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# IDENTIFIABILITY, STABILITY AND RECONSTRUCTION RESULTS OF SOURCES BY INTERIOR MEASUREMENTS

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**Abstract:** We consider the inverse problem of determining wave sources in bounded domains. We show that the interior observation on a part of the domain determines uniquely the sources if the time of observation is large enough. We further establish conditional stabilities for some particular unknown sources. We finally give a reconstructing scheme.

## 1 – Introduction

Inverse problems of distributed parameter systems is in our days an expanding field. Here we are mainly concerned with the determination of some sources using some observations. As usual in such problems the three main steps are the uniqueness (unique solvability of the problem), the stability (small perturbations of the measurements give rise to small perturbations of the sources) and finally the reconstruction (build appropriate processes in order to find a good approximation of the unknowns).

The resolution of such problems using control results of distributed systems (like the wave equation, Petrowsky systems, etc...) have been recently developed, in particular by Yamamoto and coautors [19, 3, 4, 20]. The main idea is to use some observability estimates and controllabillity results, using for instance the so-called multiplier method and the Hilbert Uniqueness Method [14], to deduce the uniqueness and the reconstruction process. For the wave equation this method successfully leads to the reconstruction of point sources in 1-dimensional domains

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by boundary observations in [3, 4, 10, 17]. In higher dimensional domains the same technique leads to the reconstruction of smoother unknown sources using boundary observations [18, 19]. In [4] the authors consider interior pointwise observations for the determination of the point sources in $]0,1[$. For the standard Petrovsky system (vibrations of beams or plates), pointwise and line observations are treated in a similar spirit in [20].

To our knowledge the determination of sources by interior measurements for the wave equation has been not yet considered. Therefore our goal is to answer to this question by adapting some results from [2, 3, 19, 17]. The main ingredients are first the existence of some observability estimates obtained in practice by some interior controllability results [14, 15] and second appropriate properties of some integral operators [19, 3]. Since the eigenvalues and eigenvectors of the Laplace equation are not explicitly known, our reconstruction process is different from the one in [3] and is more close to the one in [19].

The paper is organized as follows: In section 2 we recall the wave equation with some special sources and present the inverse problem we have in mind. In section 3 we introduce the notion of strategic subset, which means that some observability estimates hold for this domain, we further present some examples of strategic subsets. Section 4 is devoted to the proof of the uniqueness result and is based on the previous observability estimates and some properties of an integral operator between different Sobolev spaces. The conditional stability for some particular unknown sources, i.e. linear combination of Dirac functions or approximations of them (see below for the specific definitions), is deduced in section 5. Finally the reconstruction is detailed in section 6.

## 2 – Preliminaries

Let $\Omega$ be a bounded domain of $\mathbb{R}^n$, $n \geq 1$, with a Lipschitz boundary $\Gamma$. On this domain we consider the following wave equation:

$$\left\{\begin{array}{ll} \partial_t^2 u(x,t) - \Delta u(x,t) = \lambda(t)\, a(x) & \text{in } \ Q_T \ , \\ u(x,t) = 0 & \text{on } \ \Sigma_T \ , \\ u(x,0) = \partial_t u(x,0) = 0 & \text{in } \ \Omega \ , \end{array}\right. \tag{1}$$

where $Q_T := \Omega \times \,]0,T[$, $\Sigma_T := \Gamma \times ]0,T[$. Above and below $\lambda \in C^1([0,T])$ is a given function satisfying

$$\lambda(0) \neq 0 \ . \tag{2}$$

The unknow source $a$ is assumed to be in $L^2(\Omega)$ (or in $H^{-1}(\Omega)$ if $n = 1$) so that our wave equation (1) has a unique (weak) solution $u$ satisfying

$$u \in C([0,T];V) \cap C^1([0,T];H) ,$$

where for shorthness we write $V = H_0^1(\Omega)$ and $H = L^2(\Omega)$.

Our goal is to identify the unknown source $a$ from interior measurements, namely the value of $\partial_t u(x,t)$, for $0<t<T$ and all $x$ in a fixed subdomain $\omega$ of $\Omega$.

## 3 – Some observability estimates

As usual the identifiability results for the system (1) is based on some observability estimates for the unique solution $v \in C([0,T];H) \cap C^1([0,T];V')$ of

$$\text{(3)} \qquad \begin{cases} \partial_t^2 v - \Delta v = 0 \quad \text{in } \;]0,T[\;, \\ v(0) = 0\,, \quad \partial_t v(0) = a\;. \end{cases}$$

Accordingly we make the following definition:

**Definition 3.1.** A subdomain $\omega$ of $\Omega$ is said to be strategic if there $T > 0$ large enough and two positive constants $C_1$ and $C_2$ depending on $T$ such that

$$\text{(4)} \qquad C_1 \|a\|_{V'} \le \left( \int_0^T \int_\omega |v(x,t)|^2 \,dx\,dt \right)^{1/2} \le C_2 \|a\|_{V'} \,,$$

when $v$ is the unique solution of (3). □

Let us give some examples of strategic subdomains $\omega$: The one-dimensional case is quite easy and use a spectral decomposition (see [11] for a similar point of view).

