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DECAY OF SOLUTIONS OF SOME NONLINEAR EQUATIONS

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Abstract: For a class of scalar partial differential equations that incorporate convection, diffusion, and possibly dispersion in one space and one time dimension, the stability of solutions is investigated.

1 – Introduction

The topic of this paper is the class of equations

$$u_t - \alpha\, u_{xx} + \beta\, u_{xxx} + \gamma\, u_{xxxxt} + \big(g(u)\big)_x = 0\,, \qquad x \in \mathbb{R}, \quad t > 0 \tag{1.1}$$

where subscripts denote partial derivatives. The case $g(u) = \frac{u^2}{2}$ and $\gamma = 0$ is typical and has received much attention. If $\alpha > 0, \beta = 0$, this is known as Burgers equation. If $\alpha = 0, \beta > 0$, this is essentially the Korteweg–de-Vries equation. The case $\alpha > 0, \beta > 0$ is thus refereed to as KdV-Burgers equation; it also has been studied extensively as has been the case of general g. The case $g(u) = \frac{u^{p+1}}{p+1}$ and $\gamma = 1, \beta = 0$ where $p \geq 1$ is an integer is refereed to as the Rosenau–Burgers equation. Indeed, if $\alpha = 0$ then we have the Rosenau equation proposed by Rosenau [8] for treating the dynamics of dense discrette systems in order to overcome the shortenings by the KdV equation, since the KdV equation describes unidimensional propagation of waves, but wave-wave and wave-wall interactions cannot be treated by it. Such a model were studied

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by Park [9] and Chung and Ha [3] for the global existence of the solution to the IBVP. Equation (1.1) with $\alpha > 0, \gamma = 1, \beta = 0$ is called the Rosenau–Burgers equation and somehow corresponds to the KdV-Burgers equation and the Benjamin–Bona–Mahoney–Burgers equation, but it is given neither by Rosenau nor by Burgers. The Rosenau equation with the dissipative term $-\alpha u_{xx}$, or say, the Rosenau–Burgers equation arises in some natural phenomena as for example, in bore propagation and in water waves.

In section 2 of this paper, we study the problem (1.1) with $\beta = 1, \gamma = 0$:

$$\text{(E1)} \qquad u_t - \alpha\, u_{xx} + u_{xxx} + \big(g(u)\big)_x = 0\,, \qquad x \in \mathbb{R}, \quad t > 0$$

where $\alpha > 0$ and g is a C^2-class function. We give a general criterion for the existence of traveling wave solutions of the form $u(x,t) = \phi(x - ct)$.

In section 3, we study the asymptotic behaviour of the solution for the Rosenau–Burgers equation (problem (1.1) with $\beta = 0, \gamma = 1$):

$$\text{(E2)} \qquad \begin{cases} u_t - \alpha\, u_{xx} + u_{xxxxt} + \left(\frac{u^{p+1}}{p+1}\right)_x = 0\,, & x \in \mathbb{R}, \quad t > 0\ , \\ u(x,0) = u_0(x) \to 0 \quad \text{as} \quad x \to \pm\infty\ , \end{cases}$$

where $\alpha > 0$ and $p \geq 1$ is an integer.

The problem (E2) was studied by Mei [7]. He proved that if $\displaystyle\int_{\mathbb{R}} u_0(x)\,dx \neq 0$ then

$$\|u(t)\|_{L^2} \leq \frac{c}{(1+t)^4} \quad \text{and} \quad \|u(t)\|_\infty \leq \frac{c}{\sqrt{1+t}}\,, \qquad \forall\, t \geq 0\ .$$

And, if $\int_{\mathbb{R}} u_0(x)\,dx = 0$, then

$$\|u(t)\|_{L^2} \leq \frac{c}{(1+t)^{3/4}} \quad \text{and} \quad \|u(t)\|_\infty \leq \frac{c}{1+t}\,, \qquad \forall\, t \geq 0\ .$$

Hence, it is proved in [7] that 0 is the asymptotic state of the solution $u(x,t)$ for the Rosenau–Burgers equation. In this paper, we prove that the solution of the nonlinear parabolic equation $u_t - \alpha u_{xx} + \left(\frac{u^{p+1}}{p+1}\right)_x = 0$ is a better asymptotic profile for the Rosenau–Burgers equation. Furthermore, we prove that the convergence to this asymptotic profile is faster than the convergence to 0 proved in [7].

Before ending this section, we state and prove a general technical lemma which will be needed later.

Lemma 1.1.

(**i**) *If $a > 0$, $b > 0$, then we have for all $t \geq 0$*

$$\int_0^t (1+t-s)^{-a}\,(1+s)^{-b}\,ds \;\leq\; \begin{cases} c(1+t)^{-\min(a,b)} & \textit{if } \max(a,b) > 1, \\ c(1+t)^{-\min(a,b)}\log(2+t) & \textit{if } \max(a,b) = 1, \\ c(1+t)^{1-a-b} & \textit{if } \max(a,b) < 1\;. \end{cases}$$

(**ii**) *Let $0 < a < b$ with $b > 1$. Let $f\colon (0,\infty) \to \mathbb{R}$ be bounded on $[1,\infty)$ and integrable on (0,1). Then we have for all $t \geq 0$*

$$\int_0^t f(t-s)\,(1+t-s)^{-a}\,(1+s)^{-b}\,ds \;\leq\; c\,t^{-a}\;.$$

Proof:

(**i**) We give a brief outline of the proof. Let

$$I \;=\; \int_0^\infty (1+s)^{-a}\,\Big(1+|t-s|\Big)^{-b}\;,$$

where $a > 0$, $b > 0$ and $\max(a,b) > 1$.

Assuming $t > 0$, as is no essential loss of generality, since evidently $I(-t) \leq I(t)$, we may write

$$I \;=\; \int_0^{\varepsilon t} + \int_{\varepsilon t}^t + \int_t^\infty\;,$$

where $0 < \varepsilon < 1$, and ε is held fixed. Now

$$\int_0^{\varepsilon t} \;\leq\; \int_0^{\varepsilon t} (1+s)^{-a}\,\Big(t(1-\varepsilon)\Big)^{-b}\,ds \;=\; o(t^{-b})\int_0^{\varepsilon t}(1+s)^{-a}\,ds\;.$$

In case $a > 1$, this expression is $o(t^{-b})$. In case $a = 1$, it is $o(\log t \cdot t^{-b})$, which is $o(t^{-a})$ since $b > 1$. In case $a < 1$, it is

$$o(t^{-b})\int_0^{\varepsilon t} s^{-a}\,ds \;=\; o(t^{-b})\,o(t^{-a+1}) \;=\; o(t^{1-a-b})\;;$$

since $1 + \min(a,b) \leq a + b$ by virtue of the assumption that $\max(a,b) > 1$, this is in turn $o(t^{-\min(a,b)})$.

