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HOLOMORPHIC MAPPINGS OF UNIFORMLY BOUNDED TYPE AND THE LINEAR TOPOLOGICAL INVARIANTS (H_{ub}), (LB^∞) AND (DN)

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Abstract: The main aim of this paper is to establish the equalities

$$H_b(E,F) = H_{ub}(E,F)$$
$$H_b(E^*_\beta, F^*_\beta) = H_{ub}(E^*_\beta, F^*_\beta)$$

for the case where E and F are Fréchet spaces in the relation with the linear topological invariants (H_{ub}), (LB^∞) and (DN).

1 – Introduction

Let E, F be locally convex spaces. By $H(E,F)$ we denote the space of all F-valued holomorphic mappings on E. Instead of $H(E,\mathbb{C})$ we write $H(E)$. Each element of $H(E,F)$ is called an entire mapping. By $H_b(E,F)$ we denote the space of all entire mappings which are bounded on all bounded subsets of E. The mappings in $H_b(E,F)$ are called of bounded type. An entire mapping $f \in H(E,F)$ is called of uniformly bounded type if it is bounded on multiples of some neighbourhood of 0 in E. We denote by $H_{ub}(E,F)$ the space of all entire mappings of uniformly bounded type.

A locally convex space E has the property (H_{ub}) and is written shortly $E \in (H_{ub})$ if $H(E) = H_{ub}(E)$. The property (H_{ub}) has been investigated by some authors. Colombeau and Mujica have proved that $H(E) = H_{ub}(E)$ for each (DFM)-space E (Ex. 3.11 in [2], p. 163) while Nachbin has shown that $H_{ub}(E) \not\subset H(E)$ for the nuclear Fréchet space $E = H(\mathbb{C})$ (Ex. 3.12 in [2], p. 165).

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Meise–Vogt have also proved that a nuclear locally convex space E satisfies $H(E)=H_{ub}(E)$ if and only if entire mappings on E are universally extendable in the following sense, whenever E is a topological linear subspace of a locally convex space F with the topology defined by a fundamental system of continuous semi-norms induced by semi-inner products, then each $f \in H(E)$ has a holomorphic extension to F (Proposition 6.21 in [2], p. 421).

Next they have given some sufficient conditions for the equality $H(E)=H_{ub}(E)$ in terms of the linear topological invariants $(\overline{\overline{\Omega}})$ and $(\widetilde{\Omega})$ (Theorem 3.3 and 3.9 in [8]) and in the case E is a nuclear Fréchet space they have shown that $(\widetilde{\Omega}) \Rightarrow (H_{ub}) \Rightarrow (LB^\infty)$ (Remark 3.11 in [8]). By Vogt (Ex. 5.5 in [15]) the class (LB^∞) is strictly larger than the class $(\widetilde{\Omega})$. However we do not know whether one of the above implications can be reversed.

In this paper we will establish the relations

$$H_b(E,F) = H_{ub}(E,F) \tag{1}$$

and

$$H_b(E^*_\beta, F^*_\beta) = H_{ub}(E^*_\beta, F^*_\beta) \tag{2}$$

for Fréchet-valued (resp. DF-valued) entire mappings on Fréchet spaces (resp. DF spaces) in the relation with linear topological invariants (H_{ub}), (LB^∞) and (DN). Note that under various assumptions (1) has been considered by some authors [3], [4], [5], [6]. It should be noticed that if E is a Fréchet space that is not a Banach space then the scalar valued equality $H_{ub}(E) = H_b(E)$ does not imply the equality $H_{ub}(E,F) = H_b(E,F)$ for all Fréchet spaces F. It is enough to consider the case $F = E$.

Beside the introduction the article contains four sections. In the second one we recall some definitions and fix the notations. The section 3 is devoted to prove the equality (2). The main aim of section 4 is to prove that (1) holds in a special case where $F = H(\mathbb{C}, A)$, A is a Banach space. In order to obtain the result in this case we modify some techniques of Vogt (Proposition 1.3 and 1.4 in [15]) for continuous linear maps to holomorphic mappings of bounded type. From the results obtained in the section 4 as a special case we prove, in the section 5, the equality (1) under the assumption that E has the property (H_{ub}) and F has the property (DN).

2 – Preliminaries

2.1. We shall use standard notations from the theory of locally convex spaces as presented in the books of R. Meise and D. Vogt [9] and Schaefer [13]. All locally convex spaces E are assumed to be complex vector spaces and Hausdorff.

For a locally convex space E by $\mathcal{U}(E)$ we denote a neighbourhood basis of $0 \in E$. For each $U \in \mathcal{U}(E)$ by E_U we denote the Banach space associated to the neighbourhood U. Let $V \in \mathcal{U}(E)$, $V \subset U$, $\omega_{VU} \colon E_V \to E_U$ denotes the canonical map from E_V to E_U.

A locally convex space E is called to be Schwartz if for each $U \in \mathcal{U}(E)$ there exists $V \in \mathcal{U}(E)$, $V \subset U$ such that $\omega_{VU} \colon E_V \to E_U$ is compact.

For each locally convex space E, E^*_β denotes the topological dual space E^* of E equipped with the strong topology $\beta(E^*, E)$.

Now assume that E is a Fréchet space. We always consider that its locally convex structure is generated by an increasing system $(\|\cdot\|_k)_{k\geq 1}$ of semi-norms. For $k \geq 1$ E_k will denote the Banach space associated to the semi-norm $\|\cdot\|_k$.