**Lemma 3.2.** *Let $\Omega$ be the real interval $(0,1)$, then any open subinterval $\omega$ of $\Omega$ is strategic, namely for $T > 2$, the estimates (4) hold.*

**Proof:** Fix $x_0$ and $\epsilon > 0$ such that $\omega = (x_0 - \epsilon, x_0 + \epsilon)$.

By the spectral theorem, $v$ is given by

$$v(x,t) = \sum_{k=1}^{\infty} b_k \frac{\sin(k\pi t)}{k\pi} \sin(k\pi x) \,,$$

when the initial datum is given by

$$a = \sum_{k=1}^{\infty} b_k \sin(k\pi\cdot) .$$

By Ingham's inequalities [8], for $T > 2$ we then have

$$\int_0^T\int_\omega |v(x,t)|^2 \, dx \, dt \sim \sum_{k=1}^{\infty} \frac{|b_k|^2}{k^2\pi^2} \int_{x_0-\epsilon}^{x_0+\epsilon} |\sin(k\pi x)|^2 \, dx .$$

By explicit calculations we have

$$\int_{x_0-\epsilon}^{x_0+\epsilon} |\sin(k\pi x)|^2 \, dx = \epsilon - \frac{\sin(2k\pi\epsilon)\cos(2k\pi x_0)}{4k\pi} .$$

Therefore there exists $k_0$ such that for $k \geq k_0$ we have

$$\epsilon/2 \leq \int_{x_0-\epsilon}^{x_0+\epsilon} |\sin(k\pi x)|^2 \, dx \leq 2\,\epsilon .$$

And the requested estimates follow. ∎

In higher dimension we can still evoke the spectral decomposition for some special domains (see [11]), like a sphere or a truncated cone and a subdomain $\omega$ near the boundary. But these examples are covered by the method introduced by E. Zuazua based on the so-called HUM method of J.-L. Lions [14] and using the multiplier method:

**Lemma 3.3** (Zuazua, Theorem VII.2.5 of [14]). *Assume that $\Omega$ is a bounded domain of $\mathbb{R}^n$, $n \geq 2$, which has a $C^{1,1}$ boundary, or is convex, or is a polygonal domain of the plane with a Lipschitz boundary or is a polyhedral domain of the space with a Lipschitz boundary. Fix $x_0 \in \mathbb{R}^n$. For a subset $S$ of $\mathbb{R}^n$ and $\epsilon > 0$, set*

$$\mathcal{N}_\epsilon[S] := \bigcup_{x\in S} \Big\{ y \in \mathbb{R}^n : \, |y-x| < \epsilon \Big\} ,$$

$$\Gamma(x_0) := \Big\{ x \in \partial\Omega : \, (x-x_0)\cdot n(x) > 0 \Big\} ,$$

*where $n(x)$ is the unit exterior normal vector of $\partial\Omega$ at $x$. Then a subdomain $\omega$ satisfying*

$$\Omega \cap \mathcal{N}_\epsilon[\Gamma(x_0)] \subset \omega$$

*for some $\epsilon > 0$, is strategic.*

**Proof:** See sections 2.3 and 2.4 of [14] and in particular Theorem VII.2.5 of [14]. ∎

**Lemma 3.4.** *Fix $G$ a (relatively) open subset of the unit sphere $S_{n-1}$ of $\mathbb{R}^n$, with $n=2$ or $3$ with a smooth boundary. Let $\Omega = \{x = re^{i\omega} \in \mathbb{R}^n : r<1,\ \omega \in G\}$ be the associated truncated cone in $\mathbb{R}^n$. Then for all $\epsilon \in (0,1)$ the subdomain $\omega = \{x \in \Omega \colon 1-\epsilon < |x| < 1\}$ is strategic.*

**Proof:** Let us set $C = \{x \in \partial\Omega \colon |x| = 1\}$. Using the technique of Theorem 4.2 of [7] with the multiplier $m(x) = x$, there exists $T_0 > 0$ such that for all $T > T_0$, the solution $v$ of

$$\tag{5} \left\{ \begin{array}{l} \partial_t^2 v - \Delta v = 0 \quad \text{ in } \ ]0,T[\ , \\ v(0) = v_0\,, \qquad \partial_t v(0) = v_1\ . \end{array} \right.$$

satisfies

$$\tag{6} C_3\Big(\|v_0\|_V^2 + \|v_1\|_H^2\Big) \le \int_0^T\!\!\int_C \left|\frac{\partial v}{\partial n}(x,t)\right|^2 ds(x)\,dt \le C_4\Big(\|v_0\|_V^2 + \|v_1\|_H^2\Big)\ ,$$

for some positive constants $C_3, C_4$ depending on $T$. Consequently the arguments of Theorem VII.2.5 of [14] leads to the estimates (4). ∎

In the above lemma in dimension 2, the case $G = ]0, 2\pi[$ is allowed, this is an example of a domain $\Omega$ with a crack for which the results below hold.