Similarly,

$$\int_{\varepsilon t}^t \;\leq\; o(t^{-a})\int_{\varepsilon t}^t \Big(1+|t-s|\Big)^{-b}\,ds \;=\; o(t^{-a})\,o(t^{-b+1})$$

if $b\neq 1$, and so is $o(t^{-\min(a,b)})$. In case $b=1$, the integral is $o(t^{-a})o(\log t) = o(t^{-b})$ since in this case $a>1$. If $b>1$,

$$\int_t^\infty \leq o(t^{-a}) \int_t^\infty \Big(1+|t-s|\Big)^{-b} ds \ ,$$

which is $o(t^{1-a-b})$. Finally, if $b\leq 1$,

$$\int_t^\infty \leq o(t^{-b}) \int_t^\infty \Big(1+|t-s|\Big)^{-a} = o(t^{-b}) \ .$$

(ii) We have

$$\begin{aligned}\int_0^t f(t-s)\,(1+t-s)^{-a}\,(1+s)^{-b}\,ds \ &\leq \\ &\leq \int_0^t (1+t-s)^{-a}\,(1+s)^{-b}\,ds + \int_0^1 f(s)\,(1+s)^{-a}\,(1+t-s)^{-b}\,ds \\ &\leq \int_0^t (1+t-s)^{-a}\,(1+s)^{-b}\,ds + c(1+t)^{-b} \ .\end{aligned}$$

Now,

$$\begin{aligned}\int_0^t (1+t-s)^{-a}\,(1+s)^{-b}\,ds \ &= \\ &= \int_0^{\frac{t}{2}} (1+t-s)^{-a}(1+s)^{-b}\,ds + \int_{\frac{t}{2}}^t (1+t-s)^{-a}\,(1+s)^{-b}\,ds \ .\end{aligned}$$

If $a,b>1$, we get

$$\int_0^t (1+t-s)^{-a}(1+s)^{-b}\,ds \leq \frac{1}{b-1}\Big(1+\frac{t}{2}\Big)^{-a} + (a-1)\Big(1+\frac{t}{2}\Big)^{-b} \leq c(1+t)^{-a} \ .$$

If $a<1<b$, we get

$$\begin{aligned}\int_0^t (1+t-s)^{-a}\,(1+s)^{-b}\,ds &\leq \frac{1}{b-1}\Big(1+\frac{t}{2}\Big)^{-a} + (1-a)\,(1+t)^{1-a}\Big(1+\frac{t}{2}\Big)^{-b} \\ &\leq c\,(1+t)^{-a} \ .\end{aligned}$$

If $a=1<b$, we get

$$\begin{aligned}\int_0^t (1+t-s)^{-a}\,(1+s)^{-b}\,ds &\leq \frac{1}{b-1}\Big(1+\frac{t}{2}\Big)^{-1} + \log\Big(1+\frac{t}{2}\Big)\Big(1+\frac{t}{2}\Big)^{-b} \\ &\leq c(1+t)^{-1} \ . \ \blacksquare\end{aligned}$$

2 – Existence of traveling wave solutions

Consider the equation

$$u_t - \alpha\, u_{xx} + u_{xxx} + \big(g(u)\big)_x \,=\, 0\,, \qquad x\in\mathbb{R}, \quad t>0 \tag{E1}$$

where $\alpha>0$ and g is a C^2-class function.

Under certain conditions, the equation admits monotone traveling wave solutions $u(x,t)=\phi(x-ct)$ with speed c that connect the end states $\phi_\pm = \lim_{r\to\pm\infty}\phi(r)$. Such a wave profile must satisfy the third order ordinary differential equation

$$-c\,\phi' + g(\phi)' + \phi''' - \alpha\,\phi'' \,=\, 0\ .$$

An example is $g(x) = 2\,x(x-1)\,(b-x)$ with $b\geq 2$, which has the wave profile $\phi(x)=\frac{1}{1+e^x}$ for the parameter $\alpha=2b-1$ and the speed $c=0$. General profiles (non necessarily monotone) have been constructed in [1, 5]. It is known that monotone profiles exist for $g(u)=\frac{u^{p+1}}{p+1}$ and $\alpha\geq 2\sqrt{pc}$. More generally, we have the following criterion:

Theorem 2.1. *Let $g\in C^2$ be a strictly convex function. A monotone wave profile ϕ for (E1) exists if and only if:*

$$c \,=\, \frac{g(\phi_+)-g(\phi_-)}{\phi_+-\phi_-}\ , \tag{2.1}$$

$$\alpha \,\geq\, 2\sqrt{g'(\phi_-)-c}\ , \tag{2.2}$$

$$\phi_+ \,<\, \phi_-\ . \tag{2.3}$$

The profile must therefore be monotonically decreasing.

Proof:

($\Longrightarrow$) Clearly we have

$$-c\,\phi + g(\phi) + \phi'' - \alpha\,\phi' \,=\, \mathit{constant}$$

and hence $-c\phi_- + g(\phi_-) = -c\phi_+ + g(\phi_+)$ which implies (2.1).

Now, set $\psi(z)=\phi_- - \phi(-z)$ and

$$f(r) \,=\, g(\phi_-) - c\,r - g(\phi_- - r)\ . \tag{2.4}$$

Then, f is concave, and $-\alpha\psi' - \psi'' = f(\psi)$ which is the equation for a wave profile ψ of the Fisher–Kolmogorov–Petrovskii–Piskunov (F-KPP) equation $v_t - v_{xx} = f(v)$ that travels to the right with speed α and has limits $\psi_- = \phi_- - \phi_+,\ \psi_+ = 0$.

Thanks to [4], we know that such a monotone wave profile for concave f exists if and only if $\alpha \geq 2\sqrt{f'(0)} = 2\sqrt{g'(\phi_-) - c}$. In this case, $\psi_- > \psi_+$ and therefore $\phi_- > \phi_+$, since f is positive between ψ_- and ψ_+. Thus (2.2)–(2.3) are true.

($\Longleftarrow$) Define f as in (2.4), then by known results about the F-KPP equation, there exists a unique decreasing wave profile ψ with $\psi(0) = \frac{\phi_- - \phi_+}{2}$ that moves to the right with speed α. Then, $\phi(z) = \phi_- - \psi(-z)$ is a monotone wave profile for (E1) with $\phi(\pm\infty) = \phi_\pm$. ∎

3 – Asymptotic profile of Rosenau–Burgers equation

Consider the Rosenau–Burgers equation in the form

$$\text{(E2)} \qquad \begin{cases} u_t - \alpha\, u_{xx} + u_{xxxxt} + \left(\dfrac{u^{p+1}}{p+1}\right)_x = 0\,, & x \in \mathbb{R}, \quad t > 0, \\ u(x,0) = u_0(x) \to 0 \quad \text{as} \quad x \to \pm\infty\,, \end{cases}$$

where α is any given constant, $p \geq 1$ is integer.

Consider the following scalings to the variables

$$t \to \frac{t}{\varepsilon^2}\,, \qquad x \to \frac{x}{\varepsilon}\,, \qquad u \to \varepsilon^{1/p}\, u\,,$$

where $\varepsilon \ll 1$, then we obtain from (E2)

$$\text{(3.1)} \qquad u_t - \alpha\, u_{xx} + \varepsilon^4\, u_{xxxxt} + u^p\, u_x = 0\,.$$

For $\varepsilon \ll 1$, neglecting the small term $\varepsilon^4 u_{xxxxt}$ leads to the asymptotic state equation of the Rosenau–Burgers equation (E2) as follows

$$u_t - \alpha\, u_{xx} + u^p\, u_x = 0\,.$$

The solution of this parabolic equation should be a better asymptotic profile of equation (E2).

