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ASYMPTOTIC EXPANSION FOR A DISSIPATIVE BENJAMIN–BONA–MAHONY EQUATION WITH PERIODIC COEFFICIENTS

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Recommended by E. Zuazua

Abstract: In this work we study the asymptotic behavior of solutions of a dissipative BBM equation in $\mathbb{R}^N$ with periodic coefficients. We use Bloch waves decomposition to obtain a complete expansion, as $t \to +\infty$, and conclude that the solutions behave, in a first approximation, as the homogenized heat kernel.

1 – Introduction

The aim of this paper is to investigate the asymptotic behavior, for large time, of the solutions of the following Cauchy problem associated with the BBM equation

$$(1.1)\quad \begin{cases} \rho(x)\,\dfrac{\partial u}{\partial t} - \dfrac{\partial}{\partial x_j}\Big(a_{jk}(x)\,\dfrac{\partial^2 u}{\partial x_k\,\partial t}\Big) - \nu\,\dfrac{\partial}{\partial x_j}\Big(a_{jk}(x)\,\dfrac{\partial u}{\partial x_k}\Big) = 0 \quad \text{in } \mathbb{R}^N \times (0,\infty) \\ u(x,0) = \varphi(x) \quad \text{in } \ \mathbb{R}^N \end{cases}$$

where ν is a positive constant.

The coefficients $(a_{jk}(x))_{1\le j,k\le N}$ are assumed to be bounded, symmetric and periodic. The variable density ρ also is periodic and bounded. More precisely, we set $Y = [0,2\pi)^N$ and we denote by $L^\infty_\#(Y)$ the subspace of $L^\infty(\mathbb{R}^N)$ of functions which are Y-periodic, that is,

$$L^\infty_\#(Y) \;=\; \Big\{\phi \in L^\infty(\mathbb{R}^N)\colon\ \phi(x+2\pi\,p) = \phi(x),\ \ \forall\, x \in \mathbb{R}^n,\ \ \forall\, p \in \mathbb{Z}^N\Big\}\ .$$

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Also, we assume that

$$\text{(1.2)}\qquad \begin{cases} a_{jk} \in L^\infty_\#(Y)\ , \\ \exists\, \alpha > 0, \quad \text{such that } \ a_{jk}(x)\,\eta_j\,\eta_k \ge \alpha|\eta|^2, \ \ \forall\,\eta \in \mathbb{N}^N, \ \text{ a.e. } x \in \mathbb{R}^N\ , \\ a_{jk} = a_{kj} \quad \forall\, j,k = 1,2,...,N\ , \end{cases}$$

and

$$\text{(1.3)}\qquad \begin{cases} \rho \in L^\infty_\#(Y), \quad \text{i.e.} \quad \rho \text{ is } Y\text{-periodic} \\ \exists\,\rho_1,\rho_2 \in \mathbb{R}_+, \quad \text{such that } \ 0 < \rho_0 \le \rho(x) \le \rho_1, \ \text{ a.e. } x \in Y\ . \end{cases}$$

Observe that equation in (1.1) has a dissipative nature. Indeed, the energy associated to (1.1) is given by

$$E(t) \ = \ \frac{1}{2}\int_{\mathbb{R}^N} \left[\rho(x)u^2 + a_{jk}(x)\,\frac{\partial u}{\partial x_j}\,\frac{\partial u}{\partial x_k}\right] dx$$

and it is decreasing due to the coercivity condition in (1.2) and according to the law

$$\frac{dE}{dt} \ = \ -\nu\int_{\mathbb{R}^N} a_{jk}(x)\,\frac{\partial u}{\partial x_j}\,\frac{\partial u}{\partial x_k}\,dx\ .$$

Moreover, solutions of (1.1) satisfy the conservation law

$$\text{(1.4)}\qquad \frac{\partial}{\partial t}\int_{\mathbb{R}^N} u\,\rho(x)\,dx \ = \ 0\ ,$$

that is, the mass of $u = u(x,t)$ with respect to the weight $\rho(.)$ is conserved along time:

$$\text{(1.5)}\qquad m_\rho(u) \ = \ m_\rho(\varphi) \ = \int_{\mathbb{R}^N} \varphi\,\rho(x)\,dx\ .$$

An important model associated to (1.1) is the nonlinear Benjamin–Bona–Mahony equation with constant coefficients

$$\text{(1.6)}\qquad \begin{cases} u_t - \Delta u_t - \nu\,\Delta u = \nabla\cdot F(u)\,; \quad x \in \mathbb{R}^N, \ \ t > 0 \\ u(x,0) = \varphi(x) \end{cases}$$

where $\nu > 0$ is a fixed constant, $F \in C^1(\mathbb{R},\mathbb{R}^N)$ is a fixed vector field and $\nabla\cdot F(u) = \sum\limits_{j=1}^{N} \dfrac{\partial}{\partial x_j} F(u)$. The equation (1.6) is a direct generalization to higher dimensions of the equation

$$\text{(1.7)}\qquad \begin{cases} u_t - u_{xxt} - \nu\,u_{xx} + u\,u_x = 0\,; \quad x \in \mathbb{R}, \ \ t > 0 \\ u(x,0) = \varphi(x) \end{cases}$$

with $\nabla \cdot F(u) = -uu_x$ that was originally derived as a model for surface water waves in a uniform channel (see for instance, [1], [3] and [4]). It also cover cases of the following type: surfaces of long wavelength in liquids, acoustic-gravity waves in compressible fluids, hydromagnetic waves in cold plasma, acoustic waves in anharmonic crystals, etc (see [13], [14] and [18]). The wide applicability of these equations is the main reason why, during the last decades, they have attracted so much attention from mathematicians.

In this paper we derive a complete asymptotic expansion of solution of (1.1) as $t \to +\infty$. In order to understand the type of the asymptotic expansion, one may expect, it is convenient to analyse first the parabolic equation

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial}{\partial x_j}\left(a_{jk}(x)\,\dfrac{\partial u}{\partial x_k}\right) = 0, & \text{in } \mathbb{R}^N \times (0,\infty) \\ u(x,0) = \varphi(x) \quad \text{in } \mathbb{R}^N \,. \end{cases} \tag{1.8}$$

Equation (1.1) can be viewed as a perturbation of (1.8). Moreover, when $\rho \equiv 1$ it was proved in [11] that

$$t^{\frac{N}{2}(1-\frac{1}{p})}\left\| u(\cdot,t) - m(\varphi)\,G_h(\cdot,t)\right\|_p \to 0, \qquad \text{as } t \to +\infty, \;\; 1 \le p \le \infty$$

where $m(\varphi) = \displaystyle\int_{\mathbb{R}^N} \varphi\,dx$ and G_h the fundamental solution of the homogenized system

$$\begin{cases} u_t - q_{jk}\,\dfrac{\partial^2 u}{\partial x_j \partial x_k} = 0\,, & x \in \mathbb{R}^N, \;\; t > 0 \\ u(x,0) = \delta_0(x)\,. \end{cases} \tag{1.9}$$

Here and in the sequel we denote by δ_0 the Dirac delta at the origin and by $q_{i,j}$ the homogenized coefficients associated to the periodic matrix with coefficients (1.2). We observe that the homogenized coefficients q_{jk} associated to the periodic matrix a are given by (see [20], [21])

$$q_{jk} = \frac{1}{|Y|}\int_Y a_{jk}\,dy + \frac{1}{|Y|}\int_Y a_{jm}\,\frac{\partial \chi_\ell}{\partial y_m}\,dy\,, \qquad 1 \le j,k \le N$$

where χ_ℓ is the solution of the Y-periodic elliptic problem

$$\left|\begin{array}{l} -\dfrac{\partial}{\partial x_j}\left(a_{jk}\,\dfrac{\partial \chi_\ell}{\partial x_k}\right) = \dfrac{\partial a_{\ell k}}{\partial y_k} \\ \chi_\ell \text{ is } Y\text{-periodic}\,, \qquad 1 \le \ell \le N\,. \end{array}\right.$$

Note that the solution χ_ℓ is uniquely determined up to an additive constant. Moreover, the homogenized matrix $q = (q_{jk})$ is symmetric, that is,

$$q_{jk} = q_{kj}$$

and elliptic with the same constant of ellipticity α of the matrix $a_{j,k}(x)$ in (1.2), that is,

$$\sum_{j,k=1}^{N} q_{jk}\, \xi_j \xi_k \geq \alpha |\xi|^2 , \quad \forall\, \xi \in \mathbb{R}^N .$$

We refer to [8] and [9] for more details on homogenization.

In this work we study the asymptotic expansion of the solutions of the linear equation in (1.1) with periodic coefficients, using Bloch waves decomposition.

We observe that equation in (1.1) can be assumed as a model for the heat conduction involving a thermodynamic temperature $\theta = u - \nu \Delta u$ and a conductive temperature u (see [5]). We are going to prove that the solutions of (1.1) behave as a linear combination of the derivatives of the fundamental solution of the heat equation modulated by periodic functions $c_\alpha(\cdot)$. Furthermore, from the decomposition we prove it follows that the total mass of the solution is captured by the first term in the asymptotic expansion.

A similar analysis was done in some recent works: in [15] the asymptotic expansion of the solutions of the heat equation with periodic coefficients and initial data in $L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ was studied. The results in [15] are an extension of those for the heat equation with constant coefficients (see [10]). In [7] C. Conca and J.H. Ortega studied the asymptotic behavior of the solutions of a linear parabolic equations in $\mathbb{R}^N$ with periodically oscillating coefficients. In [16] R. Orive, E. Zuazua and A. Pazoto applied Bloch waves theory to study the asymptotic expansion of a linear wave equation with damping and periodic coefficients. In [17] R. Prado Raya and E. Zuazua obtained the complete asymptotic expansion of solutions, as time goes to infinite, of the both the linearized Benjamin–Bona–Mahony–Burger equation and the linear Korteweg–de-Vries–Burger equation. They also compute the second term in the asymptotic expansion of the solution to the two-dimensional Benjamin–Bona–Mahony–Burger equation, with quadratic nonlinear term.

In this work, the general notation is standard and is the same that which appears, for instance, in J.L. Lions [12].

2 – Main results

The well-posedness of (1.1) under the conditions (1.2) and (1.3) can be obtained inverting the operator

$$A \stackrel{\text{def}}{=} -\frac{\partial}{\partial x_j}\left(a_{jk}(x)\,\frac{\partial}{\partial x_k}\right)$$

and rewriting (1.1) as an abstract evolution equation in the Hilbert space $H = H^1(\mathbb{R}^N)$, with the inner product

$$(u,v)_{H^1(\mathbb{R}^N)} = \int_{\mathbb{R}^N} u\,v\,\rho(x)\,dx \;+ \sum_{j,k=1}^{N} \int_{\mathbb{R}^N} a_{jk}(x)\,\frac{\partial u}{\partial x_j}\,\frac{\partial v}{\partial x_k}\,dx$$

where the functions $(a_{jk}(x))$ and $\rho(x)$are given in (1.2) and (1.3) respectively. Under these conditions the operator associated to (1.1) is maximal and dissipative in H. Then Lummer–Philips theorem guarantees that the operator associated to (1.1) is the infinitesimal generator of a continuous semigroup. Thus, we deduce that for any initial data $\varphi \in L^2(\mathbb{R}^N)$ the problem (1.1) has a unique global weak solution $u = u(x,t)$ such that $u \in C(\mathbb{R}^+, L^2(\mathbb{R}^N))$.