Let E be a Fréchet space and $u \in E^*$. For each $k \geq 1$ we define

$$\|u\|^*_k = \sup\Big\{|u(x)| : \|x\|_k \leq 1\Big\} .$$

Now we say that E has the property (LB^∞) if

$$(LB^\infty) \qquad \begin{aligned} &\forall \{\rho_n\} \uparrow +\infty \ \ \forall p \ \ \exists q \ \ \forall n_0 \ \ \exists N_0 \geq n_0, \quad C > 0 \\ &\forall u \in E^*, \ \ \exists k \ n_0 \leq k \leq N_0 : \ \ \|u\|^{*1+\rho_k}_q \leq C\, \|u\|^*_k\, \|u\|^{*\rho_k}_p \ . \end{aligned}$$

E is said to have the property (DN) if

$$(DN) \qquad \exists p, d > 0 \ \ \forall q \ \ \exists k, C > 0 \ \ \forall x \in E : \ \ \|x\|^{1+d}_q \leq C\, \|x\|_k\, \|x\|^d_p \ .$$

The properties (LB^∞) and (DN) and some others are introduced and investigated by Vogt [15], [16], [17].

From now on, to be brief, whenever E has the property (H_{ub}) (resp. (LB^∞), (DN), ...) we write $E \in (H_{ub})$ (resp. $E \in (LB^\infty)$, $E \in (DN)$, ...).

2.2. Holomorphic mappings. Let E, F be locally convex spaces and D be a non empty open subset of E.

A mapping $f \colon D \to F$ is called Gâteaux-holomorphic if for each $x \in D, a \in E$ and $u \in F^*$ the $\mathbb{C}$-valued function of one complex variable

$$\lambda \longrightarrow u \circ f(x + \lambda a)$$

is holomorphic on some neighbourhood of 0 in $\mathbb{C}$. A mapping $f\colon D \to F$ is called holomorphic if f is Gâteaux-holomorphic and continuous. By $H(D,F)$ we denote the space of all F-valued holomorphic mappings on D, the compact-open topology on $H(D,F)$ is denoted by τ_0. For details concerning holomorphic mappings on locally convex spaces we refer to the books of Dineen [2] and Noverraz [12].

3 – DF-valued holomorphic mappings of uniformly bounded type and the linear topological invariants (LB^∞) and (DN)

In the section we investigate the connection between DF-valued holomorphic mappings of uniformly bounded type on DF-spaces and the linear topological invariants (LB^∞) and (DN). We prove the following

3.1. Theorem. *Let E be a Fréchet space. Then*

a) *E has the property (DN) if and only if $H_{ub}(E^*_\beta, F^*_\beta) = H_b(E^*_\beta, F^*_\beta)$ holds for every Fréchet space F having the property (LB^∞).*

b) *E has the property (LB^∞) if and only if $H_{ub}(F^*_\beta, E^*_\beta) = H_b(F^*_\beta, E^*_\beta)$ holds for every Fréchet space F having the property (DN).*

Proof: **a)** Assume that $E \in (DN)$, obviously $H_{ub}(E^*_\beta, F^*_\beta) \subset H_b(E^*_\beta, F^*_\beta)$. Let $f\colon E^*_\beta \to F^*_\beta$ be a holomorphic mapping of bounded type. Consider the linear map $\hat{f}\colon H_b(F^*_\beta) \to H_b(E^*_\beta)$ given by $\hat{f}(g) = g \circ f$ for all $g \in H_b(F^*_\beta)$. It is easy to that F is a subspace of $H_b(F^*_\beta)$. Hence $\hat{f}\colon F \to H_b(E^*_\beta)$ is linear and continuous. Since $E \in (DN)$ by (Theorem 3 in [10]) $H_b(E^*_\beta)$ also has the property (DN). Now from $F \in (LB^\infty)$ we infer that there exists a neighbourhood V of $0 \in F$ for which $\hat{f}(V)$ is bounded in $H_b(E^*_\beta)$ (Theorem 6.2 in [15]). This yields that

$$\sup\Big\{|\hat{f}(x)(u)|\colon\ x \in V,\ u \in B\Big\} = \sup\Big\{|f(u)(x)|\colon\ x \in V,\ u \in B\Big\} < +\infty$$

for every bounded subset $B \subset E^*_\beta$. Hence $f\colon E^*_\beta \to (F_V)^*$ is holomorphic and of bounded type (Proposition 7 in [3]).

Conversely, by (Theorem 2.1 in [15]) it sufficies to show that

$$L(\Lambda_1(\alpha), E) = LB(\Lambda_1(\alpha), E)$$

for the exponential sequence $\alpha = (\alpha_n)$ where $\alpha_n = n$ for $n \geq 1$.

Let $f\colon \Lambda_1(\alpha) \to E$ be a continuous linear map. Since f maps bounded subsets of $\Lambda_1(\alpha)$ to bounded subsets of E then $f^* \in L\Big(E^*_\beta, (\Lambda_1(\alpha))^*_\beta\Big)$ where f^* is the dual map of f. In view of $\Lambda_1(\alpha) \in (LB^\infty)$ and by applying the hypothesis we obtain that $f^* \in LB\Big(E^*_\beta, (\Lambda_1(\alpha))^*_\beta\Big)$. Hence $f \in LB(\Lambda_1(\alpha), E)$.

b) Necessity follows from a).

Conversely, by (Theorem 5.2 in [16]) it suffices to show that

$$L(E, \Lambda^\infty_\infty(\alpha)) = LB(E, \Lambda^\infty_\infty(\alpha))$$

where $\alpha_n = n$ for all $n \geq 1$ and

$$\Lambda^\infty_\infty(\alpha) = \Big\{\xi = (\xi_j)_{j\geq 1}\colon \|\xi\|_k = \sup |\xi_j| \rho_k^{\alpha_j} < +\infty \text{ for all } k \geq 1\Big\}$$

and $\{\rho_k\} \uparrow +\infty$.

