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OPTIMAL ENERGY DECAY RATE OF COUPLED WAVE EQUATIONS

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Abstract: We consider a system of coupled wave equations subject to positive viscous damping. Under the assumption that the damping function is of bounded variations, we give the asymptotic expansion of eigenvalues and eigenfunctions of the infinitesimal generator of the associated semigroup. Moreover, we prove that the eigenfunctions form a Riesz basis in the energy space.

1 – Introduction

In this paper we consider a system of coupled wave equations in the presence of viscous damping :

$$\begin{cases} u_{tt} - u_{xx} + 2\,a\,u_t + \alpha(u-v) \;=\; 0\,, & 0<x<1, \quad t>0\;, \\ v_{tt} - v_{xx} + 2\,a\,v_t + \alpha(v-u) \;=\; 0\,, & 0<x<1, \quad t>0\;, \\ u(0,t) = u(1,t) = 0\,, & t>0\;, \\ v(0,t) = v(1,t) = 0\,, & t>0\;, \end{cases} \tag{1.1}$$

where $a, \alpha \in L^\infty(0,1)$ are positive functions.

Let $\mathcal{H} = H_0^1(0,1) \times L^2(0,1) \times H_0^1(0,1) \times L^2(0,1)$. We next define the linear unbounded operator $\mathcal{A}$ by

$$\mathcal{A} = (u,z,v,w) = \Big(z,\; u_{xx} - 2\,a\,z - \alpha(u-v),\; w,\; v_{xx} - 2\,a\,w - \alpha(v-u)\Big)\;, \tag{1.2}$$

$$D(\mathcal{A}) = \mathcal{V}{\times}H_0^1(0,1){\times}\mathcal{V}{\times}H_0^1(0,1)\;, \tag{1.3}$$

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where we have put $\mathcal{V} := H_0^1(0,1) \cap H^2(0,1)$. Setting $U = (u, u_t, v, v_t)$, we transform the system (1.1) into an evolutionary equation

$$U_t = \mathcal{A}U\,, \qquad U(0) = U_0 \in \mathcal{H}\ . \tag{1.4}$$

we can prove easily that the operator $\mathcal{A}$ generates a C_0-semigroup (see Pazy [16]). Moreover defining the energy of the system by:

$$E(t) \;=\; \frac{1}{2}\int_0^1 u_x^2 + u_t^2 + v_x^2 + v_t^2 + \alpha(u-v)^2\ dx\ , \tag{1.5}$$

we find that

$$\frac{d}{dt}E(t) \;=\; -2\int_0^1 a(u_t^2 + v_t^2)\,dt \;\le\; 0\ . \tag{1.6}$$

Assume that a is nonnegative and strictly positive on some subinterval we can easily prove (see[1]) that there exist constants $C > 0$ and $\omega < 0$ such that the following exponential decay rate holds:

$$E(t) \le CE(0)\exp(2\,\omega t)\,, \qquad \forall\, t \ge 0\ . \tag{1.7}$$

The exponential stability of the system (1.1) has been established by Najafi et al [15] in the case of linear boundary feedback and by Komornik–Rao [10] in the case of nonlinear boundary feedback.

In this work, we will determine the optimal energy decay rate of the system (1.1). More precisely, denoting by $\omega(a)$ the supremum of ω satisfying (1.7), and by $\mu(a)$ the minimum of the real part of eigenvalues of $\mathcal{A}$, we will establish the relation $\mu(a) = \omega(a)$ for the coefficient a being of bounded variations. To this end, we will give the asymptotic expansion of the eigenvalues and prove that the system of eigenvectors of the operator $\mathcal{A}$ constitutes a Riesz basis in the energy space $\mathcal{H}$.

In section 2 we prove that the spectrum of the system (1.1) is the union of the spectrum of the systems:

$$\begin{cases} u_{xx} - (\lambda^2 + 2\,a\,\lambda + 2\,\alpha)\,u \;=\; 0\ , \\ u(0) = u(1) = 0\ , \end{cases} \tag{1.8}$$

and

$$\begin{cases} v_{xx} - (\mu^2 + 2\,a\,\mu)\,v \;=\; 0\ , \\ v(0) = v(1) = 0\ . \end{cases} \tag{1.9}$$

Note that the system (1.9) is well studied by Cox and Zuazua in [4]. In this work, we will apply a method used by Rao in [18] to the system (1.8).

This approach consists in constructing, without any a priori ansatz, an explicit approximation of the characteristic equation of the underlying system. In this way, we find the asymptotic form of the eigenvalues of (1.1). In section 3, we construct the root system of the system (1.1) and we prove that root vectors of the operator $\mathcal{A}$ constitue a Riesz basis in $\mathcal{H}$, therefore we identify the optimal decay rate of energy $\omega(a)$ with the supremum of the real part of the eigenvalues of the system (1.1).

The method that is used in this work can be adapted to the problem of indefinite damping:

$$\begin{cases} u_{tt} - u_{xx} + 2\,\varepsilon\, a\, u_t + \alpha(u - v) \;=\; 0\,, & 0 < x < 1, \quad t > 0\;, \\ v_{tt} - v_{xx} + 2\,\varepsilon\, b\, v_t + \alpha(v - u) \;=\; 0\,, & 0 < x < 1, \quad t > 0\;, \\ u(0) = u(1) = 0\;, \\ v(0) = v(1) = 0\;, \end{cases} \tag{1.10}$$

where a, b are functions of indefinite sign and $\varepsilon > 0$ is a small parameter. In fact in the case $a = b$ the determination of the spectrum of (1.10) can be reduced to that one of the following system:

$$\begin{cases} \varphi_{xx} = \varphi_{tt} + 2\, a\, \varepsilon\, \lambda\, \varphi_t + 2\, \alpha\, \varphi \;=\; 0\;, \\ \varphi(0) = \varphi(1) = 0\;. \end{cases} \tag{1.11}$$

In the case where $\alpha = 0$, it was proved that the system (1.11) is exponentially uniformly stable for $\varepsilon > 0$ small enough if a is of bounded variation and is "more positive then negative" (see Freitas–Zuazua [6]). Recently, this result was improved to the system (1.11) with an arbitrary function $\alpha \in L^\infty(0,1)$ by Benaddi–Rao [18] using a new asymptotic expansion of eigenfunction which take into account the potential term $\alpha\,\varphi$.