Let us finally mention that the piecewise multiplier method of Liu [15] allows to show that some "internal" subdomains are strategic:

**Lemma 3.5.** *Assume that $\Omega$ is either convex or has a $C^{1,1}$ boundary and that there exists open sets $\Omega_j \subset \Omega$ with a Lipschitz boundary $\partial\Omega_j$, and points $x_0^j \in \mathbb{R}^n$, $j = 1, ..., J$ such that $\Omega_i \cap \Omega_j = \emptyset$ if $i \neq j$. Set*

$$\Gamma_j := \left\{ x \in \partial\Omega_j \colon (x - x_0^j)\cdot n_j(x) > 0 \right\}\ ,$$

*where $n_j(x)$ is the unit exterior normal vector of $\partial\Omega_j$ at $x$. Then a subdomain $\omega$ satisfying*

$$\Omega \cap \mathcal{N}_\epsilon \left[ \left( \bigcup_{j=1}^J \Gamma_j \right) \cup \left( \Omega \setminus \bigcup_{j=1}^J \Omega_j \right) \right] \subset \omega$$

*for some $\epsilon > 0$, is strategic.*

**Proof:** By Theorems 4.2 and 2.3 of [15], there exist $T>0$ and $\delta>0$ such that any solution $v$ of (5) satisfies

$$\int_0^T\int_\omega\left(|\nabla v(x,t)|^2+\left|\frac{\partial v}{\partial t}(x,t)\right|^2\right)dx\,dt\ \geq\ \delta\Big(\|v_0\|_V^2+\|v_1\|_H^2\Big)\ . \tag{7}$$

Since the energy of our system is constant, the inverse estimate

$$\int_0^T\int_\omega\left(|\nabla v(x,t)|^2+\left|\frac{\partial v}{\partial t}(x,t)\right|^2\right)dx\,dt\ \leq\ T\Big(\|v_0\|_V^2+\|v_1\|_H^2\Big)$$

clearly holds. These two estimates and the weaker norm arguments of section 2.4 in [14] allow to obtain the estimates (4). ∎

Note that the case $J=1$ corresponds to the case of Lemma 3.3. As an example (see Remark 4.3 of [14]) we may take the rectangle $\Omega=\,]0,l_1[\times]0,l_2[$ and $\omega=\,]x_1-\epsilon,x_1+\epsilon[\times]x_2-\epsilon,x_2+\epsilon[$, for any $0<x_i<l_i$ and $\epsilon<\min_{i=1,2}\{x_i,l_i-x_i\}$.

## 4 – Uniqueness

We first recall Duhamel's principle (see for instance [19, 3]) which gives the relationship between $v$ solution of (3) and $u$ solution of (1).

**Lemma 4.1.** *Let $u\in C([0,T];V)\cap C^1([0,T];H)$ be the unique solution of (1) with unknown source $a\in L^2(\Omega)$ (or in $V'$ if $n=1$) and let $v\in C([0,T];H)\cap C^1([0,T];V')$ be the unique solution of (3) with initial speed $a$. Then*

$$u(t)=(Kv)(t)\,,\qquad\forall\,t\in\,]0,T[\ , \tag{8}$$

*where $K$ is defined by*

$$(K\psi)(t)=\int_0^t\lambda(t-s)\,\psi(s)\,ds\,,\qquad\forall\,t\in\,]0,T[\ , \tag{9}$$

*and is a bounded operator from $L^2(0,T;H)$ into itself.*

We can now recall the following result proved in [19]:

**Lemma 4.2.** *If $\lambda\in C^1([0,T])$ satisfies (2) then the bounded operator $K$ from $L^2(0,T;L^2(\omega))$ into itself defined by (9) is an isomorphism from $L^2(0,T;L^2(\omega))$ into ${}_0H^1(0,T;L^2(\omega))$, where*

$${}_0H^1(0,T;L^2(\omega))\ =\ \Big\{v\in H^1(0,T;L^2(\omega))\mid v(t=0,\cdot)=0\Big\}\ ,$$

*equipped with the norm*

$$\left(\int_0^T\!\!\int_\omega |\partial_t v|^2\,dx\,dt\right)^{1/2} .$$

Let us now give a consequence to the solution $u$ of problem (1):