Let $f\colon E \to \Lambda^\infty_\infty(\alpha)$ be a continuous linear map.
As in a) $f^* \in L((\Lambda^\infty_\infty(\alpha))^*_\beta, E^*_\beta)$. It is easy to check that $\Lambda^\infty_\infty(\alpha)$ has the property (DN) and, hence, $f^* \in LB((\Lambda^\infty_\infty(\alpha))^*_\beta, E^*_\beta)$. From an argument as in a) we obtain that $f \in LB(E, \Lambda^\infty_\infty(\alpha))$ which completes the proof of 3.1 Theorem. ∎

4 – Fréchet-valued holomorphic mappings of uniformly bounded type and the linear topological invariant (H_{ub})

The main aim of this section is to prove the following technical result which is crucial for the proof of 5.1 Theorem.

4.1. Theorem. *Let E be a Fréchet-Schwartz space having the property (H_{ub}) and A be a Banach space. Then $\forall \{\rho_n\} \uparrow +\infty$ $\exists k > 0$ $\forall p, s > 0$ $\forall r > 0$ $\forall n$ sufficiently large $\exists N_0 > n$, $C > 0$ $\forall f \in H_b(E, A)$ $\exists\, n \leq N^* \leq N_0$:*

$$\|f\|_{k,r}^{1+\rho_{N^*}} \leq C\, \|f\|_{N^*, \rho_{N^*}} \cdot \|f\|_{p,\rho_s}^{\rho_{N^*}} \tag{3}$$

where

$$\|f\|_{k,r} = \sup\Big\{\|f(x)\|\colon \|x\|_k \leq r\Big\}$$

for $f \in H_b(E, A)$.

In order to derive the proof of this theorem first we establish the stability of the property (H_{ub}) under the finite products (see 4.2 Proposition below). 4.2 Proposition is a key ingredient in the proof of 4.1 Theorem. Moreover, next we modify some techniques of Vogt (Proposition 1.3, 1.4 in [15]) which are used for establishing (1) for continuous linear maps to holomorphic mappings of bounded type.

Now we state and prove the following

4.2. Proposition. *Let E and F be Fréchet-Schwartz spaces having the property (H_{ub}). Then $E \times F$ has also the property (H_{ub}).*

Proof: Given $f \in H(E \times F)$. Consider the holomorphic mapping $f_E : E \to (H(F), \tau_0)$ associated to f. Since $F \in (H_{ub})$, by (Proposition 4.1 in [8]), $(H(F), \tau_0)_{bor}$ is a regular inductive limit of $H_b(F_\alpha)$, $\alpha \in \mathbb{N}$, the Banach space of holomorphic mappings of bounded type on F_α where F_α is the Banach space associated to the continuous semi-norm $\| \cdot \|_\alpha$ of F. First we prove that there exist p, $\alpha \geq 1$ such that

$$f_E(U_p) \subset H_b(F_\alpha) \ .$$

Indeed, otherwise, for each $p \geq 1$, $\alpha \geq 1$ there exists x_p^α such that $x_p^\alpha \in U_p$ and $f_E(x_p^\alpha) \notin H_b(F_\alpha)$. Since $\{x_p^p\}_{p\geq 1} \to 0$ and $(H(F), \tau_0)_{bor} = \lim\operatorname{ind} H_b(F_\alpha)$ is regular we can find α_0 such that

$$f_E(x_p^p) \subset H_b(F_{\alpha_0}) \quad \text{ for all } \quad p \geq 1 \ .$$

This is impossible because $f_E(x_{\alpha_0}^{\alpha_0}) \notin H_b(F_{\alpha_0})$. Thus there exists p and α such that $f_E(U_p) \subset H_b(F_\alpha)$. Similarly there exist $q > p$, $\beta > \alpha$ such that $f^F(V_\beta) \subset H_b(E_q)$ where $f^F\colon F \to (H(E), \tau_0)$ is the holomorphic mapping induced by f.

Consider the mapping

$$g\colon \ (U_q \times F_\beta) \cup (E_q \times V_\beta) \ \subset \ E_q \times F_\beta \ \to \ \mathbb{C}$$

defined by f_E and f^F. Notice that g is separately holomorphic. By a result of N.T. Van–Zeriahi (Théorème 1.1 in [11]) g extends to Gâteaux-holomorphic mapping $\tilde{g}$ on $E_q \times F_\beta$ such that f is Gâteaux-holomorphically factorized through $\tilde{g}$ by $\omega_q \times \omega_\beta : E \times F \to E_q \times F_\beta$.

By shrinking U_q and V_β we may assume that f is bounded on $U_q \times V_\beta$. Hence by the Zorn theorem $\hat{g}$ is holomorphic on $E_q \times F_\beta$.

On the other hand, since E and F are Schwartz spaces we can find $k \geq q$ and $\gamma \geq \beta$ such that the canonical maps $\omega_{qk} : E_k \to E_q$, $\omega_{\beta\gamma} : F_\gamma \to F_\beta$ are compact. Hence $\hat{g} \in H_b(E_k \times F_\gamma)$ and f is factorized through $\hat{g}$ by $\omega_k \times \omega_\gamma$. Hence $f \in H_{ub}(E \times F)$. ∎

Remark. In the above proposition, if we take $F = \mathbb{C}$ then we have $H_b(E \times \mathbb{C}) = H_{ub}(E \times \mathbb{C})$. However, $H_b(E \times \mathbb{C}) = H_b(E, H(\mathbb{C}))$, $H_{ub}(E \times \mathbb{C}) = H_{ub}(E, H(\mathbb{C}))$ and, hence, (1) holds for the case $F = H(\mathbb{C})$. But it is known that $H(\mathbb{C})$ has the property (DN). Below, in 5.1 Theorem , we shall show that (1) holds under the assumptions $E \in (H_{ub})$ and $F \in (DN)$. □

Now in order to obtain the proof of 4.1 Theorem we shall establish some equivalent conditions for which (1) holds.