In Liu et al [12] we can find a general result on the stability of nondissipative semigroups which is based on the perturbation theory (Kato [9]) and the characteristic condition of the uniform stability of semigroups (Huang [8], Prüss, [17]). It seems interesting to adapt their approach to the system (1.10) with $a \neq b$.

2 – Asymptotic analysis of the spectrum of $\mathcal{A}$

For any $U_1 = (u_1, z_1, v_1, w_1) \in \mathcal{H}$, $U_2 = (u_2, z_2, v_2, w_2) \in \mathcal{H}$, we defind the inner product in the space $\mathcal{H}$ by setting:

$$\langle U_1, U_2\rangle \;=\; \int_0^1 u_{1x}\, \bar{u}_{2x} + z_1 \bar{z}_2 + v_{1x}\, \bar{v}_{2x} + w_1 \bar{w}_2 + \alpha(u_1 - v_1)\,(\bar{u}_2 - \bar{v}_2)\; dx\;, \tag{2.1}$$

and we consider the following eigenvalue problem

$$\begin{cases} \lambda^2\, u - u_{xx} + 2\, a\, \lambda\, u + \alpha(u - v) \;=\; 0 \;, \\ \lambda^2\, v - v_{xx} + 2\, a\, \lambda\, v + \alpha(v - u) \;=\; 0 \;, \\ u(0) = u(1) = 0 \;, \\ v(0) = v(1) = 0 \;. \end{cases} \tag{2.2}$$

We first remark that the eigenvalues of (2.2) are the eigenvalues of one of the systems (1.8) or (1.9). More precisely, putting $\varphi = u - v$ and $\phi = u + v$, then we have φ is solution of the problem (1.8), and ϕ is solution of the problem (1.9).

Proposition 2.1. *Let $\alpha_0, a_1 \ge 0$ and $a_2 \ge 0$. Let α and a be two functions in $L^\infty(0,1)$ such that $\forall\, x \in [0,1]$, $a_1 \le a(x) \le a_2 < \infty$ and $\alpha > \alpha_0 > 0$. Then the complex part of the spectrum of (1.8) is symetric about the real axis and is contained in*

$$\mathcal{C} \;=\; \Big\{\lambda \in \mathbb{C}\colon\; |\lambda| \ge \sqrt{\pi^2 + 2\,\alpha_0}; \;\; -a_2 \le \operatorname{Re}\lambda \le -a_1\Big\} \;. \tag{2.3}$$

A necessary condition for the existence of real eigenvalue is:

$$a_2 \ge \sqrt{\pi^2 + 2\,\alpha_0} \;. \tag{2.4}$$

In that case the real eigenvalues λ_n are contained in the interval:

$$-\,a_2 - \sqrt{a_2^2 - \pi^2 - 2\,\alpha_0} \;\le\; \lambda_n \;\le\; -a_1 + \sqrt{a_2^2 - \pi^2 - 2\,\alpha_0} \;. \tag{2.5}$$

Proof: Let λ_n be an eigenvalue associated to the eigenfunction u_n. Then, we have

$$\begin{cases} u_{nxx} - (\lambda_n^2 + 2\, a\, \lambda_n + 2\,\alpha)\, u_n \;=\; 0 \;, \\ u_n(0) = u_n(1) = 0 \;. \end{cases} \tag{2.6}$$

Multiplying (2.6) by u_n we obtain:

$$\lambda_n^2 \int_0^1 |u_n|^2\, dx \,+\, 2\,\lambda_n \int_0^1 a|u_n|^2\, dx + \int_0^1 |u_{nx}|^2\, dx \,+\, 2 \int_0^1 \alpha |u_n|^2\, dx \;=\; 0 \;. \tag{2.7}$$

Hence

$$\lambda_n = \frac{-\displaystyle\int_0^1 a|u_n|^2\, dx \pm \sqrt{\left(\int_0^1 a|u_n|^2\right)^2 dx - \int_0^1 |u_n|^2\, dx \left(\int_0^1 |u_{nx}|^2 + 2\,\alpha \int_0^1 |u_n|^2\, dx\right)}}{\displaystyle\int_0^1 |u_n|^2\, dx} \,. \tag{2.8}$$

If λ_n is a complex eigenvalue then we have:

$$\operatorname{Re}\lambda_n = -\frac{\displaystyle\int_0^1 a|u_n|^2\,dx}{\displaystyle\int_0^1 |u_n|^2\,dx} \quad \text{and} \quad -a_2 \le \operatorname{Re}\lambda_n \le -a_1\ . \tag{2.9}$$

Furthermore we have

$$(\operatorname{Re}\lambda_n)^2 + (\operatorname{Im}\lambda_n)^2 = \frac{\displaystyle\int_0^1 |u_{nx}|^2 + 2\int_0^1 \alpha|u_n|^2\,dx}{\displaystyle\int_0^1 |u_n|^2\,dx}\ . \tag{2.10}$$