**Lemma 4.3.** *Let $u \in C([0,T];V) \cap C^1([0,T];H)$ be the unique solution of (1) with unknown source $a \in L^2(\Omega)$ (or in $V'$ if $n = 1$) and let $\omega$ be a strategic subset of $\Omega$. Then for $T > 0$ large enough it holds*

$$C_1\|a\|_{V'} \le \left(\int_0^T\!\!\int_\omega |\partial_t u|^2\,dx\,dt\right)^{1/2} \le C_2\,\|a\|_{V'} \ , \tag{10}$$

*for some positive constants $C_1, C_2$ depending on $T$.*

**Proof:** By Lemmas 4.1 and 4.2 we clearly have

$$\int_0^T\!\!\int_\omega |\partial_t u|^2\,dx\,dt \sim \int_0^T\!\!\int_\omega |v|^2\,dx\,dt \ .$$

We then conclude from the estimates (4). ∎

We are now ready to formulate the uniqueness result:

**Theorem 4.4.** *Let $u^1$ (resp. $u^2$) in $C([0,T];V)\cap C^1([0,T];H)$ be the unique solution of (1) with unknown source $a^1$ (resp. $a^2$) in $L^2(\Omega)$ (or in $V'$ if $n = 1$). Fix a strategic subset $\omega$ of $\Omega$ and $T > 0$ large enough from Definition 3.1. If*

$$\partial_t u^1 = \partial_t u^2 \quad \text{on} \ \ \omega \times (0,T) \ ,$$

*then $a^1 = a^2$.*

**Proof:** We remark that $u = u^1 - u^2$ satisfies (1) with $a = a^1 - a^2$. By the assumption we further have

$$\partial_t u = 0 \quad \text{on} \ \ \omega \times (0,T) \ .$$

Therefore by Lemma 4.3 we get $a = 0$ in $H^{-1}(\Omega)$ and then if $n > 1$, in $L^2(\Omega)$ since $H_0^1(\Omega)$ is dense in $L^2(\Omega)$. ∎

## 5 – Stability

Usually conditional stability results are only obtained for specific unknown sources [2, 3, 19, 17]. We here restrict ourselves to three kinds of unknown sources: The first case concerns the one-dimensional situation $n = 1$. In that case we assume that the unknown sources $a^l$, $l = 1, 2$ are linear combinations of Dirac functions, namely of the form:

$$\langle a^l, \phi\rangle = \sum_{k=1}^{K} \alpha_k^l\, \phi(\xi_k^l)\,, \qquad \forall\, \phi \in H^1(0, L)\ ,$$

for some positive integer $K$, some real numbers $\alpha_k^l$ different from zero and some different points $\xi_k^l$ in $\Omega = (0, L)$. This case was considered in [3, 17] for boundary observations, but the same idea yields a stability result in the case of internal observations. Namely we have the

**Theorem 5.1.** *In the setting of the first case, fix a strategic subset $\omega$ of $\Omega$ and $T > 0$ large enough from Definition 3.1. Suppose that*

$$|\alpha_k^1 - \alpha_k^2| + |\xi_k^1 - \xi_k^2| \le \epsilon\,, \qquad \forall\, k = 1, ..., K\ ,$$

*with $\epsilon > 0$ satisfying the constraints*

$$\epsilon \le \frac{1}{2} \min_{k \neq k'} |\xi_k^1 - \xi_{k'}^1|\ , \tag{11}$$

$$\epsilon \le \frac{1}{2} \min_{k} |\xi_k^1|\ , \tag{12}$$

$$\epsilon \le \frac{1}{2} \min_{k} |\xi_{jk}^1 - L|\ , \tag{13}$$

$$\epsilon \le \frac{1}{2} \min_{k} |\alpha_k^1|\ . \tag{14}$$

*Then there exists a constant $C$ depending on $T$, $\min_{k\neq k'} |\xi_k^1 - \xi_{k'}^1|$ and $\min_k |\alpha_k^1|$ such that*

$$\sum_{k=1}^{K} \Big(|\alpha_k^1 - \alpha_k^2| + |\xi_k^1 - \xi_k^2|\Big) \ \le\ C(1 + \sqrt{\epsilon})\, \|\partial_t u^1 - \partial_t u^2\|_{L^2(0,T;L^2(\omega))}\ .$$

**Proof:** By Lemma 4.3 we clearly have

$$\|a^1 - a^2\|_{V'} \le\ C\|\partial_t u^1 - \partial_t u^2\|_{L^2(0,T;L^2(\omega))}\ . \tag{15}$$

Therefore it remains to estimate from below the norm of $a^1 - a^2$ in $V'$. For that purpose we recall that

$$\|a^1 - a^2\|_{V'} = \sup_{\phi \in V, \phi \neq 0} \frac{|\langle a^1 - a^2, \phi\rangle|}{\|\phi\|_V} ,$$

and use appropriate test functions $\phi$.