First we fix some notations. Let E (resp. F) be a Fréchet space with the topology defined by an increasing system of semi-norms $(\|\cdot\|_\gamma)_{\gamma\geq 1}$ (resp. $(\|\cdot\|_k)_{k\geq 1}$). For each $k, \gamma, r > 0$ (or $\rho > 0$) and $f \in H(E,F)$ we define

$$\|f\|_{k,\gamma,r} = \sup\Big\{\|f(x)\|_k:\ \|x\|_\gamma \leq r\Big\}\ .$$

Through this section we always assume that E is a Fréchet space having the property (H_{ub}). Now we have the following

4.3. Proposition. *The following assertions are equivalent*

(i) $H_b(E,F) = H_{ub}(E,F)$.

(ii) $\forall\{\gamma(n)\}\uparrow\ \forall\{\rho_n\}\uparrow +\infty\ \exists k\ \forall r>0\ \forall n\ \exists N_0,\ C>0\ \forall f \in H_b(E,F)$

$$\|f\|_{n,\gamma(k),r} \leq C \max_{1\leq N\leq N_0} \|f\|_{N,\gamma(N),\rho_N}\ . \tag{4}$$

Proof: (i)⇒(ii) Given $\{\gamma(n)\}\uparrow$ and $\{\rho_n\}\uparrow +\infty$. Put

$$G = \Big\{f \in H_b(E,F):\ \|f\|_{n,\gamma(n),\rho_n} < +\infty,\ \forall n\Big\}\ .$$

Since $H_b(E,F) = H_{ub}(E,F)$ then G is a Fréchet space equipped with the topology defined by the system of semi-norms

$$q_m(f) = \sup\Big\{\|f\|_{n,\gamma(n),\rho_n}:\ n = 1,2,\ldots,m\Big\}$$

for $f \in G$. For each $k \in \mathbb{N}$, define

$$H_k = \Big\{f \in H_b(E,F):\ \|f\|_{n,\gamma(k),r} < +\infty \text{ for all } n, r > 0\Big\}\ .$$

H_k is a Fréchet space under the topology defined by the systems of semi-norms

$$p_{n,r}(f) = \|f\|_{n,\gamma(k),r}\ .$$

We note that $H_k \subset H_{k+1}$ for all $k \geq 1$. By the hypothesis $H_b(E,F) = H_{ub}(E,F)$ it follows that $G \subset \bigcup_{k\geq 1} H_k$. All these spaces are continuously embeded in $H_b(E,F)$.

By the factorization theorem of Grothendieck (Theorem 24.33 in [9], p. 290) there exists k such that G is continuously embeded in H_k. Hence $\forall r > 0\ \forall n\ \exists N_0, C > 0$ such that

$$p_{n,r}(f) \le C \max_{N \le N_0} q_N(f)$$

for $f \in H_b(E, F)$. This shows that (4) holds.

(ii)$\Rightarrow$**(i)** is trivial. ∎

Now we need the following result which shows that (1) holds for the case F is a Banach space.

4.4. Lemma. *Let E be a Fréchet space having the property (H_{ub}) and F a Banach space. Then*

$$H_b(E, F) = H_{ub}(E, F) \ .$$

Proof: See the proof of (i)$\Rightarrow$(iii) of Proposition 2.5 in [4].

Let A be a Banach space and $B = (b_{j,k})_{j,k \ge 1}$ a Köthe matrix. We define

$$\Lambda^\infty(B, A) := \left\{ a = (a_i)_{i \ge 1} \colon\ a_i \in A,\ \|a\|_n = \sup \|a_i\| b_{i,n} < +\infty \text{ for all } n \ge 1 \right\}.$$

$\Lambda^\infty(B, A)$ is a Fréchet space under the topology defined by the system of seminorms $(\|\cdot\|)_{n \ge 1}$.

When $A = \mathbb{C}$ we write $\Lambda^\infty(B)$ instead of $\Lambda^\infty(B, \mathbb{C})$. ∎

For a comprehensive survey on the theory of Köthe sequence spaces we refer the readers to the book of Meise–Vogt (Chapters 27-31, p. 326–403 in [9]).

Let $E \in (H_{ub})$. Then we have the following

4.5. Proposition. *Let A be a Banach space. The following assertions are equivalent*

(i) $H_b(E, \Lambda^\infty(B, A)) = H_{ub}(E, \Lambda^\infty(B, A))$.

(ii) $\forall \{\gamma(n)\} \uparrow\ \ \forall \{\rho_n\} \uparrow +\infty\ \ \exists k\ \forall r > 0\ \ \forall n\ \ \exists N_0,\ C > 0$

$$b_{j,n} \|f\|_{\gamma(k),r} \le C \max_{1 \le N \le N_0} b_{j,N}\, \|f\|_{\gamma(N),\rho_N} \tag{5}$$

for all $j \ge 1$ and for $f \in H_b(E, A)$.

Proof: (**i**)⇒(**ii**) Let $f \in H_b(E, A)$.

Put $g_j = f \otimes e_j \in H_b(E, \Lambda^\infty(B, A))$ where $\{e_j\}_{j\geq 1}$ are vectors in $\Lambda^\infty(B)$ of the form $e_j = (0, 0, ..., 0, \underset{\hat{j}}{1}, 0, ...)$. By applying 4.3 Proposition to g_j and using

$$\|g_j\|_{n,\gamma(k),r} = b_{j,n}\|f\|_{\gamma(k),r}$$

we obtain (ii).