The main result of this work is the following theorem.

Theorem 2.1 (Asymptotic Expansion). *Let the initial data $\varphi \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ with $|x|^{k+1}\,\varphi(x) \in L^1(\mathbb{R}^N)$ for some fixed integer $k \geq 0$. Let $u = u(x,t)$ be the solution of (1.1). Then, there exist periodic functions $c_\alpha(\cdot) \in L^\infty_{\neq}(Y)$, $|\alpha| \leq k$, and constants $c_{\beta,n}$, $4 \leq |\beta| \leq 2\,k$, depending on the initial data and the coefficients (a_{jk}), such that*

$$\left\|u(\cdot,t) - \sum_{|\alpha|\leq k} c_\alpha(\cdot)\left[G_\alpha(\cdot,t) + \sum_{n=1}^{p} \frac{(-t)^n}{n!} \sum_{m=0}^{p_1} \sum_{|\beta|=4n+2m} c_{\beta,n}\,G_{\alpha+\beta}(\cdot,t)\right]\right\| \leq \tag{2.1}$$

$$\leq\; C_k\,t^{-\frac{2k+N+2}{4}}$$

as $t \to \infty$, where $p = p(\alpha) = [\frac{k-|\alpha|}{2}]$, $p_1 = p_1(\alpha,n) = p(\alpha) - n$ and the functions $G_\alpha(\cdot,t)$, $\alpha \in \mathbb{N}^n$, are given by

$$G_\alpha(x,t) = \int_{\mathbb{R}^N} \xi^\alpha\, e^{-\frac{1}{\bar\rho}\sum_{j,k} q_{jk}\,\xi_j\,\xi_k\,t}\, e^{ix\cdot\xi}\,d\xi \tag{2.2}$$

where q_{jk}, $1 \leq j,k \leq N$, are the homogenized constant coefficients associated with the matrix $a = (a^\varepsilon_{jk}(x))$, $a^\varepsilon_{jk}(x) = a_{jk}(x/\varepsilon)$, as $\varepsilon \to 0$, and $\bar\rho$ is the averaged *density:*

$$\overline{\rho} = \frac{1}{(2\pi)^N} \int_Y \rho(x)\, dx \;. \tag{2.3}$$

Let us also underline that the functions G_α which appear in the asymptotic expansion (2.1) are such that $G_\alpha = (-i)^{|\alpha|} \dfrac{\partial^\alpha G}{\partial x^\alpha}$ where $G = G(x,t)$ is the fundamental solution of the underlying parabolic homogenized system

$$\left|\begin{array}{l} \overline{\rho}\, G_t - q_{jk} \dfrac{\partial^2 G}{\partial x_j\, \partial x_k} = 0\,, \qquad x \in \mathbb{R}^N, \quad t > 0 \\[2ex] G(x,0) = \delta_0(x) \end{array}\right.$$

where $\overline{\rho}$ is the averaged density given in (2.3).

Thus the convergence result (2.1) shows us that the solution u of (1.1) may be approximated by a linear combination at order k, of the derivatives of the fundamental solution of the heat equation, modulated by the functions $c_\alpha(\cdot)$. The role of these coefficients is to adapt the Gaussian asymptotic profiles to the periodicity of the medium where the solution u involves. Indeed, $c_0(\cdot)$ turns out be a constant and more precisely $c_0 = \dfrac{1}{\bar{\rho}}\, m_\rho(u)$ so that the first term in the asymptotic expansion is really the Gaussian kernel G (see Section 8 and (1.5)). In what concerns the coefficients c_β, two ingredients are involved: the moments of the initial data $\varphi \in L^1(\mathbb{R}^N; 1 + |x|^{k+1})$ and the derivatives of the periodicity of the medium.

This work is organized as follows. The next section contains the basic results on Bloch waves. In Section 4 we present some technical lemmas that we use in the Section 5. Section 5 is devoted to the asymptotic expansion and in Section 6 we prove the main result in the particular case that the density $\rho(x)$ is constant. Finally, in Section 7 we prove Theorem 2.1 for a general nonconstant density and in Section 8 we analyze the periodic functions c_α and the constants c_β entering in the asymptotic expansion.

3 – Bloch waves decomposition

In this section we recall some basic results on Bloch waves decomposition. We refer to [8] and [9] for the notations and the proofs.

Let us consider the following spectral problem parametrized by $\xi \in \mathbb{R}^n$: to find $\lambda = \lambda(\xi) \in \mathbb{R}$ and $\psi = \psi(x;\xi)$, not identically zero, such that

$$\begin{cases} A\psi(\cdot,\xi) = \lambda(\xi)\,\psi(\cdot,\xi) \quad \text{in } \mathbb{R}^n\,, \\ \psi(\cdot,\xi) \ \text{ is } \ (\xi, Y)\text{-periodic, i.e.,} \\ \psi(y+2\pi m;\,\xi) = e^{2\pi i m\cdot\xi}\,\psi(y)\,, \quad \forall\, m \in \mathbb{Z}^N, \ \ y \in \mathbb{R}^N\,, \end{cases} \tag{3.1}$$

where $i = \sqrt{-1}$ and A is the elliptic operator in divergence form

$$A \stackrel{\text{def}}{=} -\frac{\partial}{\partial x_j}\left(a_{jk}(x)\,\frac{\partial}{\partial x_k}\right). \tag{3.2}$$

We can write $\psi(x,\xi) = e^{ix\cdot\xi}\,\phi(x,\xi)$, ϕ being Y-periodic in the variable x. From (3.1) we can observe that the (ξ, Y)-periodicity is unaltered if we replace ξ by $(\xi + m)$, with $m \in \mathbb{Z}^N$. Therefore, ξ can be confined to the dual cell $Y' = \left[-\frac{1}{2}, \frac{1}{2}\right)^N$.

From the ellipticity and symmetry assumptions on the matrix $a_{j,k}(x)$ can be proved (see [20]) that for each $\xi \in Y'$ the spectral problem (3.1) admits a sequence of eigenvalues with the following properties:

$$\begin{cases} 0 \le \lambda_1(\xi) \le \cdots \le \lambda_m(\xi) \le \cdots \to +\infty \qquad \text{and} \\ \lambda_m(\xi) \ \text{ is a Lipschitz function of } \ \xi \in Y'\,, \quad \forall\, m \ge 1\,. \end{cases} \tag{3.3}$$

Besides, the corresponding sequence of eigenfunctions $\{\psi_m(\xi)\}_m$ may be normalized to constitute an orthonormal basis of $L^2_\#(Y)$. Moreover, as a consequence of the min-max principle we have that

$$\cdots \ge \lambda_m(\xi) \ge \cdots \ge \lambda_2(\xi) \ge \lambda_2^N > 0\,, \quad \forall\, \xi \in Y' \tag{3.4}$$

where λ_2^N is the second eigenvalue of the operator A, given in (3.2), in the cell Y with Neumann boundary conditions on ∂Y.

The following result provides the classical Bloch wave decomposition of $L^2(\mathbb{R}^n)$:

Proposition 3.1. *Let $g \in L^2(\mathbb{R}^n)$. The m-th Bloch coefficient of g is defined as follows:*

$$\hat{g}_m(\xi) = \int_{\mathbb{R}^n} g(x)\,e^{-ix\cdot\xi}\,\overline{\phi_m(x,\xi)}\;dx\,, \quad \forall\, m \ge 1, \ \ \xi \in Y'\,.$$

Then the following inverse formula:

$$g(x) = \int_{Y'} \sum_{m=1}^{\infty} \hat{g}_m(\xi)\,e^{ix\cdot\xi}\,\phi_m(x;\xi)\;d\xi$$

and the Parseval's identity:

$$\int_{\mathbb{R}^n} |g(x)|^2 \, dx = \int_{Y'} \sum_{m=1}^{\infty} |\hat{g}_m(\xi)|^2 \, d\xi \tag{3.5}$$

hold.

Further, for all g in the domain of A, it holds that

$$Ag(x) = \int_{Y'} \sum_{m=1}^{\infty} \lambda_m(\xi)\, \hat{g}_m(\xi)\, e^{ix\cdot\xi}\, \phi_m(x;\xi) \, d\xi$$

and, consequently,

$$\|g\|^2_{H^1(\mathbb{R}^N)} = \int_{Y'} \sum_{m=1}^{\infty} \Big(1 + \lambda_m(\xi)\Big) \, |\hat{g}_m(\xi)|^2 \, d\xi \ ,$$

$$\|g\|^2_{H^{-1}(\mathbb{R}^N)} = \int_{Y'} \sum_{m=1}^{\infty} \frac{|\hat{g}_m(\xi)|^2}{1 + \lambda_m(\xi)} \, d\xi \ . \tag{3.6}$$

Observe that Proposition 3.1 guarantees that the set $\{e^{ix\cdot\xi}\phi_m(x,\xi) \colon m \geq 1, \xi \in Y'\}$ forms a basis of $L^2(\mathbb{R}^N)$ in a generalized sense. Moreover, $L^2(\mathbb{R}^N)$ may be identified with $L^2(Y', \ell(\mathbb{N}))$ via Parseval's identity (3.5), (3.6).

The following result on the behavior of $\lambda_1(\xi)$ and $\phi_1(x,\xi)$, near $\xi = 0$, will also be necessary in this work.

Proposition 3.2. *We assume that (a_{jk}) satisfy the conditions (1.2). Then there exists $\delta_1 > 0$ such that the first eigenvalue $\lambda_1(\xi)$ is an analytic function on $B_{\delta_1} \stackrel{\text{def}}{=} \{\xi \colon |\xi| < \delta_1\}$, and there is a choice of the first eigenvector $\phi_1(\cdot,\xi)$ such that*

$$\xi \mapsto \phi_1(\cdot,\xi) \in L^\infty_\#(Y) \cap H^1_\#(\mathbb{R}^N)$$

is analytic on B_{δ_1} and

$$\phi_1(x,0) = |Y|^{-1/2} = \frac{1}{(2\pi)^{N/2}} \,, \qquad x \in \mathbb{R}^N \ .$$

Moreover,

$$\begin{cases} \lambda_1(0) = 0 \,, \qquad \partial_j \lambda_1(0) = 0, \quad 1 \leq j \leq N \ , \\ \dfrac{1}{2}\, \partial^2_{jk}\, \lambda_1(0) = q_{jk} \,, \qquad 1 \leq j,k \leq N \ , \\ \partial^\alpha \lambda_1(0) = 0 \,, \qquad \forall\, \alpha \ \text{such that } |\alpha| \text{ is odd} \ , \end{cases} \tag{3.7}$$

and

$$c|\xi|^2 \leq \lambda_1(\xi) \leq \tilde{c}|\xi|^2\,, \qquad \forall\, \xi \in Y' \tag{3.8}$$

where c and $\tilde{c}$ are positive constants.