In this situation we use the test functions $\phi$ used in Theorem 5.1 of [17] to conclude that

$$\sum_{k=1}^{K} \Big(|\alpha_k^1 - \alpha_k^2| + |\xi_k^1 - \xi_k^2|\Big) \le C(1+\sqrt{\epsilon})\, \|a^1 - a^2\|_{V'} ,$$

for some positive constant $C$ depending on $T$, $\min_{k \neq k'} |\xi_k^1 - \xi_{k'}^1|$ and $\min_k |\alpha_k^1|$.

The two above estimates lead to the conclusion. ∎

In the above setting for the determination of the locations of point sources only (i.e. $\alpha_k^l = 1$, for all $k, l$), using the results from [10], the above results may be sharpened as follows:

**Theorem 5.2.** *In the setting of the first case assume that $\alpha_k^l = 1$, for all $k = 1, ..., K,\ l = 1, 2$. Fix a strategic subset $\omega$ of $\Omega$ and $T > 0$ large enough from Definition 3.1. Suppose that*

$$|\xi_k^1 - \xi_k^2| \le \epsilon , \qquad \forall\, k = 1, ..., K ,$$

*with $\epsilon \in (0, 1/2)$ satisfying the constraint (11). Then there exists a constant $C$ depending on $T$ and $K$ such that*

$$\sum_{k=1}^{K} |\xi_k^1 - \xi_k^2| \le C\|\partial_t u^1 - \partial_t u^2\|_{L^2(0,T;L^2(\omega))} .$$

**Proof:** From the assumptions on $\xi_k^l$ and $\epsilon$ and Proposition 2 of [10], there exists a constant $C$ depending on $K$ such that

$$\sum_{k=1}^{K} |\xi_k^1 - \xi_k^2| \le C\|a^1 - a^2\|_{V'} .$$

The conclusion then follows from the estimate (15). ∎

The second case concerns the multi-dimensional case $n \geq 1$ (arbitrary) and takes for $a^l$, the $(L^2(\Omega))$ function

$$a^l(x) = \sum_{k=1}^{K} \alpha_k\, \delta^{-1}\, \varphi\Big(\frac{x-\xi_k^l}{\delta}\Big), \qquad l = 1, 2 \ ,$$

with a fixed positive integer $K$, fixed real numbers $\alpha_k \in \mathbb{R}$, and different points $\xi_k^l \in \Omega$, and finally $\varphi \in \mathcal{D}(\mathbb{R}^n)$ with a support in $B(0,1)$ and $\delta > 0$ small enough such that supp $\varphi(\frac{\cdot - \xi_k^l}{\delta}) = B(\xi_k^l, \delta) \subset \Omega$ and satifying

$$B(\xi_k^1, 4\delta) \subset \Omega\,, \qquad \forall\, k = 1, ..., K \ ,$$
$$\delta < \frac{1}{5} \min_{k \neq k'} |\xi_k^1 - \xi_{k'}^1| \ .$$

This last condition implies, in particular, that the functions $\varphi(\frac{\cdot - \xi_k^1}{\delta})$ have disjoint supports.

This choice is motivated by the fact that $\delta^{-1}\varphi(\frac{x-\xi_k^l}{\delta})$ tends to the Dirac function at $\xi_k^l$ as $\delta$ goes to zero so the above choice is an approximation of the first case.

Under these assumptions we can prove the following conditional stability result:

**Theorem 5.3.** *In the setting of the second case, fix a strategic subset $\omega$ of $\Omega$ and $T > 0$ large enough from Definition 3.1. Suppose that*

$$|\xi_k^1 - \xi_k^2| \leq \epsilon\,, \qquad \forall\, k = 1, ..., K \ ,$$

*with $\epsilon > 0$ satisfying the constraints*

$$\epsilon \ \leq \ \min_{k \neq k'} |\xi_k^1 - \xi_{k'}^1| - 5\,\delta \ , \tag{16}$$

$$\epsilon \ \leq \ \delta \ . \tag{17}$$

*Then there exists a constant $C$ depending on $T$ and $\delta$ such that*

$$\sum_{k=1}^{K} |\xi_k^1 - \xi_k^2| \ \leq \ C \|\partial_t u^1 - \partial_t u^2\|_{L^2(0,T;L^2(\omega))} \ .$$