Concerning the parabolic equation

$$\text{(3.2)} \qquad \begin{cases} v_t - \alpha\, v_{xx} + v^p\, v_x = 0\,, \\ v(x,0) = v_0(x) \to 0 \quad \text{as} \quad x \to \pm\infty \end{cases}$$

we have the following result:

Theorem 3.1 ([10, 11]). *Suppose that $v_0(x) \in H^2(\mathbb{R}) \cap L^1(\mathbb{R})$. Then, there exists a positive constant δ_0 such that if $\|v_0\|_{L^1} + \|v_0\|_{H^2} \leq \delta_0$, then the problem (3.2) has a unique global solution $v(x,t)$ with*

$$v \in C(\mathbb{R}_+, H^2(\mathbb{R}) \cap L^1(\mathbb{R})) \cap L^2(\mathbb{R}_+, H^1(\mathbb{R}))$$

and

$$\|\partial_x^j v(t)\|_{L^q} = \mathcal{O}(1) \left(\|v_0\|_{L^1} + \|v_0\|_{H^2}\right) (1+t)^{-\frac{(j+1)q-1}{2q}}, \quad 1 \leq q \leq \infty \,. \tag{3.3}$$

Furthermore, if $v_0 \in L^1(\mathbb{R}) \cap H^6(\mathbb{R})$, then

$$\|\partial_x^j v_t(t)\|_{L^1} = \mathcal{O}(1) \left(\|v_0\|_{L^1} + \|v_0\|_{H^6}\right) (1+t)^{-1-\frac{j}{2}}, \quad j = 0,1,2,3,4 \,. \tag{3.4}$$

The main result in this section is the following:

Theorem 3.2. *Suppose that*

$$w_0(x) = \int_{-\infty}^{x} \Big(u_0(y) - v_0(y)\Big)\, dy \in W^{3,1}(\mathbb{R})$$

and

$$v_0(x) \in L^1(\mathbb{R}) \cap H^6(\mathbb{R}) \,.$$

Let $\alpha := \|v_0\|_{L^1} + \|v_0\|_{H^6}$, then there exists a positive constant δ_0 such that if

$$\|w_0\|_{W^{3,1}} + \alpha \leq \delta_0 \,,$$

then the Cauchy problem (E2) has a unique global solution $u(x,t)$ with $u(x,t) - v(x,t) \in C(\mathbb{R}_+, H^1(\mathbb{R}))$ and satisfies

(i) *if $p = 1$, for any $\eta > 0$ we have*

$$\|(u-v)(t)\|_{L^2} \leq c(1+t)^{-\frac{3}{4}+\eta} \,, \tag{3.5}$$

$$\|(u-v)_x(t)\|_{L^2} \leq c(1+t)^{-1+\eta} \,, \tag{3.6}$$

$$\|(u-v)(t)\|_{L^\infty} \leq c(1+t)^{-\frac{7}{8}+\eta} \,. \tag{3.7}$$

(ii) *If $p \geq 2$, then we have*

$$\|(u-v)(t)\|_{L^2} \leq c(1+t)^{-\frac{3}{4}} \,, \tag{3.8}$$

$$\|(u-v)_x(t)\|_{L^2} \leq c(1+t)^{-\frac{5}{4}} \,, \tag{3.9}$$

$$\|(u-v)(t)\|_{L^\infty} \leq c(1+t)^{-1} \,. \tag{3.10}$$

As a corollary, we obtain

Corollary 3.3. *Under the hypotheses of theorem 3.2, we have for* $2 \le q \le \infty$ *the decay rates*

$$\|(u-v)(t)\|_{L^q} \le \begin{cases} c(1+t)^{-\frac{7}{8}+\frac{1}{4q}+\eta} & \text{if } p=1, \\ c(1+t)^{-1+\frac{1}{2q}} & \text{if } p \ge 2\ . \end{cases} \tag{3.11}$$

The result in the corollary follows from the interpolation inequality

$$\|f\|_{L^q} \le \|f\|_{L^\infty}^{\frac{q-2}{q}} \|f\|_{L^2}^{\frac{2}{q}}, \quad 2 \le q \le \infty\ .$$

The rest of the paper is devoted to the proof of theorem 3.2.

From (E2) and (3.2), we have

$$(u-v)_t - \alpha(u-v)_{xx} - u_{xxxxt} + \left(\frac{u^{p+1}}{p+1} - \frac{v^{p+1}}{p+1}\right)_x = 0\ . \tag{3.12}$$

Since $v(\pm\infty,t) = 0$, and we expect $u(\pm\infty,t) = 0$, $u_x(\pm\infty,t) = 0$, then after integrating (3.12) over $\mathbb{R}$, we get

$$\frac{d}{dt}\int_{\mathbb{R}} \Big(u(x,t) - v(x,t)\Big)\, dx = 0\ . \tag{3.13}$$

Integration of (3.13) over $[0,t]$ and thanks to the assumptions we get

$$\int_{\mathbb{R}} \Big(u(x,t) - v(x,t)\Big)\, dx = \int_{\mathbb{R}} \Big(u_0(x) - v_0(x)\Big)\, dx = 0\ . \tag{3.14}$$

Thus we have

$$w_{xt} - \alpha\, w_{xxx} + w_{xxxxxt} - v_{xxxxt} + \left(\frac{(v+w_x)^{p+1}}{p+1} - \frac{v^{p+1}}{p+1}\right)_x = 0\ ,$$

where we set

$$w(x,t) = \int_{-\infty}^{x} \Big(u(y,t) - v(y,t)\Big)\, dy \tag{3.15}$$

that is $w_x(x,t) = u(x,t) - v(x,t)$.

The integration over $(-\infty, x]$ with respect to x of the above equation yields

$$\begin{cases} w_t - \alpha\, w_{xx} + w_{xxxxt} = H_p(w)\ , \\ w(0,x) = w_0(x)\ , \end{cases} \tag{3.16}$$

where

$$\begin{aligned} H_p(w) &:= v_{xxxt} - \frac{1}{p+1}\Big((v+w_x)^{p+1} - v^{p+1}\Big) \\ &= v_{xxxt} - \frac{1}{p+1}\sum_{j=0}^{p}\binom{j}{p+1} v^j\, w_x^{p+1-j}\,, \qquad p \ge 1\,. \end{aligned}$$

We have

$$|H_p(w)| \le |v_{xxxt}| + \frac{1}{p+1}\sum_{j=0}^{p}\binom{j}{p+1} |v^j\, w_x^{p+1-j}|\,,$$

$$\begin{aligned} |\partial_x H_p(w)| &\le |v_{xxxxt}| + |w_x^p\, w_{xx}| \\ &+ \frac{1}{p+1}\sum_{j=1}^{p}\binom{j}{p+1}\Big\{ j\,|v^{j-1}\, v_x\, w_x^{p+1-j}| + (p+1-j)\,|v\, j\, w_x^{p-j}\, w_{xx}|\Big\}\,. \end{aligned} \tag{3.17}$$

Taking the Fourier transform of (3.16), we get

$$\widehat{w}_t + \frac{\alpha\,\xi^2}{1+\xi^4}\,\widehat{w} = \frac{\widehat{H_p(w)}}{1+\xi^4}$$

which admits as solution

$$\widehat{w}(\xi,t) = e^{-a(\xi)t}\,\widehat{w}_0(\xi) + \int_0^t e^{-a(\xi)(t-s)}\,\frac{\widehat{H_p(w)}(\xi,s)}{1+\xi^4}\,ds\,,$$

where we set $a(\xi) := \frac{\alpha\,\xi^2}{1+\xi^4}$.

The inverse Fourier transform of the above resultant equation yields

$$\begin{aligned} w(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi x}\, e^{-a(\xi)t}\,\widehat{w}_0(\xi)\,d\xi \\ &+ \frac{1}{2\pi}\int_0^t\!\!\int_{-\infty}^{\infty} e^{i\xi x}\, e^{-a(\xi)(t-s)}\,\frac{\widehat{H_p(w)}(\xi,s)}{1+\xi^4}\,d\xi\,ds\,. \end{aligned} \tag{3.18}$$

The differentiation with respect to x of (3.18) gives

$$\begin{aligned} \partial_x^j w(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} (i\xi)^j\, e^{i\xi x}\, e^{-a(\xi)t}\,\widehat{w}_0(\xi)\,d\xi \\ &- \frac{1}{2\pi}\int_0^t\!\!\int_{-\infty}^{\infty} (i\xi)^j\, e^{i\xi x}\, e^{-a(\xi)(t-s)}\,\frac{\widehat{H_p(w)}(\xi,s)}{1+\xi^4}\,d\xi\,ds\,. \end{aligned} \tag{3.19}$$

Now, we define the solution spaces as follows, for any positive integer $p \ge 1$ and given $\delta > 0$:

$$\mathcal{S}_p^\delta := \Big\{ w \in C(\mathbb{R}_+, H^2(\mathbb{R}))\ \big|\ A_p(w) \le \delta \Big\}\,,$$

where

$$A_1(w) \;=\; \sup_{0\le t\le\infty}\left\{\sum_{j=0}^{1}(1+t)^{\frac{2j+1}{4}-\eta}\,\|\partial_x^j w(t)\|_{L^2} \;+\; (1+t)^{1-\eta}\,\|w_{xx}(t)\|_{L^2}\right\},$$

$$A_p(w) \;=\; \sup_{0\le t\le\infty}\sum_{j=0}^{2}(1+t)^{\frac{2j+1}{4}}\,\|\partial_x^j w(t)\|_{L^2}\,, \qquad p\ge 2\;.$$

Rewriting (3.18) as the operational form $w = Sw$, we need to prove that S is a contraction mapping from $\mathcal{S}_p^\delta$ into $\mathcal{S}_p^\delta$ where $\delta>0$ is a positive constant.