As a consequence of this proposition we have:

Proposition 3.3. *Assume the same hypotheses of Proposition 3.2 and let $f(\xi) = \dfrac{\lambda_1(\xi)}{1+\lambda_1(\xi)}\cdot$ Then there exists $\delta > 0$ such that $f(\xi)$ is analytic on B_δ. Furthermore, $f(\xi)$ satisfies*

$$c_1\,\frac{|\xi|^2}{1+|\xi|^2} \;\leq\; f(\xi) \;\leq\; c_2\,\frac{|\xi|^2}{1+|\xi|^2}\,, \qquad c_1, c_2 > 0\;,$$

and

$$\begin{cases} f(0) = \partial_i f(0) = 0\,, \quad i = 1,2,\ldots,N\;, \\ \partial^2_{ij}\, f(0) = 2\,q_{ij}\,, \quad i,\; j = 1,\ldots,N\;, \\ \partial^\beta f(0) = 0\,, \quad \forall\,\beta \;\; \text{such that } |\beta| \text{ is odd}\;. \end{cases} \tag{3.9}$$

These results follows directly from the computations of the derivatives of $f(\xi)$ at $\xi = 0$.

4 – Technical Lemmas

In this section we are going to present a basic lemma on asymptotic analysis and some technical results which will be useful in the proof of the asymptotic expansion.

Definition 4.1. Given $f, g \in C^1(\mathbb{R};\mathbb{R})$, we say that f and g are of the same order as $t \to +\infty$ and we denote it by $f \sim g$ when

$$\lim_{t\to\infty} \frac{f(t)}{g(t)} = 1\;. \;\square$$

Lemma 4.1. *Let $C > 0$. Then*

$$\int_{B_\delta} e^{-C|\xi|^2 t}\, |\xi|^k\, d\xi \;\sim\; C_k\, t^{-\frac{k+N}{2}}\;, \tag{4.1}$$

as $t \to +\infty$ for all $k \in \mathbb{N}$, where C_k is a positive constant which may be computed explicitly.

On the other hand, if $q = q_{jk}$ is a symmetric positive matrix satisfying (2.6) then

$$\int_{Y'} e^{-\sum\limits_{j,k} q_{jk}\xi_j\xi_k\, t}\, \xi^\beta\, d\xi \sim C_{|\beta|}\, t^{-\frac{|\beta|+N}{2}} \tag{4.2}$$

as $t \to +\infty$ for all multi index $\beta \in (\mathbb{N}\cup\{0\})^N$ and for a suitable positive constant $C_{|\beta|}$ which may be computed as well.

Lemma 4.2. *Let $\varphi \in L^1(\mathbb{R}^n)$ be a function such that $|x|^k\,\varphi \in L^1(\mathbb{R}^n)$. Then its first Bloch coefficient $\hat{\varphi}_1(\xi)$ belongs to $C^k(B_\delta)$, where B_δ is the neighborhood of $\xi = 0$ where the first Bloch wave $\phi_1(x,\xi)$ is analytic.*

Proof: Since

$$\hat{\varphi}_1(\xi) = \int_{\mathbb{R}^N} \varphi(x)\, e^{-ix\cdot\xi}\, \overline{\phi_1}(x;\xi)\; dx\ ,$$

for all $\alpha \in (\mathbb{N}\cup\{0\})^N$ with $|\alpha| \le k$, we have

$$\frac{\partial^{|\alpha|}\hat{\varphi}_1}{\partial\xi^\alpha}(\xi) = \sum_{\beta\le\alpha} \binom{\alpha}{\beta} \int_{\mathbb{R}^N} \varphi(x)\,(-i)^{|\beta|}\, x^\beta e^{ix\cdot\xi}\, \frac{\partial^{|\alpha-\beta|}\overline{\phi_1}}{\partial\xi^{\alpha-\beta}}\,(x;\xi)\; dx\ ,$$

where $\beta \le \alpha$ means that $\beta_j \le \alpha_j$ for all $j = 1,...,N$, and

$$\binom{\alpha}{\beta} = \prod_{k=1}^{N} \binom{\alpha_j}{\beta_j}\ .$$

Moreover, Proposition 3.2 gives us that the function $\xi \to \phi_1(x;\xi)$ is analytic with values in $L^\infty_\#(Y)$, which guarantees that

$$\begin{aligned}\left|\frac{\partial^{|\alpha|}\hat{\varphi}_1}{\partial\xi^\alpha}(\xi)\right| &\le \sum_{\beta\le\alpha}\binom{\alpha}{\beta} c_\beta \int_{\mathbb{R}^N} |\varphi(x)x^\beta|\, dx \\ &\le \sum_{\beta\le\alpha}\binom{\alpha}{\beta} c_\beta \int_{\mathbb{R}^N} (1+|x|^k)\,|\varphi(x)|\, dx\ ,\end{aligned}$$

for $0 \le |\alpha| \le k$, where c_β is a positive constant. Thus, using again Proposition 3.2, we have that the map $\xi \to e^{i\xi\cdot x}\,\partial_\xi^\alpha\phi_1(x;\xi)$ is continuous, and the result follows. ∎

Lemma 4.3. *Consider the function*

$$G(x) = \int_{Y'} g(\xi)\, e^{ix\cdot\xi}\, w(x;\xi)\, d\xi\ , \quad x \in \mathbb{R}^N\ , \tag{4.3}$$

where $g \in L^2(Y')$ and $w \in L^\infty(Y'; L^2_\#(Y))$. Then $G \in L^2(\mathbb{R}^N)$ and

$$\|G\|^2_{L^2(\mathbb{R}^N)} = \int_{Y'} |g(\xi)|^2 \, \|w(\,\cdot\,;\xi)\|^2_{L^2(Y)} \, d\xi \,.$$

Proof: To check this result we expand $w(x;\xi)$ as a function of x in the orthonormal basis $\{\phi_m(x;\xi)\}_{m=1}^\infty$ where $\xi \in Y'$ is a parameter:

$$w(x;\xi) = \sum_{m=1}^\infty a_m(\xi)\, \phi_m(x;\xi) \,.$$

Introducing this expression in (4.3), we get

$$G(x) = \int_{Y'} g(\xi) \sum_{m=1}^\infty a_m(\xi)\, e^{ix\cdot\xi}\, \phi_m(x;\xi)\; d\xi \,.$$

Applying the Parseval's identity of Proposition 3.1, it follows that

$$\|G\|^2 = \int_{Y'} |g(\xi)|^2 \sum_{m=1}^\infty |a_m(\xi)|^2 \; d\xi \,.$$

The proof of the lemma is complete using the Parseval's identity in $L^2(Y)$:

$$\|w(\,\cdot\,;\xi)\|^2_{L^2(Y)} = \sum_{m=1}^\infty |a_m(\xi)|^2 \,, \qquad \forall\, \xi \in Y' \,. \blacksquare$$

5 – Asymptotic expansion for constant density

In this section and in the next, C_k will indicate different positive constants which depend on the integer k. In these two sections we develop the asymptotic expansion for the case when the density function $\rho(x) \equiv \rho_1 = \rho_0 = 1$. The general case is studied in Section 7.

First we compute the Bloch coefficients of the solution u of (1.1). Next, we show that the terms corresponding to the eigenvalue $\lambda_m(\xi)$, $m \geq 2$, are negligible since they decay exponentially as $t \to +\infty$.

Lemma 5.1. *Let $u = u(x,t)$ be the solution of (1.1) with $\varphi \in L^2(\mathbb{R}^N)$. Then*

$$u(x,t) = \sum_{m=1}^\infty \int_{Y'} \hat{\varphi}_0^m(\xi)\; e^{-\frac{\lambda_m(\xi)}{1+\lambda_m(\xi)}\,t}\, e^{ix\cdot\xi}\, \phi_m(x,\xi)\; d\xi \tag{5.1}$$

where $\hat{\varphi}_0^m(\xi)$ are the Bloch coefficients associated to the initial data $\varphi(x)$.

Proof: Since $u(x,t) \in L^2(\mathbb{R}^n)$ for all $t>0$, we have that

$$u(x,t) \,=\, \sum_{m=1}^{\infty} \int_{Y'} \hat{u}^m(\xi,t)\, e^{ix\cdot\xi}\, \phi_m(x,\xi)\; d\xi \tag{5.2}$$

where $\hat{u}^m(\xi,t)$ are the Bloch coefficients of $u(x,t)$. On the other hand, from Proposition 3.1 we obtain

$$\sum_{m=1}^{\infty} \int_{Y'} \Big[\hat{u}_t^m(\xi,t) + \lambda_m(\xi)\,\hat{u}_t^m(\xi,t) + \lambda_m(\xi)\,\hat{u}^m(\xi,t)\Big]\, e^{ix\cdot\xi}\, \phi_m(x,\xi)\; d\xi \;=\; 0 \;.$$

Thus, since system $\{\phi_m(x,\cdot)\}$ is orthogonal, it follows, that for each $m \geq 1$, $\hat{u}^m(\xi,t)$ satisfies the following ordinary differential equation:

$$\begin{cases} \hat{u}_t^m(\xi,t) + \lambda_m(\xi)\,\hat{u}_t^m(\xi,t) + \lambda_m(\xi)\,\hat{u}^m(\xi,t) \;=\; 0 \quad \text{ in } \; Y' \times (0,+\infty) \\ \hat{u}^m(\xi,0) = \hat{\varphi}_0^m(\xi) \;. \end{cases} \tag{5.3}$$

Solving the differential equation (5.3) we find

$$\hat{u}^m(\xi,t) = \hat{\varphi}_0^m(\xi)\, e^{-\frac{\lambda_m(\xi)}{1+\lambda_m(\xi)}\,t}\,, \qquad m \geq 1 \;. \tag{5.4}$$

From (5.4) and (5.2) the result follows. ∎

Lemma 5.2. *Let $\hat{u}^m(\xi,t)$, $m \geq 2$, be the Bloch coefficients, associated with the solution $u = u(x,t)$ of (1.1), given in (5.4). Then, there exists a positive constant γ such that*

$$\sum_{m=2}^{\infty} \int_{Y'} |\hat{u}^m(\xi;t)|^2\, d\xi \;\leq\; e^{-\gamma t}\, \|\varphi\|^2_{L^2(\mathbb{R}^N)} \qquad \textit{for all } \; t>0 \;.$$

Proof: From (3.6) it follows that, for $m \geq 2$,

$$\lambda_m(\xi) \geq \lambda_2^N > 0 \qquad \text{for all } \; \xi \in Y' \;.$$

Therefore

$$\frac{\lambda_m(\xi)}{1+\lambda_m(\xi)} \geq \frac{\lambda_2^N}{1+\lambda_m(\xi)} \qquad \text{for all } \; m \geq 2 \;\text{ and }\; \xi \in Y' \;. \tag{5.5}$$

In order to obtain the result, we consider two cases:

Case 1: $\lambda_2^N \le \lambda_m(\xi) \le 1$.

In this case, from (5.5) it results that

$$\frac{\lambda_m(\xi)}{1+\lambda_m(\xi)} \ge \frac{\lambda_2^N}{1+\lambda_m(\xi)} \ge \frac{\lambda_2^N}{2} \quad \text{for all } \ m \ge 2 \ .$$

Case 2: $\lambda_m(\xi) > 1$.

If $\lambda_m(\xi) \ge 1$ then

$$\frac{\lambda_m(\xi)}{1+\lambda_m(\xi)} \ge \frac{1}{2} \quad \text{for all } \ m \ge 2$$

since $1+\lambda_m(\xi) \le 2\,\lambda_m(\xi)$.