**Proof:** As in Theorem 5.1 it suffices to estimate from below the norm of $a^1 - a^2$ in $V'$.

If $n = 1$, we take

$$\phi_k(x) = \phi_1\left(\frac{x - \xi_k^1}{\delta}\right) \quad \text{in } \ \Omega \ ,$$

where $\phi_1$ is a fixed function defined by

$$\phi_1(\hat{x}) \ = \begin{cases} -4 - \hat{x} & \text{if } \ -4 < \hat{x} \leq -2 \ , \\ \hat{x} & \text{if } \ -2 < \hat{x} \leq 2 \ , \\ -\hat{x} + 4 & \text{if } \ 2 < \hat{x} \leq 4 \ , \\ 0 & \text{else } . \end{cases}$$

For that choice, by the above conditions for any $k = 1, ..., K$, we have

$$\langle a^1 - a^2, \phi_k \rangle \ = \ \alpha_k \, \delta^{-1} \int_{\mathbb{R}} \left( \varphi\left(\frac{x - \xi_k^1}{\delta}\right) - \varphi\left(\frac{x - \xi_k^2}{\delta}\right)\right) \phi_1\left(\frac{x - \xi_k^1}{\delta}\right) dx \ ,$$

and by a change of variable we get

$$\langle a^1 - a^2, \phi_k \rangle \ = \ \alpha_k \int_{-1}^{1} \varphi(y) \left( \phi_1(y) - \phi_1\left(\frac{\xi_k^2 - \xi_k^1 + \delta y}{\delta}\right)\right) dy \ .$$

By the finite increment theorem and the fact that $|\xi_k^1 - \xi_k^2| < \delta$, we then obtain

$$\langle a^1 - a^2, \phi_k \rangle \ = \ \frac{\alpha_k}{\delta} (\xi_k^1 - \xi_k^2) \int_{-1}^{1} \varphi(y) \, dy \ .$$

The conclusion follows from the fact that $\|\phi_k\|_V = \frac{C_1}{\sqrt{\delta}}$ for some $C_1 > 0$.

If $n = 2$ we first take

$$\phi_k(x_1, x_2) \ = \ \phi_1\left(\frac{x_1 - \xi_{k1}^1}{\delta}\right) \phi_2\left(\frac{x_2 - \xi_{k2}^1}{\delta}\right) \quad \text{in } \ \Omega \ ,$$

where $\phi_1$ was defined above and $\phi_2$ is defined by

$$\phi_2(\hat{x}) \ = \begin{cases} 2 + \dfrac{\hat{x}}{2} & \text{if } \ -4 < \hat{x} \leq -2 \ , \\ 1 & \text{if } \ -2 < \hat{x} \leq 2 \ , \\ 2 - \dfrac{\hat{x}}{2} & \text{if } \ \ 2 < \hat{x} \leq 4 \ , \\ 0 & \text{else } . \end{cases}$$

The same arguments as before then yield

$$\langle a^1 - a^2, \phi_k\rangle \;=\; \frac{\alpha_k}{\delta}(\xi^1_{k1} - \xi^2_{k1}) \int_{\mathbb{R}^2} \varphi(y_1, y_2)\, dy_1\, dy_2 \;.$$

Since $\|\phi_k\|_V = C_2$ for some $C_2 > 0$ independent of $\delta$ we conclude that

$$|\xi^1_{k1} - \xi^2_{k1}| \;\leq\; C\,\delta\, \|a^1 - a^2\|_{V'} \;,$$

for some $C > 0$ independent of $\delta$.

Exchanging the rule of $x_1$ and $x_2$ we may conclude

$$|\xi^1_{k2} - \xi^2_{k2}| \;\leq\; C\,\delta\, \|a^1 - a^2\|_{V'} \;,$$

for some $C > 0$ independent of $\delta$.

The proof is similar for $n \geq 3$. ∎

The third case we want to treat is the case when $n = 1$ and the function $a^l$, $l = 1, 2$ is given by

$$a^l(x) \;=\; \frac{1}{|x - \xi^l|^\beta} \;,$$

for a point $\xi^l \in \Omega$ and some $0 < \beta < \frac{1}{2}$.

Under these assumptions we can prove the following conditional stability result:

**Theorem 5.4.** *Fix a strategic subset $\omega$ of $\Omega = (0, L)$ and $T > 0$ large enough from Definition 3.1. There exists a positive constant $\kappa$ depending on $\beta$ such that if*

$$|\xi^1 - \xi^2| \;\leq\; \kappa(L - \xi^1)^{2-\beta} \;, \tag{18}$$

*then there exists a constant $C$ depending on $T$, $\beta$ and $L - \xi^1$ such that*

$$|\xi^1 - \xi^2| \;\leq\; C\|\partial_t u^1 - \partial_t u^2\|_{L^2(0,T;L^2(\omega))} \;.$$