We have

Theorem 3.4. *Under the hypotheses of theorem 3.2, there exists a positive constant δ_1 such that if*

$$\|w_0\|_{W^{3,1}} + \alpha \;<\; \delta_1$$

then Cauchy problem (3.16) has a unique global solution $w(x,t)\in C(\mathbb{R}_+, H^2(\mathbb{R}))$. Furthermore, we have the following estimates:

(i) *If $p=1$, then for any $\eta>0$ we have*

$$\sum_{i=0}^{1}(1+t)^{\frac{2j+1}{4}-\eta}\,\|\partial_x^j w(t)\|_{L^2} + (1+t)^{1-\eta}\,\|w_{xx}(t)\|_{L^2} \;\le\; c\Big(\|w_0\|_{W^{3,1}}+\alpha\Big)\;. \tag{3.20}$$

(ii) *If $p\ge 2$, then we have*

$$\sum_{j=0}^{2}(1+t)^{\frac{2j+1}{4}}\,\|\partial_x^j w(t)\|_{L^2} \;\le\; c\Big(\|w_0\|_{W^{3,1}} + \alpha\Big)\;. \tag{3.21}$$

Since $u(x,t)-v(x,t) = w_x(x,t)$, once we prove theorem 3.1, then theorem 3.2 can be easily proved. Hence, we prove theorem 3.1 in the rest of the paper. We need to this end two lemmas and two well-known estimates quoted from [7]:

$$\int_{-\infty}^{\infty}\frac{|\xi|^j\,e^{-ca(\xi)t}}{(1+\xi^4)\,(1+|\xi|)^j}\,d\xi \;\le\; c(1+t)^{-\frac{j+1}{2}}\,, \qquad j=0,1,2,3,4\;, \tag{3.22}$$

$$\left\|\frac{1}{2\pi}\int_{-\infty}^{\infty}(i\xi)^j\,e^{i\xi x}\,e^{-a(\xi)t}\,\widehat{w}_0(\xi)\,d\xi\right\|_{L^2} \;\le\; c\,\|w_0\|_{W^{j+1,1}}\,(1+t)^{-\frac{2j+1}{4}} \tag{3.23}$$

for $j=0,1,2$.

Lemma 3.5. *Let $w_1(x,t), w_2(x,t) \in \mathcal{S}_p^\delta$, then we have*

$$\sup_{\xi\in\mathbb{R}}\Big|\widehat{H_1(w_1)}(\xi,s) - \widehat{H_1(w_2)}(\xi,s)\Big| \;\le\; c(\alpha+\delta)\, A_1(w_1-w_2)\,(1+s)^{-1+\eta}\;,$$

$$\sup_{\xi\in\mathbb{R}}\Big|\widehat{H_p(w_1)}(\xi,s) - \widehat{H_p(w_2)}(\xi,s)\Big| \;\le\; c(\alpha+\delta)^p A_p(w_1-w_2)\,(1+s)^{-\frac{3}{2}}\,, \qquad p\ge 2\;,$$

$$\sup_{\xi\in\mathbb{R}}|\xi|\,\Big|\widehat{H_1(w_1)}(\xi,s) - \widehat{H_1(w_2)}(\xi,s)\Big| \;\le\; c(\alpha+\delta)\, A_1(w_1-w_2)\,(1+s)^{-\frac{5}{4}+\eta}\;,$$

$$\sup_{\xi\in\mathbb{R}}|\xi|\,\Big|\widehat{H_p(w_1)}(\xi,s) - \widehat{H_p(w_2)}(\xi,s)\Big| \;\le\; c(\alpha+\delta)^p A_p(w_1-w_2)\,(1+s)^{-2}\,, \quad p\ge 2\;.$$

Lemma 3.6. *Let $w(x,t) \in \mathcal{S}_p^\delta$, then*

(i) *if $p=1$*

$$\int_0^t \left\| \frac{1}{2\pi}\int_{-\infty}^{\infty} (i\xi)^j\, e^{i\xi x}\, e^{-a(\xi)(t-s)}\,\frac{\widehat{H_1(w)}(\xi,s)}{1+\xi^4}\,d\xi\right\|_{L^2} ds \;\le$$
$$\le\; c\Big(\alpha+(\alpha+\delta)^2\Big)\,(1+t)^{-\frac{2j+1}{4}+\eta}\,, \qquad j=0,1\;,$$

$$\int_0^t \left\| \frac{1}{2\pi}\int_{-\infty}^{\infty} (i\xi)^2\, e^{i\xi x}\, e^{-a(\xi)(t-s)}\,\frac{\widehat{H_1(w)}(\xi,s)}{1+\xi^4}\,d\xi\right\|_{L^2} ds \;\le$$
$$\le\; c\Big(\alpha+(\alpha+\delta)^2\Big)\,(1+t)^{-1+\eta}\;;$$

(ii) *if $p\ge 2$, then we have*

$$\int_0^t \left\| \frac{1}{2\pi}\int_{-\infty}^{\infty} (i\xi)^j\, e^{i\xi x}\, e^{-a(\xi)(t-s)}\,\frac{\widehat{H_p(w)}(\xi,s)}{1+\xi^4}\,d\xi\right\|_{L^2} ds \;\le$$
$$\le\; c\Big(\alpha+(\alpha+\delta)^{p+1}\Big)\,(1+t)^{-\frac{2j+1}{4}}\,, \qquad j=0,1,2\;.$$

Proof of Theorem 3.1: Rewriting (3.18) in the form $w = Sw$, we need to prove that there exists a positive constant δ_1 such that the operator S is a contraction mapping from $\mathcal{S}_p^{\delta_1}$ into $\mathcal{S}_p^{\delta_1}$.

We claim that S maps $\mathcal{S}_p^\delta$ into itself. Indeed, for any $w_1(x,t) \in \mathcal{S}_p^\delta$ and denoting $w = Sw_1$ we will prove that $w \in \mathcal{S}_p^\delta$ for some small $\delta>0$. Thanks to lemma 3.6, for any positive integer p there exists a constant c_1 such that

$$A_p(w) \;\le\; c_1\Big(\|w_0\|_{W^{3,1}} + \alpha + (\alpha+\delta)^{p+1}\Big)\;.$$

Let $n = \max\left\{2+2^{p+1}, \frac{1}{c_1}\right\}$, and choose $\delta_2 \le \frac{1}{n\,c_1}$, then for $\|v_0\|_{W^{3,1}} \le \frac{\delta_2}{nc_1}$, $\alpha \le \frac{\delta_2}{nc_1}$ and $\delta < \delta_2$ we have

$$\begin{aligned} A_p(w) &\le c_1 \left(\frac{\delta_2}{n\,c_1} + \frac{\delta_2}{n\,c_1} + \left(\frac{\delta_2}{n\,c_1} + \delta_2\right)^{p+1} \right) \\ &\le c_1 \frac{(2+2^{p+1})\,\delta_2}{n\,c_1} \le \delta_2 \,. \end{aligned}$$

Hence $S\colon \mathcal{S}_p^\delta \to \mathcal{S}_p^\delta$ for some small $\delta < \delta_2$.