Now, letting $\beta = \min\left\{\frac{\lambda_2^N}{2}, \frac{1}{2}\right\}$, we deduce that

$$0 < e^{-\frac{\lambda_m(\xi)}{1+\lambda_m(\xi)}t} \le e^{-\beta t} \quad \text{for all } \ m \ge 2, \ \ \xi \in Y' \ .$$

Consequently,

$$|\hat{u}^m(\xi,t)| \ = \ |\hat{\varphi}_0^m(\xi)\, e^{-\frac{\lambda_m(\xi)}{1+\lambda_m(\xi)}t}| \ \le \ |\hat{\varphi}_0^m(\xi)|\, e^{-\beta t} \quad \text{for all } \ m \ge 2, \ \ \xi \in Y' \ .$$

Thus,

$$\begin{aligned}\sum_{m=2}^{\infty} \int_{Y'} |\hat{u}^m(\xi,t)|^2\, d\xi \ &\le \ \sum_{m=2}^{\infty} \int_{Y'} |\hat{\varphi}_0^m(\xi)|^2\, e^{-2\beta t}\, d\xi \\ &\le \ e^{-2\beta t} \sum_{m=2}^{\infty} \int_{Y'} |\hat{\varphi}_0^m(\xi)|^2\, d\xi \ .\end{aligned}$$

Parseval's identity implies that

$$\sum_{m=2}^{\infty} \int_{Y'} |\hat{u}^m(\xi,t)|^2\, d\xi \ \le \ e^{-2\beta t} \|\varphi\|^2_{L^2(\mathbb{R}^N)} \quad \text{for all } \ t > 0 \ .$$

Taking $\gamma = 2\beta$ the conclusion of the lemma follows. ∎

In the next lemma we prove that the term corresponding to $\lambda_1(\xi)$ also goes to zero exponentially, as $t \to +\infty$, whenever $\xi \in \{\xi \in Y' \colon |\xi| > \delta\}$, with $\delta > 0$ sufficiently small.

Lemma 5.3. *Let $\hat{u}^1 = \hat{u}^1(\xi,t)$ be the first Bloch coefficient of the solution u of (1.1) given in (5.4). Then, there exists a positive constant γ_1, such that*

$$\int_{Y'\setminus B_\delta} |\hat{u}^1(\xi,t)|^2\, d\xi \ \le \ e^{-\gamma_1 t} \|\varphi\|^2_{L^2(\mathbb{R}^N)} \quad \text{for all } \ t > 0 \tag{5.6}$$

where $B_\delta = \{\xi \in Y' \colon |\xi| < \delta\}$ with δ such that $0 < \delta < 1/2$.

Proof: Let $0 < \delta < \frac{1}{2}\cdot$ From Proposition 3.2, we know that

$$\lambda_1(\xi) \geq c|\xi|^2 \quad \text{for all } \ \xi \in Y' \tag{5.7}$$

where c is a positive constant.

On the other hand, proceeding as in the proof of Lemma 5.2, we can consider two cases:

First case: $\lambda_1(\xi) \geq 1, \quad \xi \in Y' \backslash B_\delta$.

In this case

$$\frac{\lambda_1(\xi)}{1+\lambda_1(\xi)} \geq \frac{\lambda_1(\xi)}{2\,\lambda_1(\xi)} = \frac{1}{2}\ .$$

Second case: $0 \leq \lambda_1(\xi) < 1, \quad \xi \in Y' \backslash B_\delta$.

From (5.7) it follows that

$$\frac{\lambda_1(\xi)}{1+\lambda_1(\xi)} \geq \frac{\lambda_1(\xi)}{2} \geq \frac{c|\xi|^2}{2} > c\,\frac{\delta^2}{2}\,, \qquad \xi \in Y' \backslash B_\delta\ .$$

Thus, letting $\tilde{\beta} = \min\left\{\frac{1}{2}\,, c\,\frac{\delta^2}{2}\right\}$ we have

$$\frac{\lambda_1(\xi)}{1+\lambda_1(\xi)} \geq \tilde{\beta} > 0 \quad \text{ for all } \ \xi \in Y \backslash B_\delta\ .$$

Therefore, from (5.4) with $m = 1$ we have

$$\begin{aligned}
\int_{Y'\backslash B_\delta} |\hat{u}^1(\xi,t)|^2\,d\xi &= \int_{Y'\backslash B_\delta} |\hat{\varphi}_0^1(\xi)|^2\, e^{-\frac{2\lambda_1(\xi)}{1+\lambda_1(\xi)}\,t}\,d\xi \\
&\leq e^{-2\tilde{\beta}t} \int_{Y'\backslash B_\delta} |\hat{\varphi}_0^1(\xi)|^2\,d\xi \\
&\leq e^{-2\tilde{\beta}t} \int_{Y'} |\hat{\varphi}_0^1(\xi)|^2\,d\xi \\
&\leq e^{-2\tilde{\beta}t} \sum_{m=1}^{\infty} \int_{Y'} |\hat{\varphi}_0^m(\xi)|^2\,d\xi \ = \ e^{-2\tilde{\beta}t} \|\varphi\|^2_{L^2(\mathbb{R}^N)}
\end{aligned}$$

for all $t > 0$.

Taking $\gamma_1 = 2\tilde{\beta}$ we conclude the proof of Lemma 5.3. ∎

Remark 5.1. Due to Parseval's identity, it follows from Lemma 5.2 and Lemma 5.3 that

$$\begin{aligned}\|u(t)\|^2_{L^2(\mathbb{R}^N)} &= \int_{Y'} |\hat{u}^1(\xi,t)|^2\, d\xi \,+\, \sum_{m=2}^{+\infty} \int_{Y'} |\hat{u}^m(\xi,t)|^2\, d\xi \\ &\le \int_{B_\delta} |\hat{u}^1(\xi,t)|^2\, d\xi \,+\, e^{-\gamma t}\|\varphi\|^2_{L^2(\mathbb{R}^N)}\end{aligned}$$

where γ is a positive constant and $B_\delta = \{\xi \in Y' \colon |\xi| < \delta\}$, $0 < \delta < 1/2$. Thus, to prove Theorem 2.1 it is sufficient to analyse the first term in the Bloch expansion and, more precisely, its projection to $\xi \in B_\delta$, since the other one decay exponentially as $t \to +\infty$. □

According to Remark 5.1, our analysis may be restricted to consider

$$I(x,t) = \int_{B_\delta} \hat{\varphi}^1_0(\xi)\, e^{-\frac{\lambda_1(\xi)}{1+\lambda_1(\xi)}\, t}\, e^{ix\cdot\xi}\, \phi_1(x;\xi)\, d\xi\ . \tag{5.8}$$

Our next step is to prove that (5.8) may be replaced by

$$J(x,t) = \sum_{|\alpha|\le k} c_\alpha \int_{B_\delta} \xi^\alpha\, e^{-\frac{\lambda_1(\xi)}{1+\lambda_1(\xi)}\, t}\, e^{ix\cdot\xi}\, \phi_1(x,\xi)\, d\xi \tag{5.9}$$

where $c_\alpha = \dfrac{1}{\alpha!} D^\alpha \hat{\varphi}^1_0(0)$, $|\alpha| \le k$, are the Taylor coefficients of the expansion of $\hat{\varphi}^1_0(\xi)$ around $\xi = 0$. To do this, we assume that the initial data $\varphi^0 \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ and is such that $|x|^{k+1}\varphi(x) \in L^1(\mathbb{R}^N)$ for some $k \in \mathbb{N}$. Under these conditions, there exists $0 < \delta_0 < 1/2$ such that the first Bloch coefficient $\hat{\varphi}^1_0(\xi)$ of the initial data belongs to $C^{k+1}(B_{\delta_0})$ (see Lemma 4.2).

In the sequel, we will fix $\delta = \min\{\delta_0; \delta_1\}$, where δ_1 was given in Proposition 3.2.

Lemma 5.4. *There exists a positive constant $C_k = C_k(\varphi) > 0$ such that*

$$\|I(\cdot,t) - J(\cdot,t)\|^2_{L^2(\mathbb{R}^N)} \sim C_k\, t^{-\frac{2k+2+N}{2}} \qquad \text{as } \ t \to +\infty\ .$$

where $J(x,t)$ was defined in (5.9).

Proof: Parseval's identity implies that

$$\|I(\cdot,t) - J(\cdot,t)\|^2_{L^2(\mathbb{R}^N)} = \int_{B_\delta} \left| \hat{\varphi}^1_0(\xi) \,-\, \sum_{|\alpha|\le k} c_\alpha\, \xi^\alpha \right|^2 e^{-\frac{2\lambda_1(\xi)}{1+\lambda_1(\xi)}}\, d\xi \tag{5.10}$$

and Proposition 3.3 give us that

$$\frac{\lambda_1(\xi)}{1+\lambda_1(\xi)} \geq \frac{c_1|\xi|^2}{1+\delta^2} = \gamma_1|\xi|^2\,, \qquad \xi \in B_\delta\ , \tag{5.11}$$

where $\gamma_1 = \dfrac{c_1}{1+\delta^2}\cdot$

Thus, putting (5.10) and (5.11) together, we obtain

$$\|I(\cdot,t) - J(\cdot,t)\|^2_{L^2(\mathbb{R}^N)} \leq \int_{B_\delta} \Big| \hat{\varphi}^1_0(\xi) - \sum_{|\alpha|\leq k} c_\alpha\,\xi^\alpha \Big|^2 e^{-2\,\gamma_1|\xi|^2\,t}\,d\xi\ . \tag{5.12}$$

On the other hand, since $\hat{\varphi}^1_0 \in C^{k+1}(B_\delta)$ we have from Taylor's expansion

$$\Big| \hat{\varphi}^1_0(\xi) - \sum_{|\alpha|\leq k} c_\alpha\,\xi^\alpha \Big| \leq C_k|\xi|^{k+1}, \qquad \text{for all}\ \ \xi \in B_\delta \tag{5.13}$$

where $C_k > 0$ is constant.

Therefore, estimates (5.12) and (5.13)imply that

$$\|I(\cdot,t) - J(\cdot,t)\|^2_{L^2(\mathbb{R}^N)} \leq C_k^2 \int_{B_\delta} |\xi|^{2(k+1)}\,e^{-2\gamma_1|\xi|^2 t}\,d\xi$$

and, from Lemma 4.1 we obtain

$$\|I(\cdot,t) - J(\cdot,t)\|^2_{L^2(\mathbb{R}^N)} \sim C_k^2\,t^{-\frac{2k+2+N}{2}}\,, \qquad t \to \infty \tag{5.14}$$

with $C_k > 0$ a positive constant.

