$, it follows

$$a_{m+1}(x) - a_m(x) \leq a_m(x) - a_{m-1}(x), \quad m \geq 1 .$$

That is, for each real $x > \alpha_G$, the sequence $\{a_m(x)\}_m$ is concave.

If there exist a real α in $]0,1[$ such that $\frac{G(x)}{G(\alpha(x-w_G)+w_G)}$ is non decreasing for $x > \alpha_G$, then attending that $\alpha^{m-1}(x - w_G) + w_G < \alpha^m(x - w_G) + w_G$ we get

$$\log \frac{G\Big(\alpha^{m-1}(x-w_G)+w_G\Big)}{G\Big(\alpha^{m}(x-w_G)+w_G\Big)} \leq \log \frac{G\Big(\alpha^{m}(x-w_G)+w_G\Big)}{G\Big(\alpha^{m+1}(x-w_G)+w_G\Big)}$$

and so the desired result is proved.

The proof of iii) is similar. ∎

Any d.f. in the class introduced in Proposition 2 is called *Log–semiconcave* and this class will be denoted by $\mathcal{L}^*$.

From this proposition we deduce that if G belongs to $\mathcal{L}$ for some α than G is log-semiconcave for the same α. However the converse of this particular implication is false as we show in the following example.

Example 4. The d.f. G defined by

$$\log G(x) = \begin{cases} -1.9 \exp\Big(-(x+1)/1.9\Big), & x < -1, \\ 0.9\,x - 1, & -1 \leq x < 0, \\ x - 1, & 0 \leq x < 0.5, \\ -4^{-x}, & x \geq 0.5 , \end{cases}$$

is a log-semiconcave d.f. for $\alpha \geq 1$ but $\frac{G(x)}{G(x+\alpha)}$ is decreasing in $]-1,-0.5[$ for all α in $]0,1.5[$. Further $\frac{G(x)}{G(x+\alpha)}$ is non decreasing in $\mathbb{R}$, for all $\alpha \geq 1.5$. □

We remark that there are distributions which are not log-semiconcaves. One example is

$$G(x) = \begin{cases} \dfrac{1}{1-x}, & x < -1, \\ 1 - \dfrac{1}{2}\exp\Big(-(x+1)/2\Big), & x \geq -1 . \end{cases}$$

As a conclusion we present the following table which summarize what we said before about these four classes of limit laws for the maximum of independent random variables.

	$\dfrac{k_{n+1}}{k_n} \to 1,\ n \to +\infty$	$\dfrac{k_{n+1}}{k_n} \not\to 1,\ n \to +\infty$
i.i.d. marginal d.f.	MS Gnedenko (1943)	$\dfrac{k_{n+1}}{k_n} \to r > 1$ MSS Grinevich(1992)
non i.d. marginal d.f. (in general)	M Mejzler(1956)	$k_n \to +\infty$ $\mathcal{L}$ $\mathcal{L} \subset \mathcal{L}^*$

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