By Poincaré's inequality, we have:

$$|\lambda_n|^2 = (\operatorname{Re}\lambda_n)^2 + (\operatorname{Im}\lambda_n)^2 \ge \pi^2 + 2\,\alpha_0\ . \tag{2.11}$$

If λ_n is a real eigenvalue, we have

$$0 \le \left(\frac{\displaystyle\int_0^1 a|u_n|^2}{\displaystyle\int_0^1 |u_n|^2\,dx}\right)^2 - \frac{\displaystyle\int_0^1 |u_{nx}|^2\,dx}{\displaystyle\int_0^1 |u_n|^2\,dx} - 2\,\alpha \le a_2^2 - \pi^2 - 2\,\alpha_0\ . \tag{2.12}$$

This gives (2.4). The proof is complete. ∎

Now, we carry out the study of the high frequencies of the problem (2.6). We will use a method used in Rao [18]. We denote by $BV(0,1)$ the set of functions of bounded variations. We consider the following initial value problem

$$\begin{cases} y_{xx} - (\lambda^2 + 2\,a\,\lambda + 2\,\alpha)\,y = 0\ , \\ y(0,\lambda) = 0\,, \quad y_x(0,\lambda) = 1\ . \end{cases} \tag{2.13}$$

We have the following result:

Proposition 2.2. *Let $a \in BV(0,1)$ and $y(x,\lambda)$ the solution of the problem (2.13). Then for all $\lambda \in \mathcal{C}$, sufficienly large, we have:*

$$\left| y(x,\lambda) - \frac{\sinh\Big(\lambda\,x + \displaystyle\int_0^x a(s)\,ds\Big)}{\lambda} \right| \le \frac{C_0}{|\lambda|^2}\ , \tag{2.14}$$

$$\left| y_x(x,\lambda) - \cosh\Big(\lambda\,x + \int_0^x a(s)\,ds\Big) \right| \le \frac{C_0}{|\lambda|} \tag{2.15}$$

where $C_0 > 0$ is a constant independent of λ.

Proof: By the theory of ordinary differential equations (see Naimark [14]), we know that $y(x,\lambda)$ is analytic with respect to λ. Furthermore λ_n is an eigenvalue of (1.9) if and only if λ_n is a root of $y(1,\lambda)=0$, and its algebraic multiplicity is the nullity order of λ_n as a zero of the function $\lambda \to y(1,\lambda)$.

Let $z(x,\lambda) = \dfrac{1}{\lambda}\sinh \lambda x$ be the solution of the undamped initial value problem ($a \equiv 0$). By the variation of constants formula we have:

$$y(x,\lambda) \;=\; z + \int_0^x 2\,a(s)\,\lambda\,y(s)\,z(x-s)\;ds\;. \tag{2.16}$$

Hence

$$y_x(x,\lambda) \;=\; z_x + \int_0^x 2\,a(s)\,\lambda\,y(s)\,z_x(x-s)\;ds\;. \tag{2.17}$$

Since $|\sinh \lambda x| \le \cosh |a_2| := C_1$ and $|\cosh \lambda x| \le C_1$, thanks to Gronwall's inequality, we deduce that

$$|y(x,\lambda)| \;\le\; \frac{C_1}{|\lambda|}\,\exp\Bigl(2\,C_1 \int_0^1 |a(s)|\,ds\Bigr) \;:=\; \frac{C_2}{|\lambda|}\;. \tag{2.18}$$

Inserting (2.18) into (2.17) we conclude that

$$|y_x(x,\lambda)| \;\le\; C_1 + C_1 C_2 \int_0^1 2\,|a(s)|\,ds \;:=\; C_3\;. \tag{2.19}$$

Now we construct an approximate solution of the problem (2.13). Using an idea of Rao [18], we consider the case where a is a constant. In that case, the characteristic equation of (2.13) is given by

$$\tau^2 - (\lambda^2 + 2\,a\,\lambda + 2\,\alpha) \;=\; 0\;.$$

Thus we have:

$$\tau_\pm \;=\; \pm\sqrt{\lambda^2 + 2\,a\,\lambda + 2\,\alpha} \;=\; \pm\lambda\Bigl(1 + \frac{a}{\lambda} + O\Bigl(\frac{1}{|\lambda|}\Bigr)\Bigr)\,. \tag{2.20}$$

By neglecting the high order term, we set

$$\theta(x) = \lambda\,x + \int_0^x a(s)\,ds\,, \qquad v(x) = \frac{1}{\lambda + a(0)}\,\sinh\theta(x)\;. \tag{2.21}$$

Furthermore, since the functions $\sinh\theta(x)$ and $\cosh\theta(x)$ are uniformly bounded for $\lambda \in \mathcal{C}$, we deduce that there exists $C_4 > 0$ independent of λ, such that

$$|v| \le \frac{C_4}{|\lambda|} \quad \text{and} \quad |v_x| \le C_4\;. \tag{2.22}$$

Let us consider the following problem:

$$\begin{cases} v_{xx} - (\lambda^2 + 2\,a\,\lambda + 2\,\alpha)\,v \;=\; f \;, \\ v(0) = 0\,, \qquad v_x(0) = 1 \;. \end{cases} \tag{2.23}$$

where

$$f \;=\; \frac{1}{\lambda + a(0)} \Big((a^2 - 2\,\alpha)\,\sinh\theta(x) + a'\cosh\theta(x) \Big) \;. \tag{2.24}$$

By the variation of constants formula we have

$$\begin{aligned} v(x) - y(x) \;&=\; \int_0^x f(s)\,y(x-s)\,ds \;, \\ v_x(x) - y_x(x) \;&=\; \int_0^x f(s)\,y_x(x-s)\,ds \;. \end{aligned}$$