In a second step, we compute the Taylor expansion of $\phi_1(x,\xi)$ around $\xi = 0$ and we prove that all terms entering in (5.9), which are denoted by

$$\begin{aligned} J_\alpha(x,t) &= \int_{B_\delta} \xi^\alpha\,e^{-\frac{\lambda_1(\xi)}{1+\lambda_1(\xi)}t}\,e^{ix\cdot\xi}\,\phi_1(x,\xi)\ d\xi\ , \\ &\qquad \alpha \in (\mathbb{N}\cup 0)^N,\ \ |\alpha| \leq k,\ \ (x,t) \in \mathbb{R}^N \times \mathbb{R}^+\ , \end{aligned} \tag{5.15}$$

can be approximated in the L^2-setting by a linear combination of the form

$$\frac{1}{(2\pi)^N} \sum_{|\gamma|\leq k-|\alpha|} d_\gamma(x) \int_{B_\delta} \xi^\gamma\,e^{-\frac{\lambda_1(\xi)}{1+\lambda_1(\xi)}t}\,e^{ix\cdot\xi}\,d\xi \tag{5.16}$$

where d_γ are periodic functions defined by

$$d_\gamma(x) = \frac{1}{\gamma!}\,D^\gamma_\xi\,\phi_1(x,0)\,, \qquad |\gamma| \leq k\ . \tag{5.17}$$

This can be done because $\phi_1(\cdot,\xi) \in L^\infty_\#(Y) \cap H^1_\#(Y)$ is an analytic function of ξ (see Proposition 3.2) in B_δ. ∎

Lemma 5.5. *There exists a constant $C_k > 0$ such that*

$$\left\| J_\alpha(\cdot,t) \;-\; \sum_{|\gamma|\leq k-|\alpha|} d_\gamma(\cdot)\, I_{\alpha+\gamma}(\cdot,t) \right\|^2_{L^2(\mathbb{R}^N)} \sim\; C_k\, t^{-\frac{2k+2+N}{2}}\,, \qquad t\to\infty$$

where $I_\gamma(x,t) = \displaystyle\int_{B_\delta} \xi^\gamma\, e^{-\frac{\lambda_1(\xi)}{1+\lambda_1(\xi)}\,t}\, e^{ix\cdot\xi}\, d\xi$ *and* $|\alpha| \leq k$.

Proof: Let

$$R_{k,\alpha}(x,\xi) \;=\; \phi_1(x,\xi) \;-\; \sum_{|\gamma|\leq k-|\alpha|} d_\gamma(x)\,\xi^\gamma \tag{5.18}$$

where $d_\gamma(\cdot)$ is defined in (5.17) and $\alpha \in (\mathbb{N}\cup 0)^N$ with $|\alpha| \leq k$. Since $\phi_1(\cdot,\xi)$ is an analytic function with respect to ξ in the B_δ and values in $L^2_\#(Y)$ we have, for all $\xi \in B_\delta$,

$$\|R_{k,\alpha}(\cdot,\xi)\|_{L^2(\mathbb{R}^N)} \;\leq\; C_k |\xi|^{k+1-|\alpha|}\ . \tag{5.19}$$

Thus

$$R_{k,\alpha} \in L^\infty(B_\delta; L^2_\#(Y))\ . \tag{5.20}$$

Now, we consider the function F given by

$$\begin{aligned}
F(x,t) \;&=\; J_\alpha(x,t) \;-\; \sum_{|\gamma|\leq k-|\alpha|} d_\gamma(x)\, I_{\alpha+\gamma}(x,t)\\
&= \int_{B_\delta} \xi^\alpha\, e^{-\frac{\lambda_1(\xi)}{1+\lambda_1(\xi)}\,t} \left[\phi_1(x,\xi) \;-\; \sum_{|\gamma|\leq k-|\alpha|} d_\gamma(x)\xi^\gamma\right] e^{ix\cdot\xi}\; d\xi\\
&= \int_{B_\delta} \xi^\alpha\, e^{-\frac{\lambda_1(\xi)}{1+\lambda_1(\xi)}\,t}\, R_{k,\alpha}(x,\xi)\, e^{ix\cdot\xi}\; d\xi\ .
\end{aligned}$$

From Lemma 4.3, (5.19) and (5.20) we obtain, using (5.11),

$$\begin{aligned}
\|F(\cdot,t)\|^2_{L^2(\mathbb{R}^N)} \;&=\; \int_{B_\delta} \left|\xi^\alpha\, e^{-\frac{\lambda_1(\xi)}{1+\lambda_1(\xi)}\,t}\right|^2 \|R_{k,\alpha}(\cdot,\xi)\|^2_{L^2(Y)}\; d\xi\\
&\leq\; C_k^2 \int_{B_\delta} |\xi|^{2|\alpha|}\, e^{-\frac{2\lambda_1(\xi)}{1+\lambda_1(\xi)}\,t}\, |\xi|^{2k+2-2|\alpha|}\; d\xi\\
&\leq\; C_k^2 \int_{B_\delta} |\xi|^{2k+2}\, e^{-2\gamma_1|\xi|^2\,t}\, d\xi\ .
\end{aligned}$$

Consequently, from Lemma 4.1 it follows that

$$\|F(\cdot,t)\|^2_{L^2(\mathbb{R}^N)} \le C_k\, t^{-\frac{2k+2+N}{2}} \quad \text{as} \quad t \to +\infty\,. \tag{5.21}$$

The proof of Lemma 5.5 is complete. ∎

Next, we are going to study the integral I_γ which appears in the statement of Lemma 5.5. We consider

$$\sum_{|\alpha|\le k} \tilde{d}_\alpha\, \xi^\alpha \tag{5.22}$$

the power expansion of the function $f(\xi) = \dfrac{\lambda_1(\xi)}{1+\lambda_1(\xi)}$, $\xi \in B_\delta$, around $\xi = 0$ and observe that, according to Proposition 3.3, we have

$$f(0) = \frac{\partial f}{\partial \xi_j}(0) = 0\,, \qquad j = 1, 2, ..., N\,. \tag{5.23}$$

and

$$\frac{\partial^2 f}{\partial \xi_j \partial \xi_k}(0) = 2\, q_{jk}\,, \qquad j,k = 1, 2, ..., N\,. \tag{5.24}$$

In view of (5.22), (5.23) and (5.24), the map

$$r(\xi) \;=\; f(\xi) \;-\sum_{j,k}^{N} q_{jk}\, \xi_j\, \xi_k \tag{5.25}$$

is analytic in B_δ (see Proposition 3.3). Moreover,

$$e^{-f(\xi)t} \;=\; e^{-\sum\limits_{j,k} q_{jk}\xi_j\xi_k\, t}\, e^{-r(\xi)t} \;=\; e^{-\sum\limits_{j,k} q_{jk}\xi_j\xi_k\, t} \left(\sum_{n=0}^{\infty} \frac{t^n}{n!}\,(-r(\xi))^n\right). \tag{5.26}$$

Now, if we define for $p \in \mathbb{N}$

$$\widetilde{I}_\gamma(x,t) \;=\; \int_{B_\delta} \xi^\gamma\, e^{-\sum\limits_{j,k} q_{jk}\xi_j\xi_k\, t} \left(\sum_{n=0}^{p} \frac{t^n}{n!}\,(-r(\xi))^n\right) e^{ix\cdot\xi}\; d\xi \tag{5.27}$$

and replacing (5.25) in $I_\gamma(x,t)$ defined in Lemma 5.5, (5.26) allows us to study the asymptotic behavior of

$$I_\gamma(x,t) - \widetilde{I}_\gamma(x,t) = \int_{B_\delta} \xi^\gamma\, e^{-\sum\limits_{j,k} q_{jk}\xi_j\xi_k\, t} \left(e^{-r(\xi)t} - \sum_{n=0}^{p} \frac{t^n}{n!}\,(-r(\xi))^n\right) e^{ix\cdot\xi}\, d\xi\,. \tag{5.28}$$

Lemma 5.6. *Let* $2p \geq k - |\gamma| - 1$. *Then, there exists a constant* $C_k > 0$ *such that*

$$\|I_\gamma(\cdot,t) - \widetilde{I}_\gamma(\cdot,t)\|^2_{L^2(\mathbb{R}^N)} \sim C_k\, t^{-\frac{2k+N+2}{2}}, \qquad \text{as } t \to \infty\,.$$

where $\widetilde{I}_\gamma$ *was defined in (5.27).*

Proof: Parseval's identity and (5.28) imply that

$$\begin{aligned} \|I_\gamma(\cdot,t) - \widetilde{I}_\gamma(\cdot,t)\|^2_{L^2(\mathbb{R}^N)} &= \\ &= \int_{B_\delta} |\xi|^{2|\gamma|}\, e^{-2\sum_{j,k} q_{jk}\xi_j\xi_k\, t} \left| e^{-r(\xi)t} - \sum_{n=0}^{p} \frac{(-t)^n}{n!}\, r(\xi)^n \right|^2 d\xi \end{aligned} \tag{5.29}$$

and, since the function e^z, $z \in \mathbb{R}$, is analytic, there exists a positive constant C_p satisfying

$$\left| e^{-r(\xi)t} - \sum_{n=0}^{p} \frac{(-t)^n}{n!}\, r(\xi)^n \right| \leq C_p |r(\xi)|^{p+1}\, t^{p+1}\,, \qquad t > 0, \quad \xi \in B_\delta\,. \tag{5.30}$$

On the other hand, recalling that $f(\xi)$ is analytic in B_δ and the definition of $r(\xi)$, from the fact that $D^\alpha f(0) = 0$ when $|\alpha|$ is odd (see Proposition 3.3), it follows that

$$r(\xi) = \sum_{m=1}^{\infty} \sum_{|\alpha|=2+2m} \frac{1}{\alpha!}\, \partial_\xi^\alpha f(0)\, \xi^\alpha\,, \qquad \xi \in B_\delta\,. \tag{5.31}$$

Thus, we can obtain a positive constant C such that

$$|r(\xi)| \leq C|\xi|^4\,, \qquad \xi \in B_\delta\,. \tag{5.32}$$

Now, returning to (5.29) and using (5.30) and (5.32) together with Lemma 4.1 we obtain

$$\begin{aligned} \|I_\gamma(\cdot,t) - \widetilde{I}_\gamma(\cdot,t)\|^2_{L^2(\mathbb{R}^N)} &\leq C_p^2 \int_{B_\delta} |\xi|^{2|\gamma|}\, e^{-2\sum_{j,k} q_{jk}\xi_j\xi_k\, t}\, |r(\xi)|^{2p+2}\, t^{2p+2}\, d\xi \\ &\leq C_p^2\, t^{2p+2} \int_{B_\delta} e^{-2\sum_{j,k} q_{jk}\xi_j\xi_k\, t}\, |\xi|^{8p+8+2|\gamma|}\, d\xi \\ &\sim C_p^2\, C_k\, t^{2p+2-\frac{8p+8+2|\gamma|+N}{2}} \qquad \text{as } t \to \infty\,, \end{aligned}$$

for $|\gamma| \leq k$ and p to be chosen. Letting p such that

$$2p \geq k - |\gamma| - 1$$

the proof of the lemma is complete. ∎

The next step is to study the asymptotic behavior of $r(\xi)$ defined in (5.25). But, before doing it, observe that if we consider the Taylor expansion of $r(\xi)$, around $\xi=0$, we obtain

$$
\begin{aligned}
(r(\xi))^n &= \left(\sum_{\beta\geq 0} \frac{1}{\beta!}\,\partial_\xi^\beta\, r(0)\,\xi^\beta\right)^n \\
&= \left(\sum_{m=0}^{\infty} \sum_{|\beta|=4+2m} \frac{1}{\beta!}\,\partial_\xi^\beta\, f(0)\,\xi^\beta\right)^n \qquad (5.33)\\
&= \sum_{m=0}^{\infty} \sum_{\beta=4n+2m} c_{\beta,n}\,\xi^\beta\,,
\end{aligned}
$$

because $D^\beta f(0)=0$ for $|\beta|<4$ and for $|\beta|$ odd.

This fact suggest for $\widetilde{I}_\gamma(x,t)$ (see definition in (5.27)) an approximation of type

$$
\int_{B_\delta} \xi^\gamma\, e^{-\sum_{j,k} q_{jk}\xi_j\xi_k t} \sum_{n=1}^{p} \frac{(-t)^n}{n!} \sum_{m=0}^{p_1} \sum_{|\beta|=4+2m} c_{\beta,n}\,\xi^\beta\, e^{ix\cdot\xi}\, d\xi\,,
$$

where $p_1=p_1(\gamma,n)$, $p_1\geq n$, to be chosen later.