Now, let us prove that S is a contraction in $\mathcal{S}_p^\delta$. Suppose that $w_1(x,t), w_2(x,t) \in \mathcal{S}_p^\delta$ $(\delta < \delta_2)$, then we have by (3.18): for $j = 0,1,2$ and $p \ge 1$

$$\partial_x^j(Sw_1 - Sw_2) = \frac{1}{2\pi} \int_0^t \int_{-\infty}^{\infty} (i\xi)^j e^{i\xi x} e^{-a(\xi)(t-s)} \frac{\widehat{H_p(w_1)}(\xi,s) - \widehat{H_p(w_2)}(\xi,s)}{1+\xi^4} \, d\xi \, ds \,.$$

We estimate the term $\|Sw_1 - Sw_2\|_{L^2}$: we have

$$\begin{aligned} &\int_0^t \left\| \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi x} \frac{e^{-a(\xi)(t-s)}}{1+\xi^4} \Big(\widehat{H_p(w_1)}(\xi,s) - \widehat{H_p(w_2)}(\xi,s) \Big) \, d\xi \right\|_{L^2} ds = \\ &\qquad = \int_0^t \left(\int_{-\infty}^{\infty} \frac{e^{-a(\xi)(t-s)}}{(1+\xi^4)^2} \left| \widehat{H_p(w_1)}(\xi,s) - \widehat{H_p(w_2)}(\xi,s) \right|^2 d\xi \right)^{\frac12} ds \\ &\qquad \le \int_0^t \sup_{\xi\in\mathbb{R}} \left| \widehat{H_p(w_1)}(\xi,s) - \widehat{H_p(w_2)}(\xi,s) \right| \left(\int_{-\infty}^{\infty} \frac{e^{-a(\xi)(t-s)}}{(1+\xi^4)^2} \, d\xi \right)^{\frac12} ds \,. \end{aligned}$$

Using (3.22), lemma 1.1 and lemma 3.5, we obtain

$$\begin{aligned} &\int_0^t \left\| \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi x} \frac{e^{-a(\xi)(t-s)}}{1+\xi^4} \left(\widehat{H_1(w_1)}(\xi,s) - \widehat{H_1(w_2)}(\xi,s) \right) d\xi \right\|_{L^2} ds \le \\ &\qquad \le c(\alpha+\delta)\, A_1(w_1 - w_2) \int_0^t (1+s)^{-(1-\eta)} \, (1+t-s)^{-\frac14} \, ds \\ &\qquad \le c(\alpha+\delta)\, A_1(w_1 - w_2)\, (1+t)^{-(1/4-\eta)} \,, \qquad 0 < \eta \le 1/2 \,, \end{aligned}$$

$$\begin{aligned} &\int_0^t \left\| \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi x} \frac{e^{-a(\xi)(t-s)}}{1+\xi^4} \left(\widehat{H_p(w_1)}(\xi,s) - \widehat{H_p(w_2)}(\xi,s) \right) d\xi \right\|_{L^2} ds \le \\ &\qquad \le c(\alpha+\delta)^p A_p(w_1 - w_2) \int_0^t (1+s)^{-\frac32} \, (1+t-s)^{-\frac14} \, ds \\ &\qquad \le c(\alpha+\delta)^p A_p(w_1 - w_2)\, (1+t)^{-\frac14} \,, \qquad p \ge 2 \,. \end{aligned}$$

That is, we obtain

$$\|Sw_1 - Sw_2\|_{L^2} \le$$

$$\le \begin{cases} c(\alpha+\delta)\, A_1(w_1-w_2)\,(1+t)^{-(1/4-\eta)}\,, & 0<\eta<1/2, \quad \text{for } p=1\,,\\ c(\alpha+\delta)^p A_p(w_1-w_2)\,(1+t)^{-\frac{1}{4}}\,, & \text{for } \ p\ge 2\,. \end{cases}$$

Similarly, by using (3.22), lemma 1.1 and lemma 3.5, we have the estimates for $\|\partial_x(Sw_1-Sw_2)\|_{L^2}$ and $\|\partial_x^2(Sw_1-Sw_2)\|_{L^2}$ as follows:

$$\|\partial_x(Sw_1 - Sw_2)\|_{L^2} \le$$

$$\le \begin{cases} c(\alpha+\delta)\, A_1(w_1-w_2)\,(1+t)^{-(3/4-\eta)}\,, & 0<\eta<1/2, \quad \text{for } p=1\,,\\ c(\alpha+\delta)^p A_p(w_1-w_2)\,(1+t)^{-\frac{3}{4}}\,, & \text{for } \ p\ge 2\,, \end{cases}$$

and

$$\|\partial_x^2(Sw_1 - Sw_2)\|_{L^2} \le$$

$$\le \begin{cases} c(\alpha+\delta)\, A_1(w_1-w_2)\,(1+t)^{-(1-\eta)}\,, & 0<\eta<1/2, \quad \text{for } p=1\,,\\ c(\alpha+\delta)^p A_p(w_1-w_2)\,(1+t)^{-\frac{5}{4}}\,, & \text{for } \ p\ge 2\,. \end{cases}$$

Hence, we deduce that, for some constant c_1:

$$A_p(Sw_1-Sw_2) \;\le\; c_1(\alpha+\delta)^p A_p(w_1-w_2)\;.$$

Let $n=\max\left\{\frac{2}{c_1},2\right\}$ and choose $\delta\le\delta_3<\frac{1}{nc_1}$, then for $\alpha<\delta_3$ and $A_p(w_2)<\delta_3$ we deduce that

$$A_p(Sw_1-Sw_2) < A_p(w_1-w_2)\;,$$

that is $S\colon \mathcal{S}_p^\delta \to \mathcal{S}_p^\delta$ is a contraction for small $\delta<\delta_3$.

Finally, let $\delta_1<\min\{\delta_2,\delta_3\}$, then we have proved that S is a contraction from $\mathcal{S}_p^{\delta_1}$ to $\mathcal{S}_p^{\delta_1}$, and consequently by Banach's fixed point theorem, S has a unique fixed point in $\mathcal{S}_p^{\delta_1}$, and then we have the existence of a unique global solution. ∎

Proof of Lemma 3.5: For $p=1$, we have

$$\sup_{\xi\in\mathbb{R}}\left|\widehat{H_1(w_1)}(\xi,s)-\widehat{H_1(w_2)}(\xi,s)\right| \;\le$$

$$\begin{aligned}
&\le \|v(s)\|_{L^2}\,\|(w_{1x}-w_{2x})(s)\|_{L^2}\\
&\quad+\frac{1}{2}\|(w_{1x}+w_{2x})(s)\|_{L^2}\,\|(w_{1x}-w_{2x})(s)\|_{L^2}\\
&\le c\Big(\alpha(1+s)^{-\frac14}+\delta(1+s)^{-\frac34+\eta}\Big)\,A_1(w_1-w_2)\,(1+s)^{-\frac34+\eta}\\
&\le c(\alpha+\delta)\,A_1(w_1-w_2)\,(1+s)^{-1+\eta}\;,
\end{aligned} \tag{3.24}$$

and

$$
\begin{aligned}
\sup_{\xi\in\mathbb{R}} |\xi| \left|\widehat{H_1(w_1)}(\xi,s) - \widehat{H_1(w_2)}(\xi,s)\right| \ \le & \\
&\le \int_{-\infty}^{\infty} \left|H_{1x}(w_1)(x,s) - H_{1x}(w_2)(x,s)\right| dx \\
&\le \ \|v_x(s)\|_{L^2} \, \|(w_{1x} - w_{2x})(s)\|_{L^2} \ + \ \|v(s)\|_{L^2} \, \|(w_{1xx} - w_{2xx})(s)\|_{L^2} \\
&\quad + \|w_{1x}(s)\|_{L^2} \, \|(w_{1xx} - w_{2xx})(s)\|_{L^2} \ + \ \|w_{2xx}(s)\|_{L^2} \, \|(w_{1x} - w_{2x})(s)\|_{L^2} \\
&\le \ c\Big\{ \Big(\alpha(1+s)^{-\frac{3}{4}} + \delta(1+s)^{-1+\eta}\Big) \, A_1(w_1 - w_2) \, (1+s)^{-\frac{3}{4}+\eta} \\
&\quad + \Big(\alpha(1+s)^{-\frac{1}{4}} + \delta(1+s)^{-\frac{3}{4}+\eta}\Big) \, A_1(w_1 - w_2) \, (1+s)^{-1+\eta} \Big\} \\
&\le \ c(\alpha+\delta) \, A_1(w_1 - w_2) \, (1+s)^{-\frac{5}{4}+\eta} \ .
\end{aligned}
\tag{3.25}
$$