Thanks to (2.18), (2.19) and (2.24) we obtain that

$$|v(x) - y(x)| \;\le\; \frac{C_1 \cdot C_2}{|\lambda|\,|\lambda + a(0)|} \int_0^1 \Big(|a^2 - 2\,\alpha| + |a'| \Big)\,dx \;, \tag{2.25}$$

$$|v_x(x) - y_x(x)| \;\le\; \frac{C_1 \cdot C_3}{|\lambda + a(0)|} \int_0^1 \Big(|a^2 - 2\,\alpha| + |a'| \Big)\,dx \;. \tag{2.26}$$

Consequently, we obtain

$$\left| y(x) - \frac{\sinh\Big(\lambda x + \displaystyle\int_0^x a(s)\,ds\Big)}{\lambda + a(0)} \right| \;\le\; \frac{C_1 \cdot C_2 \cdot (T_a + \|a\|_\infty^2 + 2\,\alpha)}{|\lambda|\,|\lambda + a(0)|} \;, \tag{2.27}$$

$$\left| y_x(x) - \frac{\lambda + a(x)}{\lambda + a(0)} \cosh\Big(\lambda x + \int_0^x a(s)\,ds\Big) \right| \le \frac{C_1 \cdot C_3 \cdot (T_a + \|a\|_\infty^2 + 2\,\alpha)}{|\lambda + a(0)|} \;, \tag{2.28}$$

where T_a denotes the total variations of a. Since for λ large enough we have:

$$\frac{\lambda + a(x)}{\lambda + a(0)} \;=\; 1 + O\Big(\frac{1}{|\lambda|^2}\Big) \;, \tag{2.29}$$

$$\frac{1}{\lambda + a(0)} \;=\; \frac{1}{\lambda} + O\Big(\frac{1}{|\lambda|^2}\Big) \;. \tag{2.30}$$

We deduce that there exists a constant $C_0 > 0$ such that (2.14)–(2.15) hold. This achieves the proof. ∎

Let N be the smallest integer greater than $\dfrac{4\,C_0}{\pi}$. We define the sets

$$\Pi_N \;=\; \Big\{ z\,;\; |z + a_0| \le N\pi + \frac{\pi}{2} \Big\} \;, \tag{2.31}$$

$$\Pi_{\pm n} \;=\; \left\{ z\,;\; |z+a_0 \mp i\,n\,\pi| = \frac{2\,C_0}{n\,\pi} \right\}, \qquad \text{for} \;\; n>N \;, \tag{2.32}$$

where we have put:

$$a_0 \;=\; \int_0^1 a(x)\,dx \;.$$

By Lemma 5.2 in Cox–Zuazua [4], we have $|\sinh(\lambda+a_0)| > \dfrac{C_0}{|\lambda|}$ for all $\lambda\in\Pi_n$.

Theorem 2.1. *Let $a\in BV(0,1)$. There exists a finite number of eigenvalues $\lambda_n\in\Pi_N$ and one simple eigenvalue in the region enclosed by Π_n for each $n>N$.*

Proof: Let $n>N$. By (2.14) we have

$$\left| y(1,\lambda) - \frac{\sinh(\lambda+a_0)}{\lambda} \right| \;\le\; \frac{C_0}{|\lambda|^2} \;<\; \left| \frac{\sinh(\lambda+a_0)}{\lambda} \right|, \qquad \forall\,\lambda\in\Pi_n \;. \tag{2.33}$$

By Rouché's theorem, $y(1,\lambda)$ has the same number of roots as the function $\lambda \to \dfrac{\sinh(\lambda+a_0)}{\lambda}$ in the region enclosed by Π_n. In particular, we have

$$\lambda_{\pm n} \;=\; -a_0 \pm i\,n\,\pi + O\Big(\frac{1}{n}\Big) \;. \tag{2.34}$$

As the spectrum of (1.8) is discret and Π_N is compact, there exists at most a finite number of eigenvalues $\lambda_n\in\Pi_N$. This achieves the proof. ∎

Theorem 2.2. *Let $a\in BV(0,1)$. Setting*

$$\xi(x) \;=\; \int_0^x a(s)\,ds \;-\; x\,a_0 \;, \tag{2.35}$$

we have

$$\lambda_{\pm n}\cdot u_{\pm n}(x) \;=\; \sinh\Big(\xi(x) \pm i\,n\,\pi\,x\Big) + O\Big(\frac{1}{n}\Big) \;, \tag{2.36}$$

$$u_{\pm nx}(x) \;=\; \cosh\Big(\xi(x) \pm i\,n\,\pi\,x\Big) + O\Big(\frac{1}{n}\Big) \;. \tag{2.37}$$

Proof: Using (2.34) and (2.14) we obtain:

$$\begin{aligned} \lambda_{\pm n}\cdot u_{\pm n}(x) \;&=\; \lambda_{\pm n}\, y(x,\lambda_{\pm n}) \\ &=\; \sinh\Big(\xi(x) \pm i\,n\,\pi\,x\Big) + O\Big(\frac{1}{n}\Big) \;. \end{aligned}$$

Similarly, using (2.15) and (2.34) we get:

$$\begin{aligned} u_{\pm n x}(x) &= y_x(x, \lambda_{\pm n}) \\ &= \cosh\Big(\xi(x) \pm i\, n\, \pi\, x\Big) + O\Big(\frac{1}{n}\Big) . \end{aligned}$$

The proof is complete. ∎

Now we consider the eigenvalue problem (1.9) defined by

$$(2.38) \qquad \begin{cases} v_{mxx} - (\mu_m^2 + 2\, a\, \mu_m)\, v_m = 0 , \\ v_m(0) = v_m(1) = 0 . \end{cases}$$