Lemma 5.7. *Let*

$$
\widetilde{I}_{\gamma *}(x,t) = \int_{B_\delta} \xi^\gamma\, e^{-\sum_{j,k} q_{jk}\xi_j\xi_k\, t} \left(\sum_{n=1}^{p} \frac{(-t)^n}{n!}\, r(\xi)^n\right) e^{ix\cdot\xi}\, d\xi
$$

and

$$
J_{\gamma^*}(x,t) = \int_{B_\delta} \xi^\gamma\, e^{-\sum_{j,k} q_{jk}\xi_j\xi_k\, t} \left[\sum_{n=1}^{p} \frac{(-t)^n}{n!} \sum_{m=0}^{p_1} \sum_{|\beta|=4+2m} c_{\beta,n}\,\xi^\beta\right] e^{ix\cdot\xi}\, d\xi\,,
$$

with $c_{\beta,n}=\frac{1}{\beta!}\,\partial_\xi^\beta(r(\xi)^n)(0)$.

Then

$$
\|\widetilde{I}_{\gamma *}(\cdot,t) - J_{\gamma^*}(\cdot,t)\|^2_{L^2(\mathbb{R}^N)} \sim C_k\, t^{-\frac{2k+N+2}{2}} \quad \textit{as } t\to+\infty
$$

where C_k is a positive constant and $|\gamma|\leq k$.

Proof: From Parseval's Theorem we can write

$$
\|\widetilde{I}_{\gamma *}(\cdot,t) - J_{\gamma^*}(\cdot,t)\|^2_{L^2(\mathbb{R}^N)} =
$$

$$
= \int_{B_\delta} \left|\xi^\gamma\, e^{-\sum_{j,k} q_{jk}\xi_j\xi_k\, t} \left[\sum_{n=1}^{p} \frac{(-t)^n}{n!} \left[r(\xi)^n - \sum_{m=0}^{p_1} \sum_{|\beta|=4+2m} c_{\beta,n}\,\xi^\beta\right]\right]\right|^2 d\xi\,.
$$

On the other hand, from the analyticity of $r(\xi)$, we obtain a positive constant C_n such that

$$\left| r(\xi)^n - \sum_{m=0}^{p_1} \sum_{|\beta|=4+2m} c_{\beta,n}\, \xi^\beta \right| \le C_n |\xi|^{4n+2p_1+2}, \quad \forall\, \xi \in B_\delta \ ,$$

where $c_{\beta,n} = \dfrac{1}{\beta!}\, \partial_\xi^\beta (r(\xi))^n(0)$ and $p_1 = p_1(\gamma, n) \ge n$ to be chosen later.

Thus,

$$\begin{aligned}
\|\widetilde{I}_{\gamma *}(\cdot,t) - J_{\gamma^*}(\cdot,t)\|^2_{L^2(\mathbb{R}^N)} \ &\le \\
&\le \int_{B_\delta} |\xi|^{2|\gamma|}\, e^{-2\sum\limits_{j,k} q_{jk}\xi_j\xi_k\, t} \left[\sum_{n=1}^{p} \frac{t^n}{n!} \left| r(\xi)^n - \sum_{m=0}^{p_1} \sum_{|\beta|=4+2m} c_{\beta,n}\, \xi^\beta \right| \right]^2 d\xi \\
&\le \int_{B_\delta} |\xi|^{2|\gamma|}\, e^{-2\sum\limits_{j,k} q_{jk}\xi_j\xi_k\, t} \left(\sum_{n=1}^{p} \frac{t^n}{n!}\, C_n |\xi|^{4n+2p_1+2} \right)^2 d\xi \ .
\end{aligned}$$

Consequently, from Lemma 4.1, for $|\gamma| \le k$ it follows that,

$$\begin{aligned}
\|\widetilde{I}_{\gamma *}(\cdot,t) - J_{\gamma^*}(\cdot,t)\|^2_{L^2(\mathbb{R}^N)} \ &\le \\
&\le \ C_p \sum_{n=1}^{p} t^{2n} \int_{B_\delta} |\xi|^{2|\gamma|+2(4n+2p_1+2)}\, e^{-2\sum\limits_{j,k} q_{jk}\xi_j\xi_k\, t}\, d\xi \\
&\sim \ C_p\, C_k \sum_{n=1}^{p} t^{-\frac{2|\gamma|+8n+4p_1+4+N}{2}+2n}
\end{aligned}$$

as $t \to +\infty$, where C_p is a positive constant.

Now, choosing $p_1 = p_1(\gamma, n)$ such that

$$2\, p_1 \ \ge \ k - |\gamma| - 2n - 1 \ ,$$

the fact that $p = p(k)$ implies

$$\|\widetilde{I}_{\gamma *}(\cdot,t) - J_{\gamma^*}(\cdot,t)\|^2_{L^2(\mathbb{R}^N)} \sim C_k\, t^{-\frac{2k+N+2}{2}}, \quad \text{as} \ \ t \to \infty \ . \ \blacksquare$$

Remark 5.2. Although the heat kernel is defined as a integral in $\mathbb{R}^N$ and in our case only in B_δ (see Lemma 5.7) we observe that the difference between these two integrals decay exponentially in $L^2(\mathbb{R}^N)$ due to Parseval's identity and the coercivity of (q_{jk}) in $\mathbb{R}^N$ (see (2.6)), i.e.

$$\left\| \int_{\mathbb{R}^N \setminus B_\delta} \xi^\beta \, e^{-\sum\limits_{j,k} q_{jk}\xi_j\xi_k\, t} \, e^{ix\cdot\xi} \, d\xi \right\|_{L^2(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N \setminus B_\delta} \left| \xi^\beta \, e^{-\sum\limits_{j,k} q_{jk}\xi_j\xi_k\, t} \right|^2 d\xi \right)^{1/2} \leq C\, e^{-\alpha\delta^2 t}$$

for $t \geq t_0$, where C is a positive constant which depends on $\beta \in \mathbb{N}^n$ and t_0 (t_0 a fixed positive number). Here $\alpha > 0$ is the constant of coercivity. □

6 – Proof of Theorem 2.1 with $\rho \equiv 1$

Let

$$H(x,t) = \sum_{|\alpha| \leq k} \tilde{c}_\alpha(x) \left[G_\alpha(x,t) + \sum_{n=1}^{p} \frac{(-t)^n}{n!} \sum_{m=0}^{p_1} \sum_{|\beta|=4+2m} c_{\beta,n}\, G_{\alpha+\beta}(x,t) \right]$$

where $\tilde{c}_\alpha(x) = \sum\limits_{\gamma \leq \alpha} d_\gamma(x) c_{\alpha-\gamma}$, $|\alpha| \leq k$ and

$$G_\alpha(x,t) = \int_{\mathbb{R}^N} \xi^\alpha \, e^{-\sum\limits_{j,k} q_{jk}\xi_j\xi_k\, t} \, e^{ix\cdot\xi} \, d\xi \ .$$

Let $u(x,t)$ be the solution of (1.1). Using the expression (5.1) for $u(x,t)$ and the definition of $I(x,t)$, given in (5.8), we obtain the estimate

$$\begin{aligned}
&\|u(\cdot,t) - H(\cdot,t)\|_{L^2(\mathbb{R}^N)} \leq \\
&\quad \leq \|u(\cdot,t) - I(\cdot,t)\|_{L^2(\mathbb{R}^N)} + \|I(\cdot,t) - H(\cdot,t)\|_{L^2(\mathbb{R}^N)} \\
&\quad \leq \left\| \sum_{m=2}^{\infty} \int_{Y'} \hat{\varphi}_0^m(\xi)\, e^{-\frac{\lambda_m(\xi)}{1+\lambda_m(\xi)} t}\, e^{ix\cdot\xi}\, \phi_m(x,\xi)\, d\xi \right\|_{L^2(\mathbb{R}^N)} \\
&\quad + \left\| \int_{Y'\setminus B_\delta} \hat{\varphi}_0^1(\xi)\, e^{-\frac{\lambda_1(\xi)}{1+\lambda_1(\xi)} t}\, e^{ix\cdot\xi}\, \phi_1(x,\xi)\, d\xi \right\|_{L^2(\mathbb{R}^N)} + \|I(\cdot,t) - H(\cdot,t)\|_{L^2(\mathbb{R}^N)} \qquad (6.1) \\
&\quad = \left(\sum_{m=2}^{\infty} \int_{Y'} |\hat{u}^m(\xi,t)|^2 \, d\xi \right)^{1/2} + \left(\int_{Y'\setminus B_\delta} |\hat{u}^1(\xi,t)|^2 \, d\xi \right)^{1/2} \\
&\quad + \|I(\cdot,t) - H(\cdot,t)\|_{L^2(\mathbb{R}^N)}
\end{aligned}$$

where we have used Parseval's Theorem for Bloch waves(Proposition 3.1).

Applying Lemma 5.2 and Lemma 5.3 it follows that

$$\|u(\cdot,t) - H(\cdot,t)\|_{L^2(\mathbb{R}^N)} \le 2\,\|\varphi\|_{L^2(\mathbb{R}^N)}\, e^{-\gamma_2 t} + \|I(x,t) - H(x,t)\|_{L^2(\mathbb{R}^N)}\,, \quad t>0\,, \tag{6.2}$$

where $\gamma_2 = 2\min\{\gamma,\gamma_1\}$, γ and γ_1 are the constants that appear in Lemma 5.2 and 5.3 respectively.

Now, from Lemma 5.4 we obtain

$$\begin{aligned} \|I(\cdot,t)-H(\cdot,t)\|_{L^2(\mathbb{R}^N)} &\le \|I(\cdot,t)-J(\cdot,t)\|_{L^2(\mathbb{R}^N)} + \|J(\cdot,t)-H(\cdot,t)\|_{L^2(\mathbb{R}^N)} \\ &\le C_k\, t^{-\frac{2k+2+N}{4}} + \|J(\cdot,t) - H(\cdot,t)\|_{L^2(\mathbb{R}^N)} \end{aligned} \tag{6.3}$$

as $t \to \infty$.

We also have

$$\begin{aligned} \|J(\cdot,t)-H(\cdot,t)\|_{L^2(\mathbb{R}^N)} &= \Big\| \sum_{|\alpha|\le k} c_\alpha\, J_\alpha(\cdot,t) - H(\cdot,t) \Big\|_{L^2(\mathbb{R}^N)} \\ &\le \Big\| \sum_{|\alpha|\le k} c_\alpha \Big(J_\alpha(\cdot,t) - \sum_{|\gamma|\le k-|\alpha|} d_\gamma(\cdot)\, I_{\gamma+\alpha}(\cdot,t) \Big) \Big\|_{L^2(\mathbb{R}^N)} \\ &\quad + \Big\| \sum_{|\alpha|\le k} c_\alpha \sum_{|\gamma|\le k-|\alpha|} d_\gamma(\cdot)\, I_{\gamma+\alpha}(\cdot,t) - H(\cdot,t) \Big\|_{L^2(\mathbb{R}^N)} \\ &\le C_k \sum_{|\alpha|\le k} \Big\| J_\alpha(\cdot,t) - \sum_{|\gamma|\le k-|\alpha|} d_\gamma(\cdot)\, I_{\gamma+\alpha}(\cdot,t) \Big\|_{L^2(\mathbb{R}^N)} \\ &\quad + \Big\| \sum_{|\alpha|\le k} c_\alpha \sum_{|\gamma|\le k-|\alpha|} d_\gamma(\cdot)\, I_{\gamma+\alpha}(\cdot,t) - H(\cdot,t) \Big\|_{L^2(\mathbb{R}^n)}\,, \end{aligned}$$

where $C_k = \sup_{|\alpha|\le k} |c_\alpha|$.