**Proof:** As before we need to estimate from below the norm of $a^1 - a^2$ in $V'$.

We fix $\delta = \frac{L-\xi}{2}$ and take as test function $\phi_\xi$:

$$\phi_\xi(x) \;=\; \phi\Big(\frac{x - \xi}{\delta}\Big) \;,$$

where $\phi$ is a fixed function belonging to $C^2(\mathbb{R})$ with a support in $[0, 1]$ and such that $\phi(\hat{x}) > 0$ for any $\hat{x} \in (0, 1)$. By changes of variable we have

$$\langle a^1 - a^2, \phi_\xi\rangle \;=\; \int_{\mathbb{R}} \frac{1}{|y|^\beta} \left(\phi\Big(\frac{y}{\delta}\Big) - \phi\Big(\frac{\xi^2 - \xi^1 + y}{\delta}\Big)\right) dy \;.$$

Using Taylor's expansion we then get

$$\langle a^1 - a^2, \phi_\xi \rangle \;=\; c_1\,\frac{\xi^2-\xi^1}{\delta} + c_2\,\frac{(\xi^2-\xi^1)^2}{\delta^2}\ ,$$

where we have set

$$c_1 \;=\; \int_{\mathbb{R}} \frac{1}{|y|^\beta}\ \phi'\Big(\frac{y}{\delta}\Big)\,dy\ ,$$

$$c_2 \;=\; \int_{\mathbb{R}} \frac{1}{|y|^\beta}\ \phi''(\theta(y))\,dy\ ,$$

where $\theta(y)$ is a point between $\frac{y}{\delta}$ and $\frac{\xi^2-\xi^1+y}{\delta}$. Remark that by integration by parts we have

$$c_1 \;=\; \beta\,\delta^{1-\beta}\int_0^1 \phi(y)\,y^{-\beta-1}\,dy \;=\; c_1'\,\delta^{1-\beta}$$

which is positive due to our assumptions on $\phi$. This allows to write

$$|\langle a^1 - a^2, \phi_\xi\rangle| \;\geq\; \frac{|\xi^2-\xi^1|}{\delta^2}\,\Big(c_1'\delta^{2-\beta} - |c_2|\,|\xi^2-\xi^1|\Big)\ .$$

Therefore taking

$$\kappa \;=\; \frac{c_1'}{|c_2|\,2^{3-\beta}}\ ,$$

the assumption (18) and the above estimate yield

$$|\langle a^1 - a^2, \phi_\xi\rangle| \;\geq\; \frac{c_1'}{2\,\delta^\beta}\,|\xi^2-\xi^1|\ .$$

The conclusion follows from the standard identity $\|\phi_\xi\|_V = \frac{C_1}{\sqrt{\delta}}$ for some $C_1 > 0$. ∎

## 6 – Reconstruction

For the reconstruction of the sources from interior measurements we follow the point of view of [19] which consists in using the following exact controllability result:

**Lemma 6.1.** *Fix a strategic subset $\omega$ of $\Omega$ and $T > 0$ large enough from Definition 3.1. Then for every $\phi \in V$, there exists a unique control $v \in L^2(0,T;L^2(\omega))$*

*such that the (weak) solution* $\psi \in C([0,T];H) \cap C^1([0,T];V')$ *of*

$$
(19) \qquad \begin{cases} \partial_t^2\psi - \Delta\psi = \chi_{\omega\times(0,T)} v & \text{in} \quad Q_T \ , \\ \psi = 0 & \text{on} \quad \Sigma_T \ , \\ \psi(x,0) = \phi(x), \quad \partial_t\psi(x,0) = 0 & \text{in} \quad \Omega \ , \end{cases}
$$

*satisfies*

$$
(20) \qquad \psi(\cdot,T) \;=\; \partial_t\psi(\cdot,T) \;=\; 0 \ .
$$

**Proof:** This is a direct consequence of the estimates (4) and of the Hilbert Uniqueness Method of Lions [14, Th. VII.2.5]. Note that $\psi$ is only a weak solution of the system (19) with the final conditions (20) in the sense that $\psi$ is the unique solution of (using the transposition method)

$$
(21) \qquad \int_{Q_T} \psi f \, dx\, dt \;=\; -\langle \phi, \partial_t\varphi(0)\rangle_{V-V'} \;+\; \int_0^T\!\!\int_\omega v\varphi \; dx\, dt
$$

for all $f \in L^1(0,T;H),\ \varphi_0 \in H,\ \varphi_1 \in V'$, where $\varphi \in C([0,T];H) \cap C^1([0,T];V')$ is the unique solution of

$$
\begin{cases} \partial_t^2\varphi = \Delta\varphi + f \quad \text{in} \ \ ]0,T[ \ , \\ \varphi(T) = \varphi_0 \, , \qquad \partial_t\varphi(T) = \varphi_1 \ . \ \blacksquare \end{cases}
$$

In view of Lemma 6.1 we can define a bounded linear operator $\Pi : V \to L^2(0,T;L^2(\omega))$, by

$$
\phi \to v \ ,
$$

where $v$ is the control from the above Theorem driving the system (19) to rest at time $T$.