By applying the same method, we obtain the following developement for all $m > M$ where M is an integer depending only on $a(x)$:

$$(2.39) \qquad \mu_{\pm m} = -a_0 \pm i\, m\, \pi + O\Big(\frac{1}{m}\Big) ,$$

$$(2.40) \qquad \mu_{\pm m}\, v_{\pm m}(x) = \sinh\Big(\xi(x) \pm i\, m\, \pi\, x\Big) + O\Big(\frac{1}{m}\Big) ,$$

$$(2.41) \qquad v_{\pm m x}(x) = \cosh\Big(\xi(x) \pm i\, m\, \pi\, x\Big) + O\Big(\frac{1}{m}\Big) .$$

We notice that for $|n| > \sup(N, M)$, there exist, in Π_n, two eigenvalues λ_n and μ_n of algebraic multiplicity 1. We will prove that these two eigenvalues are distinct:

Proposition 2.3. *Let n be a sufficiently large integer. We have*

$$(2.42) \qquad \lambda_n \neq \mu_n .$$

Proof: Assume that $\lambda_n = \mu_n$. Let u_n and v_n be eigenfunctions associated to λ_n and μ_n. We have

$$(2.43) \qquad \begin{cases} v_{nxx} - (\mu_n^2 + 2\, a\, \mu_n)\, v_n = 0 , \\ v_n(0) = v_n(1) = 0 . \end{cases}$$

Multiplying (2.43) with u_n and integrating by parts, we obtain that

$$(2.44) \qquad \int_0^1 v_n \cdot \Big[\partial_{xx} - (\lambda_n^2 + 2\, a\, \lambda_n + 2\, \alpha)\Big]\, u_n\, dx + 2 \int_0^1 \alpha\, u_n \cdot v_n\, dx = 0 .$$

Since u_n is a solution of (1.8), we have

$$(2.45) \qquad \int_0^1 \alpha\, u_n \cdot v_n\, dx = 0 .$$

On the other hand by (2.36) and (2.40) we have

$$\int_0^1 \alpha\, u_n \cdot v_n\, dx \;=\; \frac{1}{n^2\,\pi^2} \int_0^1 \alpha \Big|\sinh\Big(\xi(x) \pm i\, n\, \pi\, x\Big)\Big|^2 dx \,+\, O\Big(\frac{1}{n^3}\Big) \;=\; 0\;. \tag{2.46}$$

It follows that

$$\int_0^1 \alpha \Big|\sinh\Big(\xi(x) \pm i\, n\, \pi\, x\Big)\Big|^2 dx \;=\; O\Big(\frac{1}{n}\Big)\;. \tag{2.47}$$

A straight forward computation shows that

$$\begin{aligned}\int_0^1 \alpha \Big|\sinh\Big(\xi(x) \pm i\, n\, \pi\, x\Big)\Big|^2 dx \;&=\; \int_0^1 \alpha \Big(\sinh^2 \xi(x) + \sin^2(n\, \pi\, x)\Big)\, dx \\ &> \frac{1}{2} \int_0^1 \alpha\, dx \;>\; \frac{\alpha_0}{2} \;>\; 0\;.\end{aligned} \tag{2.48}$$

This leads to a contradiction. Thus, we have proved that for each $n > \sup(N, M)$, the region enclosed by Π_n contain two distincts eigenvalues μ_n, λ_n. ∎

3 – System of root vectors

Let λ_n and μ_m be two eigenvalues of the operator $\mathcal{A}$. We know that their algebraic multiplicity is equal to one for $|n| > N$ and $|m| > M$. We index the eigenvalues λ_n et μ_m ($|n| > N,\ |m| > M$) of high frequencies following the asymptotic expansions (2.34) and (2.39). We denote by $\tilde{\lambda}_k$ for $1 \le k \le K$ and $\tilde{\mu}_l$ for $1 \le l \le L$ the eigenvalues of low frequencies. Hence we write the spectrum of $\mathcal{A}$:

$$\sigma(\mathcal{A}) \;=\; \Big\{\lambda_n\colon\ |n|>N\Big\} \cup \Big\{\tilde{\lambda}_k\colon\ 1\le k\le K\Big\} \cup \Big\{\mu_m\colon\ |m|>M\Big\} \cup \Big\{\tilde{\mu}_l\colon\ 1\le l\le L\Big\}\;.$$

Let λ_n be an eigenvalue of (1.8) with the corresponding eigenfunction u_n and μ_m be an eigenvalue of (1.9) with the corresponding eigenfunction v_n. Then, λ_n is an eigenvalue of $\mathcal{A}$ associated to the eigenvector $\phi_n^- = (u_n, \lambda_n u_n, -u_n, -\lambda_n u_n)$, and μ_m is an eigenvalue of $\mathcal{A}$ associated to the eigenfunction $\phi_m^+ = (v_n, \mu_m v_m, v_m, \mu_m v_m)$. We denote by s_k the algebraic multiplicity of $\tilde{\lambda}_k$ and by $\{\tilde{\phi}_{k,j}^-\}_0^{s_k-1}$ the associated Jordan chain. Respectively, we denote by q_l the algebraic multiplicity of $\tilde{\mu}_l$ and by $\{\tilde{\phi}_{l,j}^+\}_0^{q_l-1}$ the associated Jordan chain. The root vectors of $\mathcal{A}$ are given by

$$\begin{aligned}&\Big\{\tilde{\phi}_{k,j}^-\colon\ 0\le j\le s_k-1\,;\ 1\le k\le K\Big\} \cup \Big\{\phi_n^-\colon\ |n| > N\Big\} \cup \\ &\qquad \cup \Big\{\tilde{\phi}_{l,j}^+\colon\ 0\le j\le q_l-1\,;\ 1\le l\le L\Big\} \cup \Big\{\phi_m^+\colon\ |m| > M\Big\}\;.\end{aligned} \tag{3.1}$$