For $p \ge 2$, we have

$$
\begin{aligned}
\sup_{\xi\in\mathbb{R}} \left|\widehat{H_p(w_1)}(\xi,s) - \widehat{H_p(w_2)}(\xi,s)\right| \ \le & \\
&\le \int_{-\infty}^{\infty} \left|H_p(w_1)(x,s) - H_p(w_2)(x,s)\right| dx \\
&\le \int_{-\infty}^{\infty} \frac{1}{p+1} \sum_{j=0}^{p} \binom{j}{p+1} \left|v^j w_{1x}^{p+1-j} - v^j w_{2x}^{p+1-j}\right| dx \\
&\le \ \frac{1}{p+1} \Big\{ \binom{p}{p+1} \|v(s)\|_{L^\infty}^{p-1} \, \|v(s)\|_{L^2} \, \|(w_{1x} - w_{2x})(s)\|_{L^2} \\
&\quad + \sum_{j=0}^{p-1} \binom{j}{p+1} \|v(s)\|_{L^\infty}^{j} \, \|(w_{1x} - w_{2x})(s)\|_{L^2} \sum_{i=0}^{p-j} \|w_{1x}(s)\|_{L^2}^{i} \, \|w_{2x}(s)\|_{L^2}^{p-j-i} \Big\} \\
&\le \ \frac{c}{p+1} \Big\{ \binom{p}{p+1} \alpha^p + \sum_{j=0}^{p-1} \binom{j}{p+1} (p-j) \, \alpha^j \, \delta^{p-j} \Big\} A_p(w_1 - w_2) \, (1+s)^{-\frac{p+1}{2}} \\
&\le \ c(\alpha+\delta)^p A_p(w_1 - w_2) \, (1+s)^{-\frac{3}{2}}
\end{aligned}
\tag{3.26}
$$

and

$$
\begin{aligned}
\sup_{\xi\in\mathbb{R}} |\xi| \Big|\widehat{H_p(w_1)}(\xi,s) - \widehat{H_p(w_2)}(\xi,s)\Big| \;\leq\\
&\leq \int_{-\infty}^{\infty} \Big|\partial_x H_p(w_1)(x,s) - \partial_x H_p(w_2)(x,s)\Big|\, dx \\
&\leq \Big\|(w_{1xx}w_{1x}^p - w_{2xx}w_{2x}^p)(s)\Big\|_{L^1} \\
&\quad + \frac{1}{p+1}\binom{p}{p+1}\Big\{p\,\Big\|v^{p-1}(s)\, v_x(s)\,(w_{1x}-w_{2x})(s)\Big\|_{L^1} \\
&\quad + \Big\|v^p(s)(w_{1xx}-w_{2xx})(s)\Big\|_{L^1}\Big\} \\
&\quad + \frac{1}{p+1}\sum_{j=1}^{p-1}\binom{j}{p+1}\Big\{j\Big\|v^{j-1}(s)\, v_x(s)\,(w_{1x}^{p+1-j}-w_{2x}^{p+1-j})(s)\Big\|_{L^1} \\
&\quad + (p+1-j)\,\Big\|v^j(s)\,(w_{1x}^{p-j}w_{1xx} - w_{2x}^{p-j}w_{2xx})(s)\Big\|_{L^1}\Big\}\,.
\end{aligned}
$$

Since

$$
\begin{aligned}
\Big\|(w_{1xx}w_{1x}^p - w_{2xx}w_{2x}^p)(s)\Big\|_{L^1} \;\leq\\
&\leq \|w_{1xx}(s)\|_{L^2}\,\|(w_{1x}-w_{2x}(s)\|_{L^2}\sum_{i=0}^{p-1}\|w_{1x}(s)\|_{L^2}^i\,\|w_{2x}(s)\|_{L^2}^{p-1-i} \\
&\quad + \|w_{2x}(s)\|_{L^2}^p\,\|(w_{1xx}-w_{2xx})(s)\|_{L^2} \\
&\leq c\,\delta^p A_p(w_1-w_2)\,(p+1)\,(1+s)^{-2}
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{1}{p+1}\binom{p}{p+1}\Big\{p\,\Big\|v^{p-1}(s)\, v_x(s)\,(w_{1x}-w_{2x})(s)\Big\|_{L^1} + \Big\|v^p(s)\,(w_{1xx}-w_{2xx})(s)\Big\|_{L^1}\Big\} \;\leq\\
&\leq c\Big\{p\,\alpha^p A_p(w_1-w_2)\,(1+s)^{-\frac{p+2}{2}} + \alpha^p A_p(w_1-w_2)\,(1+s)^{-\frac{p+2}{2}}\Big\} \\
&\leq c\,\alpha^p(p+1)\,A_p(w_1-w_2)\,(1+s)^{-2}
\end{aligned}
$$

and

$$\frac{1}{p+1}\sum_{j=1}^{p-1}\binom{j}{p+1} j\left\| v^{j-1}(s)\, v_x(s)\, (w_{1x}^{p+1-j} - w_{2x}^{p+1-j})(s)\right\|_{L^1} \le$$

$$\le \frac{c}{p+1}\sum_{j=1}^{p-1}\binom{j}{p+1} j\,\alpha^j (1+s)^{-\frac{j-1}{2}-1} A_p(w_1-w_2)$$

$$\times\ (1+s)^{-\frac{3}{4}}\sum_{i=0}^{p-j}\delta^{p-j}(1+s)^{-\frac{3(p-j)}{4}}$$

$$\le\ c\,A_p(w_1-w_2)\,(1+s)^{-2}\frac{1}{p+1}\sum_{j=1}^{p-1}\binom{j}{p+1} j(p-j+1)\,\alpha^j\,\delta^{p-j}$$

and

$$\frac{1}{p+1}\sum_{j=1}^{p-1}\binom{j}{p+1}(p+1-j)\left\| v^j(s)(w_{1x}^{p-j}w_{1xx} - w_{2x}^{p-j}w_{2xx})(s)\right\|_{L^1} \le$$

$$\le\ \frac{c}{p+1}\sum_{j=1}^{p-1}\binom{j}{p+1}(p+1-j)\,\alpha^j\,(1+s)^{-\frac{j}{2}}$$

$$\times\ \Big\{\delta^{p-j}A_p(w_1-w_2)\,(1+s)^{-\frac{3p+5-3j}{4}}$$

$$+\ \delta^{p-j}A_p(w_1-w_2)\,(1+s)^{-2}\,(1+s)^{-\frac{3(p-j-1)}{4}}(p-j)\Big\}$$

$$\le\ c\,A_p(w_1-w_2)(1+s)^{-2}\frac{1}{p+1}\sum_{j=1}^{p-1}\binom{j}{p+1}(p+1-j)^2\,\alpha^j\,\delta^{p-j}\ ,$$