We further use the adjoint $K^\star$ of the operator $K$ as (bounded) operator from ${}_0H^1(0,T;L^2(\omega))$ into $L^2(0,T;L^2(\omega))$ and which is given by (see section 5 of [19])

$$
(K^\star\eta)(x,t) \;=\; \lambda(0)\,(\partial_t\eta)(x,t) + \int_t^T \Big(\lambda'(s-t)\,(\partial_t\eta)(x,s) + \lambda(s-t)\,\eta(x,s)\Big)\, ds \ , \qquad 0 < t < T \ ,
$$

for all $\eta \in {}_0H^1(0,T;L^2(\omega))$. By the assumption (2) we even have (see [19])

$$
R(K^\star) = L^2(0,T;L^2(\omega)) \ .
$$

Consequently for all $\psi \in L^2(0,T;L^2(\omega))$ there exists a unique $\eta \in {}_0H^1(0,T;L^2(\omega))$ solution of

$$K^\star \eta \,=\, \psi \ ,$$

equivalently, $\eta$ is solution of the Volterra equation of the second kind

$$\lambda(0)\,(\partial_t \eta)(x,t) + \int_t^T \Big(\lambda'(s-t)\,(\partial_t\eta)(x,s) + \lambda(s-t)\,\eta(x,s)\Big)\,ds \,=\, \psi(x,t) \ , \qquad x\in\omega,\ \ 0<t<T \ .$$

We then define the mapping $\Phi$ from $L^2(0,T;L^2(\omega))$ to ${}_0H^1(0,T;L^2(\omega))$ by

$$\psi \to \eta \,:=\, \Phi\psi \ ,$$

when $\eta$ is solution of the above integral equation. This means that

$$K^\star\Phi = Id \quad \text{on} \ \ L^2(0,T;L^2(\omega)) \ . \tag{22}$$

Now we can formulate our reconstruction result:

**Theorem 6.2.** *Fix a strategic subset $\omega$ of $\Omega$ and $T>0$ large enough from Definition 3.1. For all $k=1,...,\infty$ we define*

$$\theta_k \,=\, \Phi\Pi\phi_k \ ,$$

*where $\{\phi_k\}_{k=1}^\infty$ is an orthonormal basis (in $L^2(\Omega)$) of the Laplace operator with Dirichlet boundary condition. Let $u\in C([0,T];V)\cap C^1([0,T];H)$ be the unique solution of (1) with unknown source $a\in L^2(\Omega)$ (or in $H^{-1}(\Omega)$ if $n=1$). Then for all $k=1,...,\infty$ we have*

$$\langle a,\phi_k\rangle \,=\, \int_0^T\!\!\int_\omega \partial_t u(x,t)\,\partial_t\theta_k(x,t)\ dx\,dt \ , \tag{23}$$

*and then $a$ may be reconstructed by*

$$a \,=\, \sum_{k=1}^\infty \langle a,\phi_k\rangle\,\phi_k \,=\, \sum_{k=1}^\infty \left(\int_0^T\!\!\int_\omega \partial_t u(x,t)\,\partial_t\theta_k(x,t)\ dx\,dt\right)\phi_k \ .$$

**Proof:** Applying the identity (21) with $\varphi = v$, where $v$ is the unique solution of (3) with initial speed $a$ we have:

$$\langle a,\phi_k\rangle \,=\, \int_0^T\!\!\int_\omega v(x,t)\,\Pi\phi_k(x,t)\ dx\,dt \ . \tag{24}$$

To conclude we need to show that

$$\int_0^T\int_\omega v(x,t)\,\Pi\phi_k(x,t)\;dx\,dt \;=\int_0^T\int_\omega \partial_t u(x,t)\,\partial_t\theta_k(x,t)\;dx\,dt\;. \tag{25}$$

Indeed by the definition of $\theta_k$ and (22) we may write

$$K^\star\theta_k = K^\star\Phi\Pi\phi_k = \Pi\phi_k\;.$$

Therefore by the above identity, the left-hand side of (25) may be transformed as follows

$$\begin{aligned}\int_0^T\int_\omega v(x,t)\,\Pi\phi_k(x,t)\;dx\,dt \;&=\; \int_0^T\int_\omega v(x,t)\,K^\star\,\theta_k(x,t)\;dx\,dt\\ &=\;(Kv,\theta_k)_{{}_0H^1(0,T;L^2(\omega))} \;=\; (u,\theta_k)_{{}_0H^1(0,T;L^2(\omega))}\;,\end{aligned}$$

and the identity (25) follows from the definition of the inner product in ${}_0H^1(0,T;L^2(\omega))$. ∎

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