Our aim is to prove that (3.1) is a Riesz basis in the energy space $\mathcal{H}$ by using the following theorem:

Theorem 3.1 (Rao [18])**.** *Let $\{\phi_n\}_0^n$ be a Riesz basis in the Hilbert space X, and let $\{g_n\}_{n_0}^\infty$ be a ω-linearly independent system. Assume that*

$$\sum_{n=n_0}^{\infty} \|\phi_n - g_n\|_X^2 < \infty \ . \tag{3.2}$$

Then $\{g_n\}_{n_0}^\infty$ is a Riesz basis in the subspace X_0 spanned by itself in X. ∎

We first prove the following preliminary result:

Proposition 3.1. *The system of root vectors (3.1) of $\mathcal{A}$ are complete and ω-linearly independent in the energy space $\mathcal{H}$.*

Proof: Putting:

$$L = i\begin{pmatrix} 0 & I & 0 & 0 \\ \partial_{xx} - \alpha I & 0 & \alpha I & 0 \\ 0 & 0 & 0 & I \\ \alpha I & 0 & \partial_{xx} - \alpha I & 0 \end{pmatrix}, \qquad T = i\begin{pmatrix} 0 & 0 & 0 & 0 \\ -2a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2a & 0 \end{pmatrix}. \tag{3.3}$$

Then we have $i\mathcal{A} = L + T$. A straightforward computation show that L is self-adjoint in $\mathcal{H}$. Since T is bounded in the energy space $\mathcal{H}$, then

$$\rho(L^{-1}TL^{-1}) \le \|L^{-1}\| \, \|T\| \, \rho(L^{-1}) \tag{3.4}$$

where ρ denotes the order of a linear bounded operator (see [7, p. 27] for definition). On the other hand, L^{-1} is compact, from (2.34) and (2.39) we deduce that the asymptotic form of the eigenvalues of L^{-1}:

$$\lambda_n(L^{-1}) = \frac{i}{\pm i\, n\, \pi + O(\frac{1}{n})} = \frac{1}{n\, \pi} + O\Big(\frac{1}{n^3}\Big) \ . \tag{3.5}$$

Then, the order ρ of L^{-1} is given by (Gohberg–Krein [7, p. 256]):

$$\rho = \lim_{n\to\infty} \frac{\log n}{\log \frac{1}{\lambda_n(L^{-1})}} = 1 \ . \tag{3.6}$$

Hence by Theorem V 8.1 (Gohberg–Krein [7, p. 257]), we deduce that the system (3.1) is complete in the energy space $\mathcal{H}$.

On the other hand, one straightforward computation show that $\mathcal{A}^{-1}$ is compact in $\mathcal{H}$. Since the operator $i\mathcal{A}^{-1}$ has no real eigenvalues, by Theorem I.5.2 (Gohberg–Krein [7, p. 23]), all $\frac{i}{\lambda} \in \mathbb{C}\backslash\mathbb{R}$ are normal points of $i\mathcal{A}^{-1}$. Let λ_0 be a point of the operator $i\mathcal{A}$. Following Thorem I 2.1 Gohberg–Krein [7, p. 9], the projector operator

$$P_{\lambda_0} \;=\; -\frac{1}{2\,i\,\pi}\int_{|\mu-\lambda_0|=\delta}(i\mathcal{A}^{-1}-\mu I)^{-1}\,d\mu\ . \tag{3.7}$$

is of finite-dimension and the range of P_{λ_0} is the subspace $\ker(i\mathcal{A}^{-1}-\lambda_0 I)^{\nu_0}$, where $\nu_0 \geq 1$ is the algebric multiplicity of λ_0. Now we consider a serie:

$$\sum_{k=1}^{K}\sum_{j=0}^{s_k-1}|c^-_{k,j}|^2+\sum_{|n|>N}|c^-_n|^2+\sum_{l=1}^{L}\sum_{j=0}^{q_l-1}|c^+_{l,j}|^2+\sum_{|m|>M}|c^+_m|^2\;<\;\infty \tag{3.8}$$

such that

$$\sum_{k=1}^{K}\sum_{j=0}^{s_k-1}c^-_{k,j}\,\tilde{\phi}^-_{k,j}+\sum_{|n|>N}c^-_n\,\phi^-_n+\sum_{l=1}^{L}\sum_{j=0}^{q_l-1}c^+_{l,j}\,\tilde{\phi}^+_{l,j}+\sum_{|m|>M}c^+_m\,\phi^+_m\;=\;0\ . \tag{3.9}$$

Applying the projector $P_{\tilde{\mu}_l}$, $1\leq l\leq L$, to (3.9), we obtain that

$$\sum_{j=0}^{q_l-1}c^+_{l,j}\,\tilde{\phi}^+_{l,j}=0 \quad \text{for}\ \ 1\leq l\leq L\ . \tag{3.10}$$