Thus, from Lemma 5.5 it results

$$\begin{aligned} &\|J(\cdot,t) - H(\cdot,t)\|_{L^2(\mathbb{R}^n)} \le \\ &\qquad \le C_k\, t^{-\frac{2k+2+N}{4}} + \Big\| \sum_{|\alpha|\le k} c_\alpha \sum_{|\gamma|\le k-|\alpha|} d_\gamma(\cdot)\, I_{\gamma+\alpha}(\cdot,t) - H(x,t) \Big\|_{L^2(\mathbb{R}^N)}\,. \end{aligned} \tag{6.4}$$

Here, we observe that

$$\sum_{|\alpha|\le k} c_\alpha \sum_{|\gamma|\le k-|\alpha|} d_\gamma(x)\, I_{\gamma+\alpha}(x,t) \;=\; \sum_{|\alpha|\le k} \tilde{c}_\alpha(x)\, I_\alpha(x,t) \tag{6.5}$$

where $\tilde{c}_\alpha(x)$ are defined in the beginning of this section.

Then

$$\begin{aligned}
&\left\| \sum_{|\alpha|\le k} c_\alpha \sum_{|\gamma|\le k-|\alpha|} d_\gamma(\cdot)\, I_{\gamma+\alpha}(\cdot,t) - H(x,t) \right\|_{L^2(\mathbb{R}^N)} = \\
&\quad = \left\| \sum_{|\alpha|\le k} \tilde{c}_\alpha(\cdot)\, I_\alpha(\cdot,t) - H(\cdot,t) \right\|_{L^2(\mathbb{R}^N)} \\
&\quad \le \left\| \sum_{\alpha|\le k} \tilde{c}_\alpha(\cdot) \left[I_\alpha(\cdot,t) - \widetilde{I}_\alpha(\cdot,t) \right] \right\|_{L^2(\mathbb{R}^N)} + \left\| \sum_{|\alpha|\le k} \tilde{c}_\alpha(\cdot) \widetilde{I}_\alpha(\cdot,t) - H(\cdot,t) \right\|_{L^2(\mathbb{R}^N)}
\end{aligned}$$

where $\widetilde{I}_\alpha(\cdot,t)$ is defined in (5.27) and $\tilde{c}_\alpha(x) \in L^\infty_{\neq}(Y)$ because $d_\gamma(x) \in L^\infty_{\neq}(Y)$.

Therefore, Lemma 5.6 implies that

$$\begin{aligned}
&\left\| \sum_{|\alpha|\le k} c_\alpha \sum_{|\gamma|\le k-|\alpha|} d_\gamma(\cdot)\, I_{\gamma+\alpha}(\cdot,t) - H(\cdot,t) \right\|_{L^2(\mathbb{R}^N)} \le \\
&\quad \le\; C_k \sum_{|\alpha|\le k} \left\| I_\alpha(\cdot,t) - \widetilde{I}_\alpha(\cdot,t) \right\|_{L^2(\mathbb{R}^N)} + \left\| \sum_{|\alpha|\le k} \tilde{c}_\alpha(\cdot) \widetilde{I}_\alpha(\cdot,t) - H(\cdot,t) \right\|_{L^2(\mathbb{R}^N)} \\
&\quad \le\; C_k\, t^{-\frac{2k+N+2}{4}} + \left\| \sum_{|\alpha|\le k} \tilde{c}_\alpha(\cdot)\, \widetilde{I}_\alpha(\cdot,t) - H(\cdot,t) \right\|_{L^2(\mathbb{R}^N)}
\end{aligned} \tag{6.6}$$

as $t \to \infty$.

On the other hand

$$\begin{aligned}
&\sum_{|\alpha|\le k} \tilde{c}_\alpha(\cdot)\, \widetilde{I}_\alpha(\cdot,t) - H(\cdot,t) \;= \\
&\qquad = \; - \sum_{|\alpha|\le k} \tilde{c}_\alpha(\cdot) \int_{\mathbb{R}^n \setminus B_\delta} \xi^\alpha\, e^{-\sum\limits_{j,k} q_{jk}\xi_j\xi_k\, t}\; e^{ix\cdot\xi}\, d\xi \\
&\qquad\quad + \sum_{|\alpha|\le k} \tilde{c}_\alpha(\cdot)\, \widetilde{I}_{\alpha *}(\cdot,t) - H_1(\cdot,t)
\end{aligned} \tag{6.7}$$

where

$$(6.8)\qquad H_1(x,t) \;=\; \sum_{|\alpha|\le k} \tilde c_\alpha(\,x\,)\,\sum_{n=1}^{p}\frac{(-t)^n}{n!}\,\sum_{m=0}^{p_1}\,\sum_{|\beta|=4+2m} c_{\beta,n}\,G_{\alpha+\beta}(x,t)\ .$$

From Lemma 5.7 and Remark 5.2 it results

$$\begin{aligned}
&\Big\|\sum_{|\alpha|\le k}\tilde c_\alpha(\cdot)\,\widetilde I_\alpha(\cdot,t) - H(\cdot,t)\Big\|_{L^2(\mathbb{R}^N)} \;\le \\
&\le\; C\,e^{-\alpha\delta^2 t} + \Big\|\sum_{|\alpha|\le k}\tilde c_\alpha(\cdot)\,\widetilde I_{\alpha*}(\cdot,t) - H_1(\cdot,t)\Big\|_{L^2(\mathbb{R}^N)} \\
&\le\; C\,e^{-\alpha\delta^2 t} + \Big\|\sum_{|\alpha|\le k}\tilde c_\alpha(\cdot)\,\Big[J_{\alpha^*}(\cdot,t) - \widetilde I_{\alpha*}(\cdot,t)\Big]\Big\|_{L^2(\mathbb{R}^N)} \\
&\quad + \Big\|\sum_{|\alpha|\le k}\tilde c_\alpha(\cdot)\,J_{\alpha^*}(\cdot,t) - H_1(\cdot,t)\Big\|_{L^2(\mathbb{R}^N)} \\
&\le\; C_k\Big[e^{-\alpha\delta^2 t} + t^{-\frac{2k+N+2}{4}}\Big] \\
&\quad + \Big\|\sum_{|\alpha|\le k}\int_{\mathbb{R}^N\setminus B_\delta}\tilde c_\alpha(x)\,\xi^\alpha\,e^{-\sum\limits_{j,k} q_{jk}\xi_j\xi_k\,t}\Big(\sum_{n=1}^{p}\frac{(-t)^n}{n!}\sum_{m=0}^{p_1}\sum_{|\beta|=4+2m} c_{\beta,n}\,\xi^\beta\Big)e^{ix\cdot\xi}\,d\xi\Big\|_{L^2(\mathbb{R}^N)}
\end{aligned}$$

where we have used the definition of J_{α^*} and H_1, and the fact that $\tilde c_\alpha(x)\in L^\infty_{\neq}(Y)$.

Consequently,

$$(6.9)\qquad \Big\|\sum_{|\alpha|\le k}\tilde c_\alpha(x)\,\widetilde I_\alpha(\cdot,t) - H(\cdot,t)\Big\| \;\le\; C_k\,t^{-\frac{2k+2+N}{4}} + \|F(\cdot,t)\|_{L^2(\mathbb{R}^N)}$$

where C_k is a positive constant, and

$$F(x,t) = \sum_{|\alpha|\le k}\tilde c_\alpha(x)\int_{\mathbb{R}^N\setminus B_\delta}\xi^\alpha\,e^{-\sum\limits_{j,k} q_{jk}\xi_j\xi_k\,t}\Bigg(\sum_{n=1}^{p}\frac{(-t)^n}{n!}\sum_{m=0}^{p_1}\sum_{|\beta|=4+2m} c_{\beta,n}\,\xi^\beta\Bigg)e^{ix\cdot\xi}\,d\xi\ .$$

Finally we estimate $F(x,t)$. We have

$$
\begin{aligned}
\|F(\cdot,t)\|_{L^2(\mathbb{R}^N)} \;&\le\\
&\le\; C_k\, t^p \sum_{|\alpha|\le k} \sum_{n=1}^{p} \left\| \int_{\mathbb{R}^N\setminus B_\delta} \xi^\alpha\, e^{-\sum\limits_{j,k} q_{jk}\xi_j\xi_k\, t} \left(\sum_{m=0}^{p_1} \sum_{|\beta|=4+2m} c_{\beta,n}\, \xi^\beta \right) e^{ix\cdot\xi}\, d\xi \right\|_{L^2(\mathbb{R}^N)}\\
&\le\; C_k\, t^p \sum_{|\alpha|\le k} \sum_{n=1}^{p} \sum_{m=0}^{p_1} \sum_{|\beta|=4+2m} \left(\int_{\mathbb{R}^N\setminus B_\delta} \left| \xi^{\alpha+\beta}\, e^{-\sum\limits_{j,k} q_{jk}\xi_j\xi_k\, t} \right|^2 d\xi \right)^{1/2}
\end{aligned}
$$

due to Parseval's Theorem, where C_k is another positive constant which depends on k.

Then, using Remark 5.2 we get

$$
\begin{aligned}
&\|F(\cdot,t)\|_{L^2(\mathbb{R}^N)} \;\le\\
&\qquad\le\; C_k\, t^p \sum_{|\alpha|\le k} \sum_{k=1}^{p} \sum_{m=0}^{p_1} \sum_{|\beta|=4+2m} \left(\int_{\mathbb{R}^N\setminus B_\delta} |\xi|^{2|\alpha|+2|\beta|}\, e^{-2\sum\limits_{j,k} q_{jk}\xi_j\xi_k\, t} d\xi \right)^{\frac{1}{2}}\\
&\qquad\le\; C_k\, t^p\, e^{-\alpha\delta^2 t}, \qquad t>0\,,
\end{aligned}
\tag{6.10}
$$

where we have indicated by C_k different positive constants.

Returning to (6.2) and using triangular inequality associated with estimates (6.3) up to (6.10), we obtain the existence of a positive constant $C_k = C(\varphi,k)$ (φ is the initial data) such that

$$
\|u(\cdot,t) - H(\cdot,t)\| \;\le\; C_k\, t^{-\frac{2k+2+N}{4}}
$$

as $t\to+\infty$.

7 – The general case $\rho=\rho(x)$

Theorem 2.1 is proved following the same steps of Sections 5 and 6 for the case $\rho\equiv 1$. However, the Bloch wave decomposition used in these sections cannot be applied for the problem (1.1), due to variable density ρ. Consequently, we need to introduce a different spectral problem.