we obtain that

$$\sup_{\xi\in\mathbb{R}}|\xi|\left|\widehat{H_p(w_1)}(\xi,s) - \widehat{H_p(w_2)}(\xi,s)\right|\ \le$$

$$\le\ c\,A_p(w_1-w_2)\,(1+s)^{-2}\Big\{\delta^p(p+1) + \alpha^p(p+1)$$

$$+\ \frac{1}{p+1}\sum_{j=1}^{p-1}\bigg(\binom{j}{p+1} j(p+1-j)\,\alpha^j\,\delta^{p-j} + \binom{j}{p+1}(p+1-j)^2\,\alpha^j\,\delta^{p-j}\bigg)\Big\} \tag{3.27}$$

$$\le\ c(\alpha+\delta)^p A_p(w_1-w_2)\,(1+s)^{-2}\ .$$

Thanks to (3.24)–(3.27), we deduce the result of lemma 3.5. ∎

Proof of Lemma 3.6: Since for $f \in H^1$ we have

$$\|f\|_{L^\infty} \le \sqrt{2}\,\|f\|_{L^2}^{\frac{1}{2}} \cdot \|f_x\|_{L^2}^{\frac{1}{2}}$$

we easily deduce that

(**i**) for $p=1$,

$$\begin{aligned}\|w(t)\|_{L^\infty} &\le \sqrt{2}\,\|w(t)\|_{L^2}^{\frac{1}{2}}\,\|w_x(t)\|_{L^2}^{\frac{1}{2}} \le \sqrt{2}\,\delta(1+t)^{-\frac{1}{2}+\eta}\ ,\\ \|w_x(t)\|_{L^\infty} &\le \sqrt{2}\,\|w_x(t)\|_{L^2}^{\frac{1}{2}}\,\|w_{xx}(t)\|_{L^2}^{\frac{1}{2}} \le \sqrt{2}\,\delta(1+t)^{-\frac{7}{8}+\eta}\ ;\end{aligned} \tag{3.28}$$

(**ii**) for $p \ge 2$:

$$\begin{aligned}\|w(t)\|_{L^\infty} &\le \sqrt{2}\,\|w(t)\|_{L^2}^{\frac{1}{2}}\,\|w_x(t)\|_{L^2}^{\frac{1}{2}} \le \sqrt{2}\,\delta(1+t)^{-\frac{1}{2}}\ ,\\ \|w_x(t)\|_{L^\infty} &\le \sqrt{2}\,\|w_x(t)\|_{L^2}^{\frac{1}{2}}\,\|w_{xx}(t)\|_{L^2}^{\frac{1}{2}} \le \sqrt{2}\,\delta(1+t)^{-1}\ .\end{aligned} \tag{3.29}$$

Thanks to (3.17) we have

$$\begin{aligned}\sup_{\xi\in\mathbb{R}} |\widehat{H_p(w)}(\xi,s)| &\le \int_{-\infty}^{\infty} |H_p(w)(x,s)|\,dx\\ &\le \int_{-\infty}^{\infty}\left\{|v_{xxxt}| + \frac{1}{p+1}\sum_{j=0}^{p}\binom{j}{p+1}|v^j w_x^{p+1-j}|\right\}dx\\ &\le \|v_{xxxt}(s)\|_{L^1} + \frac{1}{p+1}\binom{p}{p+1}\|v(s)\|_{L^\infty}^{p-1}\,\|v(s)\|_{L^2}\,\|w_x(s)\|_{L^2}\\ &\quad + \frac{1}{p+1}\sum_{j=1}^{p-1}\binom{j}{p+1}\|v(s)\|_{L^\infty}^{j}\,\|w_x(s)\|_{L^\infty}^{p-1-j}\,\|w_x(s)\|_{L^2}^2\ .\end{aligned}$$

Now, because of (3.28)–(3.29) we have

$$\begin{aligned}\sup_{\xi\in\mathbb{R}} |\widehat{H_1(w)}(\xi,s)| &\le c\Big\{\alpha(1+s)^{-\frac{5}{2}} + \delta^2(1+s)^{-(3/2-2\eta)} + \alpha\,\delta(1+s)^{-(1-\eta)}\Big\}\\ &\le c\Big\{\alpha + (\alpha+\delta)^2\Big\}\,(1+s)^{-(1-\eta)}\ ,\end{aligned} \tag{3.30}$$

by choosing η such that $0<\eta\le 1/2$, and

$$\begin{aligned}\sup_{\xi\in\mathbb{R}} |\widehat{H_p(w)}(\xi,s)| &\le c\Big\{\alpha(1+s)^{-\frac{5}{2}} + \binom{p}{p+1}\alpha^p\,\delta(1+s)^{-\frac{p+1}{2}}\\ &\quad + \sum_{j=0}^{p-1}\binom{j}{p+1}\alpha^j\,\delta^{p+1-j}(1+s)^{-\frac{2p-j+1}{2}}\Big\}\\ &\le c\Big\{\alpha + (\alpha+\delta)^{p+1}\Big\}\,(1+s)^{-\frac{3}{2}}\,, \qquad p\ge 2\ ,\end{aligned} \tag{3.31}$$

where for $p \geq 2$, we used $(1+s)^{-\frac{2p-j+1}{2}} < (1+s)^{-\frac{p+1}{2}} \leq (1+s)^{-\frac{3}{2}}$. Similarly, we have

$$\begin{aligned}
\sup_{\xi\in\mathbb{R}} |\xi|\,|\widehat{H_p(w)}(\xi,s)| \;\leq\; & \\
\leq\; & \|v_{xxxxt}(s)\|_{L^1} + \|w_x(s)\|_{L^2}^{p-1}\,\|w_x(s)\|_{L^2}\,\|w_{xx}(s)\|_{L^2} \\
&+ \frac{1}{p+1}\sum_{j=1}^{p}\binom{j}{p+1}\Big\{ j\,\|v(s)\|_{L^\infty}^{j-1}\,\|w_x(s)\|_{L^\infty}^{p-j}\,\|v_x(s)\|_{L^2}\,\|w_x(s)\|_{L^2} \\
&+ (p+1-j)\,\|v(s)\|_{L^\infty}^{j-1}\,\|w_x(s)\|_{L^\infty}^{p-j}\,\|v(s)\|_{L^2}\,\|w_{xx}(s)\|_{L^2}\Big\}\,.
\end{aligned}$$

Now, since for $p \geq 2$ we have $(1+s)^{-(p+1)} < (1+s)^{-2}$ and $(1+s)^{-\frac{2p-j+2}{2}} \leq (1+s)^{-\frac{p+2}{2}} \leq (1+s)^{-2}$, we have

$$\begin{aligned}
\sup_{\xi\in\mathbb{R}} |\xi|\,|\widehat{H_1(w)}(\xi,s)| \;&\leq\; c\Big\{\alpha(1+s)^{-3} + \delta^2(1+s)^{-2(1-\eta)} + \alpha\,\delta(1+s)^{-(3/2-\eta)}\Big\} \\
&\leq\; c\Big\{\alpha + (\alpha+\delta)^2\Big\}\,(1+s)^{-(3/2-\eta)}\,, \qquad 0<\eta\leq\frac{1}{2}\,,
\end{aligned} \tag{3.32}$$

$$\begin{aligned}
\sup_{\xi\in\mathbb{R}} |\xi|\,|\widehat{H_p(w)}(\xi,s)| \;&\leq\; c\Big\{\alpha(1+s)^{-3} + \delta^{p+1}(1+s)^{-(p+1)} \\
&\qquad + \sum_{j=1}^{p}\binom{j}{p+1}\,\alpha^j\,\delta^{p-j+1}(1+s)^{-\frac{p+2}{2}}\Big\} \\
&\leq\; c\Big\{\alpha + (\alpha+\delta)^{p+1}\Big\}\,(1+s)^{-2}\,, \qquad p\geq 2\,.
\end{aligned} \tag{3.33}$$