Since $\{\tilde{\phi}^+_{l,j}\}_0^{q_l-1}$ is a basis of $\ker(\mathcal{A}-\tilde{\mu}_l I)^{q_l-1}$ for all $1\leq l\leq L$, it follows that

$$c^+_{l,j}=0\,,\quad 0\leq j\leq q_l-1,\ \ 1\leq l\leq L\ . \tag{3.11}$$

On the other hand, the algebraic multiplicity of the eigenvalue μ_m is equal to 1 for $m>M$. Applying P_{μ_m} for $m>M$ to (3.9) we have:

$$c^+_m=0 \quad \text{for all}\ \ |m|>M\ . \tag{3.12}$$

Similarly, applying $P_{\tilde{\lambda}_n}$ for $1\leq k\leq K$ and P_{λ_n} for $|n|>N$ to (3.9) we get that

$$c^-_{k,j}=0 \quad \text{for}\ \ 0\leq j\leq s_k-1,\ \ 1\leq k\leq K \quad \text{and} \quad c^-_n=0 \quad \text{for all}\ \ |n|>N\ . \tag{3.13}$$

This achieves the proof. ∎

Now we consider the subspace $\mathcal{L}$ of $X=L^2(0,1)\times L^2(0,1)\times L^2(0,1)\times L^2(0,1)$ defined by

$$\mathcal{L}\;=\;\left\{(f,g,h,k)\in X\ \text{ such that }\int_0^1 f(x)\,dx=\int_0^1 g(x)\,dx\;=\;0\right\} \tag{3.14}$$

and we define the linear bounded operator from $\mathcal{H}$ to $\mathcal{L}$ by

$$\mathcal{F}(u,z,v,w)=(u_x,z,v_x,w)\qquad \forall\,(u,z,v,w)\in\mathcal{H}\ . \tag{3.15}$$

Proposition 3.2. *The linear operator $\mathcal{F}$ defined by (3.14)–(3.15) is an isomorphism from $\mathcal{H}$ onto $\mathcal{L}$.*

Proof: Let $(u,z,v,w)\in\mathcal{H}$, then

$$\begin{aligned}\|\mathcal{F}(u,z,v,w)\|_X^2 &= \int_0^1 |u_x|^2+|z|^2+|v_x|^2+|w|^2\,dx\\ &= \|u\|^2_{H_0^1(0,1)}+\|z\|^2_{L^2(0,1)}+\|v\|^2_{H_0^1(0,1)}+\|w\|^2_{L^2(0,1)}\,.\end{aligned}$$

Hence $\mathcal{F}$ is a linear bounded operator from $\mathcal{H}$ to $\mathcal{L}$. Let $(f,g,h,k)\in\mathcal{L}$. We can verify that

$$u=\int_0^x f(x)\,dx\;\in\;H_0^1(0,1)\,,\qquad z=g\;\in\;L^2(0,1)\;, \tag{3.16}$$

$$v=\int_0^x h(x)\,dx\;\in\;H_0^1(0,1)\,,\qquad w=k\;\in\;L^2(0,1)\;, \tag{3.17}$$

satisfy the equation $\mathcal{F}(u,z,v,w)=(f,g,h,k)$. We conclude, by Banach's theorem that $\mathcal{F}$ is an isomorphism from $\mathcal{H}$ onto $\mathcal{L}$. The proof is complete. ∎

Let $\xi\in L^\infty(0,1)$, and set Θ_n defined by $\Theta_n(x)=\xi(x)+i\,n\,\pi\,x$. We have the following system

$$\Phi_n^\pm=(\cosh\Theta_n,\;\sinh\Theta_n,\;\pm\cosh\Theta_n,\;\pm\sinh\Theta_n)\,,\qquad n\in\mathbb{Z}\;. \tag{3.18}$$

Proposition 3.3. *For all $\xi\in L^\infty(0,1)$, the system (3.18) is a Riesz basis in X.*

Proof: For $n\in\mathbb{Z}$, we set:

$$e_n^\pm=\Big(\cos n\pi x,\;\sin n\pi x,\;\pm\cos n\pi x,\;\pm\sin n\pi x\Big)\;,$$

$$M=\begin{pmatrix}\cosh\xi(x) & \sinh\xi(x) & 0 & 0\\ i\cosh\xi(x) & i\sinh\xi(x) & 0 & 0\\ 0 & 0 & \cosh\xi(x) & \sinh\xi(x)\\ 0 & 0 & i\cosh\xi(x) & i\sinh\xi(x)\end{pmatrix}\;. \tag{3.19}$$

Then we have $\Phi_n^\pm=e_n^\pm\cdot M$. Since the transformation matrix has a bounded inverse in X and since the system $\{e_n^\pm\}_{n\in\mathbb{Z}}$ is equivalent to an orthonormal basis in X, it follows that the system (3.18) is a Riesz basis in X. The proof is complete. ∎

Theorem 3.2. *Assume that $a\in BV(0,1)$. Then the root system (3.1) forms a Riesz basis in the energy space $\mathcal{H}$.*

Proof: We use an idea of Rao in [18]. Since the operator $\mathcal{F}$ is an isomorphism from $\mathcal{H}$ on to $\mathcal{L}$, it is sufficient to prove that the system

$$\begin{aligned}\left\{\mathcal{F}\tilde{\phi}^-_{j,k}:\ 0\le j\le s_k-1,\ 1\le k\le K\right\} \cup \left\{\mathcal{F}\phi^-_n:\ |n|>N\right\} \cup\\ \cup \left\{\mathcal{F}\tilde{\phi}^+_{j,l}:\ 0\le j\le q_l-1,\ 1\le l\le L\right\} \cup \left\{\mathcal{F}\phi^+:\ |m|>M\right\}\end{aligned} \tag{3.20}$$

is a Riesz basis in $\mathcal{L}$. We distinguish three cases:

Case i: $\sum_{k=1}^K s_k + \sum_{l=1}^L q_l = M+N$. From (2.36), (2.37), (2.40) and (2.41) it follows that:

$$\begin{aligned}\sum_{k=0}^{K}\sum_{j=0}^{s_k-1} \|\mathcal{F}\tilde{\phi}^-_{j,k} - \tilde{\Phi}^-_{j,k}\|^2_X + \sum_{l=0}^{L}\sum_{j=0}^{q_l-1} \|\mathcal{F}\tilde{\phi}^+_{j,l} - \tilde{\Phi}^+_{j,l}\|^2_X +\\ + \sum_{|n|>N} \|\mathcal{F}\phi^-_n - \Phi^-_n\|^2_X + \sum_{|m|>M} \|\mathcal{F}\phi^+_m - \Phi^+_m\|^2_X \ < \ \infty\ .\end{aligned} \tag{3.21}$$

Thanks to Bari's Theorem, we show that the system (3.20) is a Riesz basis in X.