(**ii**)⇒(**i**) Let $f \in H_b(E, \Lambda^\infty(B, A))$ be given. Since

$$\Lambda^\infty(B, A) = \Big\{a = (a_i)_{i\geq 1} \colon \; a_i \in A, \; \|a\|_n = \sup_i \|a_i\| b_{i,n} < +\infty \text{ for all } n \geq 1\Big\}$$

it implies that $f = (f_i)_{i\geq 1}$ where $f_i \in H_b(E, A)$. From $E \in (H_{ub})$ and $f \in H_b(E, \Lambda^\infty(B, A))$ it follows that for each $n \geq 1$ if f can be considered as a holormorphic mapping of bounded type with values in the Banach space $\Lambda^\infty(B, A)_n$ induced by the continuous semi-norm $\|\cdot\|_n$ then 4.4 Lemma implies that there exists $\gamma(n) \geq 1$ such that

$$M(n, \gamma(n), r) = \sup\Big\{\|f(x)\|_n \colon \; \|x\|_{\gamma(n)} \leq r\Big\} < +\infty$$

for all $r > 0$.

We may assume that the sequence $\{\gamma(n)\}$ is increasing.

Take some sequence $\{\rho_n\} \uparrow +\infty$ and by using (ii) for $\{\gamma(n)\} \uparrow$ and $\{\rho_n\}$ we derive that $\exists k \; \forall r > 0 \; \forall n \; \exists N_0, C > 0$:

$$b_{i,n}\|f_i\|_{\gamma(k),r} \leq C \max_{1\leq N\leq N_0} b_{i,N}\|f_i\|_{\gamma(N),\rho_N} .$$

Hence

$$\begin{aligned} \|f\|_{n,\gamma(k),r} &= \sup_i b_{i,n}\|f_i\|_{\gamma(k),r} \\ &\leq C \max_{1\leq N\leq N_0} \sup_i b_{i,N}\|f_i\|_{\gamma(N),\rho_N} \\ &= C \max_{1\leq N\leq N_0} \|f\|_{N,\gamma(N),\rho_N} . \end{aligned}$$

It follows that $f \in H_{ub}(E, \Lambda^\infty(B, A))$. ∎

Proof of 4.1 Theorem: By 4.2 Proposition , we have $E\times\mathbb{C} \in (H_{ub})$. Using 4.4 Lemma for $F = A$, we get

$$H_b(E\times\mathbb{C}, A) = H_{ub}(E\times\mathbb{C}, A) .$$

We have $H(\mathbb{C}, A)$ is topologically isomorphic to $H(\mathbb{C})\widehat{\otimes}_\epsilon A$ [14] (Also see Ex. 4.91, p. 313 in [2]). Morever, the Fréchet-nuclear space $H(\mathbb{C})$ is topologically isomorphic to $\Lambda^\infty_\infty(\alpha)$, where

$$\Lambda^\infty_\infty(\alpha) = \Big\{\xi = (\xi_j) \subset \mathbb{C}^{\mathbb{N}}\colon\ \|\xi\|_k = \sup_j |\xi_j| e^{\rho_k \alpha_j} < +\infty,\ \text{for all } k\Big\}$$

and $\alpha = (\alpha_j),\ \alpha_j = j,\ \rho = \{\rho_k\} \uparrow +\infty$.

Hence

$$H(\mathbb{C}, A) = H(\mathbb{C})\,\widehat{\otimes}_\epsilon\, A = H(\mathbb{C})\,\widehat{\otimes}_\pi\, A = \Lambda^\infty_\infty(\alpha)\,\widehat{\otimes}_\pi\, A = \Lambda^\infty(B, A)\ .$$

Now we have

$$H_b(E\times\mathbb{C}, A) = H_b(E, H(\mathbb{C}, A)) = H_b(E, \Lambda^\infty(B, A))\ .$$

Hence

$$H_b(E, \Lambda^\infty(B, A)) = H_{ub}(E, \Lambda^\infty(B, A))\ .$$

Now by applying 4.5 Proposition to the sequence $\{\gamma(n) = n\}$ and $\{\rho_k\} \uparrow +\infty$ as above we have

$$\exists k > 0\ \ \forall r > 0\ \ \forall n > k\ \ \exists N_0 > n,\quad D > 0\ \ \forall f \in H_b(E, A)$$

$$e^{\rho_n j}\|f\|_{k,r} \le D \max_{1\le N\le N_0} e^{\rho_N j}\|f\|_{N,\rho_N} \quad \text{for all}\ \ j \ge 1\ . \tag{6}$$

For each n we can choose j_0 such that for $j \ge j_0$

$$e^{(\rho_{n-1}-\rho_n)j} D < 1\ . \tag{7}$$

For $k \le N \le n-1$ and $j \ge j_0$ the following inequality holds

$$D\, e^{\rho_N j}\|f\|_{N,\rho_N} < e^{\rho_n j}\|f\|_{k,r} \tag{8}$$

for $r \ge \rho_{n-1}$.

Indeed, in the converse case, we assume that there exist $k \le N \le n-1$ and $j \ge j_0$ such that

$$e^{\rho_n j}\|f\|_{k,r} \le D\, e^{\rho_N j}\|f\|_{N,\rho_N}$$

for $r \ge \rho_{n-1}$.

It follows that

$$\frac{\|f\|_{k,r}}{\|f\|_{N,\rho_N}} \le D\cdot e^{(\rho_N-\rho_n)j} < 1\ . \tag{9}$$

However, since $N \geq k$ it implies that $U_N \subset U_k$ and

$$\left\{ \|f(x)\| \colon \frac{x}{\rho_N} \in U_N \right\} \subset \left\{ \|f(x)\| \colon \frac{x}{r} \in U_k \right\}$$

for $r \geq \rho_{n-1}$. This shows that

$$1 \leq \frac{\|f\|_{k,r}}{\|f\|_{N,\rho_N}}$$

and, hence, it contradicts to (9).