By the Parseval equality we have

$$\begin{aligned}
\int_0^t \left\| \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi x}\,\frac{e^{-a(\xi)(t-s)}}{1+\xi^4}\,\widehat{H_p(w)}(\xi,s)\,d\xi \right\|_{L^2} ds \;= & \\
= \int_0^t \left\| \frac{e^{-a(\xi)(t-s)}}{1+\xi^4}\,\widehat{H_p(w)}(\xi,s)\right\|_{L^2} ds & \\
\leq \int_0^t \sup_{\xi\in\mathbb{R}} |\widehat{H_p(w)}(\xi,s)|\left(\int_{-\infty}^{\infty}\frac{e^{-2a(\xi)(t-s)}}{(1+\xi^4)^2}\,d\xi\right)^{\frac{1}{2}} ds\,. &
\end{aligned}$$

Thanks to (3.30)–(3.31), lemma 1.1 and (3.22) we obtain

$$\begin{aligned}
&\int_0^t \left\| \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi x}\, \frac{e^{-a(\xi)(t-s)}}{1+\xi^4}\, \widehat{H_1(w)}(\xi,s)\, d\xi \right\|_{L^2} ds \;\le \\
&\qquad \le\; c\Big\{\alpha + (\alpha{+}\delta)^2\Big\} \int_0^t (1{+}s)^{-(1-\eta)}\, (1{+}t{-}s)^{-\frac{1}{4}}\, ds \\
&\qquad \le\; c\Big\{\alpha + (\alpha{+}\delta)^2\Big\}\, (1{+}t)^{-(1/4-\eta)}\,, \qquad 0<\eta\le\frac{1}{2}\,,
\end{aligned} \tag{3.34}$$

$$\begin{aligned}
&\int_0^t \left\| \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi x}\, \frac{e^{-a(\xi)(t-s)}}{1+\xi^4}\, \widehat{H_p(w)}(\xi,s)\, d\xi \right\|_{L^2} ds \;\le \\
&\qquad \le\; c\Big\{\alpha + (\alpha{+}\delta)^{p+1}\Big\} \int_0^t (1{+}s)^{-\frac{3}{2}}\, (1{+}t{-}s)^{-\frac{1}{4}}\, ds \\
&\qquad \le\; c\Big\{\alpha + (\alpha{+}\delta)^{p+1}\Big\}\, (1{+}t)^{-\frac{1}{4}}\,, \qquad p\ge 2\,.
\end{aligned} \tag{3.35}$$

Similarly, we have

$$\begin{aligned}
&\int_0^t \left\| \frac{1}{2\pi} \int_{-\infty}^{\infty} i\xi\, e^{i\xi x}\, \frac{e^{-a(\xi)(t-s)}}{1+\xi^4}\, \widehat{H_p(w)}(\xi,s)\, d\xi \right\|_{L^2} ds \;\le \\
&\qquad \le \int_0^t \sup_{\xi\in\mathbb{R}} |\widehat{H_p(w)}(\xi,s)| \left(\int_{-\infty}^{\infty} \frac{|\xi|^2\, e^{-2a(\xi)(t-s)}}{(1+\xi^4)^2}\, d\xi \right)^{\frac{1}{2}} ds\,.
\end{aligned}$$

Using (3.30)–(3.31), lemma 1.1 and (3.22), we obtain

$$\begin{aligned}
&\int_0^t \left\| \frac{1}{2\pi} \int_{-\infty}^{\infty} i\xi\, e^{i\xi x}\, \frac{e^{-a(\xi)(t-s)}}{1+\xi^4}\, \widehat{H_1(w)}(\xi,s)\, d\xi \right\|_{L^2} ds \;\le \\
&\qquad \le\; c\Big\{\alpha + (\alpha{+}\delta)^2\Big\} \int_0^t (1{+}s)^{-(1-\eta)}\, (1{+}t{-}s)^{-\frac{3}{4}}\, ds \\
&\qquad \le\; c\Big\{\alpha + (\alpha{+}\delta)^2\Big\}\, (1{+}t)^{-(3/4-\eta)}\,, \qquad 0<\eta\le\frac{1}{2}\,,
\end{aligned} \tag{3.36}$$

$$\begin{aligned}
&\int_0^t \left\| \frac{1}{2\pi} \int_{-\infty}^{\infty} i\xi\, e^{i\xi x}\, \frac{e^{-a(\xi)(t-s)}}{1+\xi^4}\, \widehat{H_p(w)}(\xi,s)\, d\xi \right\|_{L^2} ds \;\le \\
&\qquad \le\; c\Big\{\alpha + (\alpha{+}\delta)^{p+1}\Big\}\, (1{+}t)^{-\frac{3}{4}}\,, \qquad p\ge 2\,.
\end{aligned} \tag{3.37}$$

Furthermore, we have

$$\int_0^t \left\| \frac{1}{2\pi} \int_{-\infty}^{\infty} (i\xi)^2\, e^{i\xi x}\, \frac{e^{-a(\xi)(t-s)}}{1+\xi^4}\, \widehat{H_p(w)}(\xi,s)\, d\xi \right\|_{L^2} ds \;=$$

$$= \int_0^t \left\| (i\xi)^2\, \frac{e^{-a(\xi)(t-s)}}{1+\xi^4}\, \widehat{H_p(w)}(\xi,s) \right\|_{L^2} ds$$

$$\leq \int_0^t \sup_{\xi\in\mathbb{R}} \Big((1+|\xi|) \left| \widehat{H_p(w)}(\xi,s) \right| \Big) \left(\int_{-\infty}^{\infty} \frac{|\xi|^4\, e^{-2a(\xi)(t-s)}}{(1+\xi^4)\,(1+|\xi|)^4}\, d\xi \right)^{\frac{1}{2}} ds\;.$$

Using (3.30)–(3.33), lemma 1.1 and (3.22), we have

$$\int_0^t \left\| \frac{1}{2\pi} \int_{-\infty}^{\infty} (i\xi)^2\, e^{i\xi x}\, \frac{e^{-a(\xi)(t-s)}}{1+\xi^4}\, \widehat{H_1(w)}(\xi,s)\, d\xi \right\|_{L^2} ds \;\leq$$

$$\begin{aligned} &\leq\; c\Big\{\alpha + (\alpha{+}\delta)^2\Big\} \int_0^t \Big\{ (1{+}s)^{-(1-\eta)} + (1{+}s)^{-(3/2-\eta)} \Big\}\, (1{+}t{-}s)^{-\frac{5}{4}}\, ds \\ &\leq\; c\Big\{\alpha + (\alpha{+}\delta)^2\Big\}\, (1{+}t)^{-(1-\eta)}\,, \qquad 0<\eta\leq \frac{1}{2}\,, \end{aligned} \tag{3.38}$$

$$\int_0^t \left\| \frac{1}{2\pi} \int_{-\infty}^{\infty} (i\xi)^2\, e^{i\xi x}\, \frac{e^{-a(\xi)(t-s)}}{1+\xi^4}\, \widehat{H_p(w)}(\xi,s)\, d\xi \right\|_{L^2} ds \;\leq$$

$$\begin{aligned} &\leq\; c\Big\{\alpha + (\alpha{+}\delta)^{p+1}\Big\} \int_0^t \Big\{ (1{+}s)^{-\frac{3}{2}} + (1{+}s)^{-2} \Big\}\, (1{+}t{-}s)^{-\frac{5}{4}}\, ds \\ &\leq\; c\Big\{\alpha + (\alpha{+}\delta)^{p+1}\Big\}\, (1{+}t)^{-\frac{5}{4}}\,, \qquad p\geq 2\;. \end{aligned} \tag{3.39}$$

Now, thanks to (3.34)–(3.39), the proof of lemma 3.6 is complete. ∎

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