Case ii: If $\sum_{k=1}^K s_k + \sum_{l=1}^L q_l > M+N$. From Bari's theorem, we can find a subsystem of (3.20) which is quadratically close to the Riesz basis $\{\Phi^\pm_n\}_{n\in\mathbb{Z}}$, and would be also a Riesz basis in X. This contradicts the linear independence of the system (3.20).

Case iii: If $\sum_{k=1}^K s_k + \sum_{l=1}^L q_l \le M+N$. From Proposition 3.1, the system (3.20) is complete and ω-linearly independent in $\mathcal{L}$. Since the system (3.20) is quadratically close to a subsystem of the Riesz basis $\{\Phi^\pm_n\}_{n\in\mathbb{Z}}$, applying Theorem 3.1, we conclude that system (3.20) is a Riesz basis of the subspace spanned by itself. But the system (3.20) is complete in $\mathcal{L}$, hence forms a Riesz basis in the whole space $\mathcal{L}$. The proof is thus complete. ∎

Theorem 3.3. *If $a \in BV(0,1)$, then have $\mu(a) = \omega(a)$.*

Proof: The proof is similar to the one used in [2], [4] and [13]. For the sake of the complement we give a brief outline of the proof.

We know that $\mu(a) \le \omega(a)$. We will establish the reverse inequality. We expand the initial data into:

$$(u_0, z_0, v_0, w_0) \ = \ \sum_{n=0}^{\pm\infty}\sum_{j=0}^{s_n-1} \beta^-_{n,j}\phi^-_{n,j} \ + \ \sum_{m=0}^{\pm\infty}\sum_{j=0}^{q_m-1} \beta^+_{m,j}\phi^+_{m,j}\ . \tag{3.22}$$

It follows that

$$\|(u,u_t,v,v_t)\|^2_{\mathcal{H}} = \left\| \sum_{n=0}^{\pm\infty} \sum_{j=0}^{s_n-1} \beta^-_{n,j}\, S(t)\, \phi^-_{n,j} + \sum_{m=0}^{\pm\infty} \sum_{j=0}^{q_m-1} \beta^+_{m,j}\, S(t)\, \phi^+_{m,j} \right\|^2 \tag{3.23}$$

where $S(t)$ is the C_0-semigroup generated by the system (1.1). By the property of Riesz basis there exist positive constants C_1, C_2 such that

$$\begin{aligned} C_1 \left(\sum_{n=0}^{\pm\infty} \sum_{j=0}^{s_n-1} |\beta^-_{n,j}|^2 + \sum_{m=0}^{\pm\infty} \sum_{j=0}^{q_m-1} |\beta^+_{m,j}|^2 \right) &\le \|U_0\|^2_{\mathcal{H}} \\ &\le C_2 \left(\sum_{n=0}^{\pm\infty} \sum_{j=0}^{s_n-1} |\beta^-_{n,j}|^2 + \sum_{m=0}^{\pm\infty} \sum_{j=0}^{q_m-1} |\beta^+_{m,j}|^2 \right) \end{aligned}$$

for any $U_0 = (u_0, z_0, v_0, w_0) \in \mathcal{H}$. Then a straightforward computation gives that

$$\begin{aligned} \|(u,u_t,v,v_t)\|^2_{\mathcal{H}} \le C_2 \Bigg(& \sum_{n=0}^{\pm\infty} e^{2\mu(a)t} \sum_{j=0}^{s_n-1} |\beta^-_{n,j}|^2 \sum_{k=0}^{j} \left(\frac{t^{(j-k)}}{(j-k)!} \right)^2 \\ & + \sum_{n=0}^{\pm\infty} e^{2\mu(a)t} \sum_{j=0}^{q_n-1} |\beta^+_{n,j}|^2 \sum_{k=0}^{j} \left(\frac{t^{(j-k)}}{(j-k)!} \right)^2 \Bigg) . \end{aligned} \tag{3.24}$$

Recalling that at most $M+N$ eigenvalues may be of algebraic multiplicity greater then one, we conclude that there exists a positive constant C_3 such that:

$$\begin{aligned} \|(u,u_t,v,v_t)\|^2_{\mathcal{H}} &\le C_3\, C_2\, e^{2\mu(a)t} \left(\sum_{n=0}^{\pm\infty} \sum_{j=o}^{s_n-1} |\beta^-_{n,j}|^2 + \sum_{m=0}^{\pm\infty} \sum_{j=o}^{q_m-1} |\beta^+_{m,j}|^2 \right) (1 + t^{2(m+N)}) \\ &\le \frac{C_3\, C_2}{C_1}\, e^{2\mu(a)t} (1 + t^{2(M+N)})\, \|(u_0, u_{0t}, v_0, v_{0t})\|^2_{\mathcal{H}} . \end{aligned} \tag{3.25}$$

We have established our main result. ∎

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