Given $\xi \in Y'$, we consider the spectral problem of finding numbers $\lambda = \lambda(\xi) \in \mathbb{R}$ and functions $\psi = \psi(x;\xi)$ (not identically zero) such that

$$\begin{cases} A\psi(\cdot,\xi) = \lambda(\xi)\,\psi(\cdot,\xi)\rho(\cdot) \quad \text{in } \mathbb{R}^N\,, \\ \psi(\cdot,\xi) \ \text{ is } \ (\xi,Y)\text{-periodic, i.e.} \\ \psi(x+2m;\,\xi) = e^{2\pi i m\cdot\xi}\,\psi(x)\,, \qquad \forall\, m \in Z^N, \quad x \in \mathbb{R}^N\,, \end{cases} \tag{7.1}$$

where A is the elliptic operator in divergence form defined in (3.2) and ρ satisfies (1.2). If we consider $\psi(x;\xi) = e^{ix\cdot\xi}\,\phi(x;\xi)$, the variational formulation obtained for (7.1) for any $\varphi \in H^1_\#(Y)$ is given by

$$\langle A(\xi)\phi,\varphi\rangle = \int_Y a_{k\ell}(x)\left(\frac{\partial\phi}{\partial x_k} + i\,\xi_k\,\phi\right)\overline{\left(\frac{\partial\phi}{\partial x_\ell} + i\,\xi_\ell\,\phi\right)}\,dx = \lambda(\xi)\int_Y \phi\,\bar\varphi\,\rho(x)\,dx\,.$$

Since the operator associated with (7.1) is uniformly elliptic and self-adjoint, defined in a bounded domain, it is known (see Refs. 8 and 9) that the above spectral problem admits a discrete sequence of eigenvalues with the following properties:

$$\begin{cases} 0 \le \lambda_1(\xi) \le \cdots \le \lambda_m(\xi) \le \cdots \to \infty\,, \\ \lambda_m(\xi) \ \text{ is a Lipschitz function of } \ \xi \in Y'\,, \qquad \forall\, m \ge 1\,. \end{cases} \tag{7.2}$$

Besides, the corresponding eigenfunctions are such that $\psi_m(\cdot,\xi) = e^{i\xi\cdot x}\phi_m(\cdot,\xi)$, where the functions in $L^2_{\mathrm{loc}}(\mathbb{R}^N;\rho(x)\,dx)$, i.e.

$$\int_Y \phi_m\,\bar\phi_n\,\rho(x)\,dx = \delta_{mn} \qquad \text{(Kronecker's delta)}\,.$$

The eigenfunctions $\psi_m(\cdot,\xi)$ and $\phi_m(\cdot,\xi)$ are (ξ,Y)-periodic and Y-periodic, respectively. Moreover, as a consequence of the min-max principle (see Ref. 9) we have

$$\lambda_2(\xi) \ge \frac{\lambda_2^{(N)}}{\rho_1} > 0\,, \qquad \forall\, \xi \in Y'\,,$$

where $\lambda_2^{(N)}$ is the second eigenvalue of A in the cell Y with Neumann boundary condition on ∂Y for $\rho \equiv 1$ and ρ_1 is defined in (1.3).

Now, with the orthonormal basis of Bloch waves $\{e^{ix\cdot\xi}\,\phi_m(x;\xi) : m \ge 1, \xi \in Y'\}$, we have a similar Bloch wave decomposition as in Proposition 3.1.

Proposition 7.1. *Let $g \in L^2(\mathbb{R}^N)$. The m-th Bloch coefficient of g is defined as follows:*

$$\hat g_m(\xi) = \int_{\mathbb{R}^N} g(x)\,e^{-ix\cdot\xi}\,\bar\phi_m(x;\xi)\,\rho(x)\,dx\,, \qquad \forall\, m \ge 1, \quad \xi \in Y'\,.$$

Then the following inverse formula holds:

$$g(x) = \int_{Y'} \sum_{m=1}^{\infty} \hat{g}_m(\xi)\, e^{ix\cdot\xi}\, \phi_m(x;\xi)\; d\xi \;.$$

Furthermore, we have Parseval's identity:

$$\|g\|^2_{L^2(\rho)} = \int_{\mathbb{R}^N} |g(x)|^2 \rho(x)\, dx = \int_{Y'} \sum_{m=1}^{\infty} |\hat{g}_m(\xi)|^2 \; d\xi \;.$$

Finally, for all g in the domain of A, we have

$$Ag(x) = \rho(x) \int_{Y'} \sum_{m=1}^{\infty} \lambda_m(\xi)\, \hat{g}_m(\xi)\, e^{ix\cdot\xi}\, \phi_m(x.\xi)\; d\xi \;.$$

Using Proposition 7.1, Eq. (1.1) can be written as follows:

$$\int_{Y'} \sum_{m=1}^{\infty} \Big(\hat{u}^m_t(\xi,t) + \lambda_m(\xi)\, \hat{u}^m_t(\xi,t) + \lambda_m(\xi)\, \hat{u}^m(\xi,t)\Big)\, e^{ix\cdot\xi}\, \phi_m(x;\xi)\, \rho(x)\; d\xi = 0 \;.$$

Since $\{e^{ix\cdot\xi}\, \phi_m(x;\xi) \colon m \geq 1,\ \xi \in Y'\}$ form an orthonormal basis, this is equivalent to the family of the differential equations

$$\hat{u}^m_t(\xi,t) + \lambda_m(\xi)\, \hat{u}^m_t(\xi,t) + \lambda_m(\xi)\, \hat{u}^m(\xi,t) = 0\,, \quad \forall\, m \geq 1,\ \ \xi \in Y' \;.$$

Once the differential equations are solved, (1.1) is solved as in (5.3) and Lemma 5.1 holds. Then, the developments of Sections 5 and 6 apply with minor changes and Theorem 2.1 holds.

In order to understand the type of changes that the variable density $\rho(\cdot)$ causes in the fundamental solution, we are going to study the Taylor expansion of the first Bloch eigenvalue and eigenvector. For a more complete analysis the reader is referred to [6].

We observe that

$$\lambda_1(0) = 0 \quad \text{and} \quad \phi(x;0) = (2\pi)^{-N/2}\, \bar{\rho}^{\,-1/2} \;,$$

with $\bar{\rho}$ defined in (2.6) and we consider the equation

$$A(\xi)\, \phi_1(\,\cdot\,;\xi) = \lambda_2(\xi)\, \rho(\cdot)\, \phi_1(\,\cdot\,;\xi) \;, \tag{7.3}$$

where

$$A(\xi) = -\left(\frac{\partial}{\partial x_k} + i\,\xi_k\right)\left[a_{k\ell}(x)\left(\frac{\partial}{\partial x_\ell} + i\,\xi_\ell\right)\right] .$$

If we differentiate equation (7.3) with respect to ξ_k with $k = 1, ..., N$ and if we take scalar product with $\phi_1(x;\xi)$ in $\xi = 0$, we get

$$\partial_k \lambda_1(0) = 0 \ .$$

Furthermore, if we observe that

$$A\, \partial_k \phi_1(\,\cdot\,;0) \;=\; i(2\pi)^{-N/2}\, \bar{\rho}^{-1/2}\, \frac{\partial a_{k\ell}}{\partial x_\ell}$$

then

$$\partial_k \phi_1(x;0) \;=\; i(2\pi)^{-N/2}\, \bar{\rho}^{-1/2}\, \chi^k(x) \ ,$$

where χ^k is the correctors function in homogenization theory, solution of the cell problem

$$\text{(7.4)} \qquad \begin{cases} A\chi^k = \dfrac{\partial a_{k\ell}}{\partial y_\ell} \quad \text{in } Y \ , \\ \chi^k \in H^1_{\#}(Y)\,, \quad \dfrac{1}{|Y|}\displaystyle\int_Y \chi^k\, dy = 0 \ . \end{cases}$$

If we differentiate again the eigenvalue equation, we have that

$$\partial^2_{k\ell}\, \lambda_1(0) \;=\; \frac{1}{\bar{\rho}}\, \frac{1}{(2\pi)^N} \int_Y \left(2\, a_{k\ell} + a_{km}\, \frac{\partial \chi^\ell}{\partial x_m} + a_{m\ell}\, \frac{\partial \chi^k}{\partial x_m} \right) dx \;=\; \frac{2\, q_{k\ell}}{\bar{\rho}} \ ,$$

with $q_{k\ell}$ the homogenized coefficients as in previous section (see [6]).

Since $f(\xi)$ is defined in Proposition 3.3 and due to the analysis above for the eigenvalue λ_1, we obtain

$$\begin{aligned} &f(0) = \partial_k f(0) = 0\,, \qquad k = 1, ..., N \ , \\ &\partial^2_{k\ell} f(0) = \frac{2\, q_{k\ell}}{\bar{\rho}}\,, \qquad k, \ell = 1, ..., N \ . \end{aligned}$$

Then, for all $\xi \in B_\delta$ we have

$$e^{-f(\xi)t} \sim e^{-\frac{1}{\bar{\rho}} \sum\limits_{k,\ell} q_{k\ell} \xi_k \xi_\ell t} \ .$$

8 – Analysis of the periodic functions and constants entering in the asymptotic expansion

To finish this work we describe the periodic functions $c_\alpha(\cdot)$ and constants $c_{\beta,n}$, where $\beta \in (\mathbb{N} \cup \{0\})^N$, that appear in the statement of Theorem 2.1.

Computation of $c_\alpha(\cdot)$. According to (5.9), (5.17) and (6.5),

$$c_\alpha(x) = \sum_{\beta \leq \alpha} \frac{(2\pi)^N}{(\alpha-\beta)!\,\beta!}\, \partial^{\alpha-\beta} \phi_1(x;0)\, \partial^\beta \hat{\varphi}_0^1(0) \,, \tag{8.1}$$

and for the first Bloch coefficient of the initial data

$$\partial^\gamma \hat{\varphi}_0^1(0) = \int_{\mathbb{R}^N} \varphi_0(x) \sum_{\alpha \leq \gamma} \Big[(-i)^{|\gamma-\alpha|}\, x^{\gamma-\alpha}\, \partial^\alpha \phi_1(x;0)\Big]\, dx \,.$$

We observe that the higher order derivatives of λ_1 and ϕ_1 in $\xi = 0$ may be computed as in the previous section.

First, note that

$$c_0(x) = (2\pi)^N \phi_1(x;0)\, \hat{\varphi}_0^1(0)$$

and since $\phi_1(x;0) = (2\pi)^{-N/2}\, \bar{\rho}^{-1/2}$, it follows that c_0 is constant. Furthermore, according to Proposition 7.1, we have

$$\hat{\varphi}_0^1(0) = (2\pi)^{-N/2}\, \bar{\rho}^{-1/2} \int_{\mathbb{R}^N} \varphi(x)\, \rho(x)\, dx = (2\pi)^{-N/2}\, \bar{\rho}^{-1/2}\, m_\rho(\varphi) \,.$$

Thus, the first term of the asymptotic expansion of the solution of (1.1) turns out to be a constant and more precisely

$$c_0 := c_0(x) = \frac{1}{\bar{\rho}}\, m_\rho(\varphi) \,.$$

Computation of $c_{\beta,n}$. We recall that the constants $c_{\beta,n}$ were defined in (5.33) (and lemma 5.7) and satisfy

$$\sum_{m=1}^{\infty} \sum_{|\beta|=4n+2m} \xi^\beta\, c_{\beta,n} = (-1)^n \left(\sum_{m=0}^{\infty} \sum_{|\beta|=4+2m} \frac{1}{\beta!}\, \xi^\beta\, \partial^\beta f(0) \right)^n$$

where $f(\xi)$ was defined in Proposition 3.3. This shows that the constants $\partial^\beta f(0)$ depend on the derivatives of λ_1 at $\xi = 0$, computed in Section 7.

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