Therefore, for $j \geq j_0$ and $r \geq \rho_{n-1}$

$$e^{\rho_n j}\|f\|_{k,r} \leq D \max \left\{ e^{\rho_N j}\|f\|_{N,\rho_N} \colon N = 1, 2, ..., k-1, n, ..., N_0 \right\} . \tag{10}$$

Now let $f \in H_b(E, A)$ and p, s be given. If $\|f\|_{p,\rho_s} = +\infty$ then (3) holds. Now assume that $\|f\|_{p,\rho_s} < +\infty$. Let j be the smallest natural number larger or equal to j_0 such that

$$D\|f\|_{p,\rho_s} \leq e^{(\rho_n - \rho_{k-1})j}\|f\|_{k,r} .$$

Then

$$e^{(\rho_n - \rho_{k-1})(j-1)}\|f\|_{k,r} \leq D\|f\|_{p,\rho_s} \leq e^{(\rho_n - \rho_{k-1})j}\|f\|_{k,r} . \tag{11}$$

For j such that (11) holds there exists $n \leq N^* \leq N_0$ which satisfies

$$e^{\rho_{N^*} j}\|f\|_{N^*,\rho_{N^*}} = \max_{1 \leq N \leq N_0} e^{\rho_N j}\|f\|_{N,\rho_N} .$$

Indeed, otherwise there exists $1 \leq N^* \leq k-1$ such that

$$e^{\rho_{N^*} j}\|f\|_{N^*,\rho_{N^*}} = \max_{1 \leq N \leq N_0} e^{\rho_N j}\|f\|_{N,\rho_N} .$$

From (10) we infer that

$$e^{\rho_n j}\|f\|_{k,r} \leq D\, e^{\rho_{N^*} j}\|f\|_{N^*,\rho_{N^*}} \quad \text{for } r \geq \rho_{n-1} .$$

Hence

$$\|f\|_{k,r} \leq D\, e^{(\rho_{N^*} - \rho_n)j}\|f\|_{N^*,\rho_{N^*}} < \|f\|_{N^*,\rho_{N^*}}$$

holds for all $r > 0$. It is impossible.

Now from (10) we deduce

$$e^{\rho_n j}\|f\|_{k,r} \leq D\, e^{\rho_{N^*} j}\|f\|_{N^*,\rho_{N^*}}$$

or equivalently

$$\begin{aligned}\|f\|_{k,r} \;&\leq\; D\, e^{(\rho_{N^*}-\rho_n)j}\|f\|_{N^*,\rho_{N^*}} \\ &\leq\; D\, e^{\theta\cdot\frac{\rho_{N^*}-\rho_n}{\rho_n-\rho_{k-1}}(\rho_n-\rho_{k-1})(j-1)}\|f\|_{N^*,\rho_{N^*}}\end{aligned}$$

where $\theta = \dfrac{j}{j-1}\cdot$

Put $d = \theta\cdot\dfrac{\rho_{N^*}-\rho_n}{\rho_n-\rho_{k-1}}$. Then

$$\|f\|_{k,r} \;\leq\; D\Big(D\frac{\|f\|_{p,\rho_s}}{\|f\|_{k,r}}\Big)^d \,\|f\|_{N^*,\rho_{N^*}}\;. \tag{12}$$

However

$$d \;=\; \theta\,\frac{\rho_{N^*}-\rho_n}{\rho_n-\rho_{k-1}} \;\leq\; \frac{\theta}{\rho_n-\rho_{k-1}}\;\rho_{N^*} \;\leq\; \rho_{N^*} \tag{13}$$

for n sufficiently large such that $\dfrac{\theta}{\rho_n-\rho_{k-1}}\leq 1$. On the other hand,

$$1 \;\leq\; e^{(\rho_n-\rho_{k-1})(j-1)} \leq D\;\frac{\|f\|_{p,\rho_s}}{\|f\|_{k,r}}\;\cdot \tag{14}$$

By combining (12), (13) and (14) we obtain that

$$\|f\|_{k,r} \;\leq\; D\Big(D\;\frac{\|f\|_{p,\rho_s}}{\|f\|_{k,r}}\Big)^{\rho_{N^*}}\;\|f\|_{N^*,\rho_{N^*}}\;.$$

Thus

$$\|f\|_{k,r}^{1+\rho_{N^*}} \;\leq\; C\,\|f\|_{N^*,\rho_{N^*}}\,\|f\|_{p,\rho_s}^{\rho_{N^*}}$$

where $C = D^{1+\rho_{N^*}}$ which completes the proof of 4.1 Theorem. ∎

5 – Fréchet-valued holomorphic mappings of uniformly bounded type and the linear topological invariants (H_{ub}) and (DN)

Based on results obtained in Section 4 this section is devoted to study the connection between the uniform boundedness of Fréchet-valued holomorphic mappings and the linear topological invariants (H_{ub}) and (DN). The main result of this section is the following

5.1. Theorem. *Let F be a Fréchet space. Then*

$$H_b(E,F) = H_{ub}(E,F)$$

holds for all Fréchet-Schwartz space E having the property (H_{ub}) if and only if $F \in (DN)$.

Proof:

Neccesity. Take $E = \Lambda_1(\beta)$ with $\beta = (\beta_n)$, $\beta_n = n$. Then $\Lambda_1(\beta)$ is a nuclear Fréchet space and by the hypothesis $L(\Lambda_1(\beta), F) = LB(\Lambda_1(\beta), F)$. Hence by (Theorem 2.1 in [15]) $F \in (DN)$.

Sufficiency. By the hypothesis and (Theorem 2.6 in [17]) we have that F is a subspace of $A\widehat{\otimes}_\pi s \cong A\widehat{\otimes}_\pi \Lambda_\infty^\infty(\alpha) = \Lambda^\infty(B,A)$ where A is a Banach space and $s \cong \Lambda_\infty^\infty(\alpha)$, $\alpha = (\log(n+1))_{n\geq 1}$. Hence it suffices to show that

$$H_b(E, \Lambda^\infty(B,A)) = H_{ub}(E, \Lambda^\infty(B,A)) \ .$$

We shall show that the condition (ii) of 4.5 Proposition is satisfied. Indeed, take a sequence $\{\gamma_n\} \uparrow$ and $\{\rho_n\} \uparrow +\infty$ such that $\lim\limits_{n\to\infty} \dfrac{\rho_n}{n} = 0$. As in (Theorem 3.2 in [15]) we may assume that $\gamma(n) = n$ for all $n \geq 1$. By the hypothesis and by applying 4.1 Theorem for the sequence $\{\rho_n\}$ we infer that there exists k such that $\forall p,s > 0$ $\forall r > 0$ $\forall n$ sufficiently large $\exists N_0 > n$, $C > 0$ $\forall f \in H_b(E,A)$ $\exists n \leq N^* \leq N_0$:

$$\|f\|_{k,r}^{1+\rho_{N^*}} \leq C\, \|f\|_{N^*,\rho_{N^*}}\, \|f\|_{p,\rho_s}^{\rho_{N^*}} \ . \tag{15}$$

Now take $p = 1$, $s = 1$. For given n there exists n_0 sufficiently large such that for all $N \geq n_0$ we have

$$\rho_N(n-1) \leq N - n \ .$$

Applying (15) for $p = 1$, $s = 1$ and $n = n_0$ $\forall r > 0$ we can find $N_0 > n_0$, $C > 1$ $\forall f \in H_b(E,A)$ $\exists\, n_0 \leq N^* \leq N_0$:

$$\|f\|_{k,r}^{1+\rho_{N^*}} \leq C\, \|f\|_{N^*,\rho_{N^*}}\, \|f\|_{1,\rho_1}^{\rho_{N^*}} \ . \tag{16}$$

Now we need to prove

$$e^{n\alpha_j}\|f\|_{k,r} \leq C \max_{1\leq N\leq N_0} e^{N\alpha_j}\|f\|_{N,\rho_N} \quad \text{for all } \ j \geq 1 \ . \tag{17}$$

Given $j \geq 1$. Then either

$$e^{n\alpha_j}\|f\|_{k,r} \leq e^{\alpha_j}\|f\|_{1,\rho_1}$$

or, in the converse case,

$$e^{\alpha_j}\|f\|_{1,\rho_1} \leq e^{n\alpha_j}\|f\|_{k,r} \ .$$

In the first case (17) obviously holds. We consider the second. Then we have

$$\|f\|_{1,\rho_1} \leq e^{(n-1)\alpha_j}\|f\|_{k,r} \ .$$

From (16) we have

$$\begin{aligned}\|f\|_{k,r}^{1+\rho_{N^*}} &\leq C\,\|f\|_{N^*,\rho_{N^*}}\ e^{\rho_{N^*}(n-1)\alpha_j}\|f\|_{k,r}^{\rho_{N^*}}\\ &\leq C\,\|f\|_{N^*,\rho_{N^*}}\ e^{(N^*-n)\alpha_j}\|f\|_{k,r}^{\rho_{N^*}} \ .\end{aligned}$$

Hence

$$e^{n\alpha_j}\|f\|_{k,r} \leq C\,e^{N^*\alpha_j}\|f\|_{N^*,\rho_{N^*}} \ .$$

Combining all these results we see that (17) is satisfied.

By 4.5 Proposition we have

$$H_b(E, A\widehat{\otimes}_\pi\Lambda_\infty^\infty(\alpha)) = H_{ub}(E, A\widehat{\otimes}_\pi\Lambda_\infty^\infty(\alpha)) \ .$$

This completes the proof. ∎

At the end of this paper we want to give an equivalent condition for which (1) holds in the case that $E = \Lambda(B)$ is the space of Köthe sequences and F is a Fréchet space. With the notations used as above with $B = (b_{j,k})_{j,k\geq 1}$ a matrix satisfying (*) we define the sequence space $\Lambda(B)$ given by

$$\Lambda(B) = \left\{\xi = (\xi_1, \xi_2, ...) :\ \|\xi\|_k = \sum_{j=1}^{\infty}|\xi_j|\,b_{j,k} < +\infty \ \text{ for all } k \geq 1\right\} .$$

$\Lambda(B)$ is a Fréchet space with the topology defined by the system of semi-norms $(\|\cdot\|_k)$. If we consider the Schauder basis $\{e_j\}_{j\geq 1}$ in $\Lambda(B)$ of the form

$$e_j = \Big(0, 0, ..., 0, \underset{\hat{j}}{1}, 0, ...\Big)$$

then $\{e_j\}_{j\geq 1}$ is an absolute basis of $\Lambda(B)$ and

$$\|e_j\|_k = b_{j,k}$$

for $j, k \geq 1$.

Now we prove the following

5.2. Proposition. *Let $\Lambda(B) \in (H_{ub})$ and F be a Fréchet space. The following are equivalent*

(i) $H_b(\Lambda(B), F) = H_{ub}(\Lambda(B), F)$;

(ii) $\forall \{\gamma(n)\} \uparrow \ \ \forall \{\rho_n\} \uparrow +\infty \ \ \exists k \ \ \forall r > 0 \ \ \forall n \ \ \exists\, N_0 > 0,\ C > 0$

$$\frac{\|x\|_n r^p}{b_{j_1,\gamma(k)} \cdots b_{j_p,\gamma(k)}} \ \le\ C \max_{1 \le N \le N_0} \frac{\|x\|_N \rho_N^p}{b_{j_1,\gamma(N)} \cdots b_{j_p,\gamma(N)}} \tag{18}$$

for $x \in F,\ j_1, ..., j_p \ge 1,\ p \ge 1.$

Proof: (**i**)$\Rightarrow$(**ii**) Let $\{\gamma(n)\} \uparrow$ and $\{\rho_n\} \uparrow +\infty$ be given. By 4.3 Proposition we can find k satisfying (4). For $j_1, ..., j_p \ge 1,\ p \ge 1,\ x \in F$ we define $f \in H_b(\Lambda(B), F)$ given by

$$f(\xi) \;=\; \xi_{j_1} ... \xi_{j_p}\, x$$

where $\xi = (\xi_1, ..., \xi_{j_1}, ..., \xi_{j_2}, ..., \xi_{j_p}, ...) \in \Lambda(B)$. Then

$$\frac{\|x\|_n r^p}{b_{j_1,\gamma(k)} \cdots b_{j_p,\gamma(k)} p^p} \;=\; \|f\|_{n,\gamma(k),r} \;\le\; C \max_{1 \le N \le N_0} \|f\|_{N,\gamma(N),\rho_N}\ ,$$

$$\frac{\|x\|_n r^p}{b_{j_1,\gamma(k)} \cdots b_{j_p,\gamma(k)} p^p} \;\le\; C \max_{1 \le N \le N_0} \frac{\|x\|_N \rho_N^p}{b_{j_1,\gamma(N)} \cdots b_{j_p,\gamma(N)} p^p}\ .$$

Hence we have (18).

(**ii**)$\rightarrow$(**i**) Let $f \in H_b(E, F)$. Since $\Lambda(B) \in (H_{ub})$ it follows that for each $n \ge 1$ then exists $\gamma(n)$ such that

$$M(n, \gamma(n), \rho) \;=\; \sup\Big\{\|f(\xi)\|_n \colon\ \|\xi\|_{\gamma(n)} \le \rho\Big\} \;<\; +\infty$$

for all $\rho > 0$. We may assume that $\{\gamma(n)\} \uparrow$. Fix a sequence $\{\rho_n\} \uparrow$. Write the Taylor expansion of f at $0 \in \Lambda(B)$

$$f(\xi) \;=\; \sum_{p \ge 0} P_p\, f(\xi) \;=\; \sum_{p \ge 0} \sum_{j_1, ..., j_p \ge 1} \widehat{P_p} f(e_{j_1}, ..., e_{j_p})\, \xi_{j_1} ... \xi_{j_p}\ .$$

Using (ii) for the sequence $\{\gamma(n)\} \uparrow$ defined as above we can find k such that (18) holds. On the other hand, in (18) we can take $r=1$. Now we have

$$
\begin{aligned}
\|f(\xi)\|_n &\le \sum_{p\ge 0}\ \sum_{j_1,\dots,j_p\ge 1} \|\widehat{P_p f}(e_{j_1},\dots,e_{j_p})\|_n\ |\xi_{j_1}|\cdots|\xi_{j_p}| \\
&\le \sum_{p\ge 0}\ \sum_{j_1,\dots,j_p\ge 1} \frac{\|\widehat{P_p f}(e_{j_1},\dots,e_{j_p})\|_n}{b_{j_1,\gamma(k)}\cdots b_{j_p,\gamma(k)}}\ b_{j_1,\gamma(k)}|\xi_{j_1}|\cdots b_{j_p,\gamma(k)}|\xi_{j_p}| \\
&\le \sum_{p\ge 0}\ \sup_{j_1,\dots,j_p\ge 1} \frac{\|\widehat{P_p f}(e_{j_1},\dots,e_{j_p})\|_n}{b_{j_1,\gamma(k)}\cdots b_{j_p,\gamma(k)}} \sum_{j_1,\dots,j_p\ge 1} b_{j_1,\gamma(k)}|\xi_{j_1}|\cdots b_{j_p,\gamma(k)}|\xi_{j_p}| \\
&\le C\sum_{p\ge 0}\ \sup_{j_1,\dots,j_p\ge 1}\left(\max_{1\le N\le N_0}\frac{\|\widehat{P_p f}(e_{j_1},\dots,e_{j_p})\|_N\,\rho_N^p}{b_{j_1,\gamma(N)}\cdots b_{j_p,\gamma(N)}}\right)\|\xi\|^p_{\gamma(k)} \qquad (19)\\
&\le C\sum_{p\ge 0}\rho^p_{N_0}\,\frac{1}{\rho^p}\sup_{j_1,\dots,j_p\ge 1}\left(\max_{1\le N\le N_0}\frac{\|\widehat{P_p f}(e_{j_1},\dots,e_{j_p})\|_N\,\rho^p}{b_{j_1,\gamma(N)}\cdots b_{j_p,\gamma(N)}}\right)\|\xi\|^p_{\gamma(k)} \\
&\le C\sum_{p\ge 0}\frac{\rho^p_{N_0}}{\rho^p}\max_{1\le N\le N_0}\ \sup_{j_1,\dots,j_p\ge 1}\left\|\widehat{P_p f}\left(\frac{\rho e_{j_1}}{b_{j_1,\gamma(N)}},\dots,\frac{\rho e_{j_p}}{b_{j_p,\gamma(N)}}\right)\right\|_N \|\xi\|^p_{\gamma(k)} \\
&\le C\sum_{p\ge 0}\frac{\rho^p_{N_0}}{\rho^p}\max_{1\le N\le N_0}\left(\frac{p^p}{p!}\,\|f\|_{N,\gamma(N),\rho}\right)\|\xi\|^p_{\gamma(k)}\ .
\end{aligned}
$$

Now let $\|\xi\|_{\gamma(k)}\le R$ for arbitrary $R>0$. From (19) we derive that

$$\|f\|_{n,\gamma(k),R}\ \le\ C\max_{1\le N\le N_0} M(N,\gamma(N),\rho)\sum_{p\ge 0}\frac{\rho^p_{N_0}\cdot p^p}{p!}\ \frac{R^p}{\rho^p}\ <\ +\infty$$

for ρ sufficiently large and the conclusion follows. ∎

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