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VALUE DISTRIBUTION OF A WRONSKIAN

Indrajit Lahiri and Abhijit Banerjee

Abstract: We discuss the value distribution of a Wronskian generated by a meromorphic function having maximum deficiency sum and as an application we solve Drasin's problem for meromorphic functions of unrestricted order.

1 – Introduction and Definitions

In value distribution theory, one studies the frequency with which a meromorphic function takes on different values in the complex plane. A common technical difficulty in value distribution theory is the presence of the 'exceptional set' in the second fundamental theorem of Nevanlinna [7, p. 31]. This exceptional set can be eliminated for functions of finite order i.e., for functions that do not grow faster than a certain rate. Unfortunately, this exceptional set cannot be eliminated in general and this is often the reason that certain results are only proved for functions of finite order. Toda [14] developed an integration trick that more or less removes the obstacle of the exceptional set in studying meromorphic functions of infinite order. This integration trick of Toda is the principal motivation of this paper and we use the same to extend a result of Singh–Kulkarni [13] and a result of Yang–Wang [18] of the value distribution theory to meromorphic functions of unrestricted order. Since the natural extension of a derivative is a differential polynomial, in the paper we also extend the result of Singh–Kulkarni [13] to a special type of linear differential polynomials viz., the Wronskians. Before discussing the main problem undertaken in the paper let us explain some basic definitions of the value distribution theory (cf. [7]).

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Let f be a meromorphic function defined in the open complex plane $\mathbb{C}$. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $n(t, a; f)$ $(\overline{n}(t, a; f))$ the number of a-points (distinct a-points) of f in $|z| \leq t$, where an ∞-point is a pole of f. We put

$$N(r, a; f) = \int_0^r \frac{n(t, a; f) - n(0, a; f)}{t}\, dt + n(0, a; f) \log r$$

and

$$\overline{N}(r, a; f) = \int_0^r \frac{\overline{n}(t, a; f) - \overline{n}(0, a; f)}{t}\, dt + \overline{n}(0, a; f) \log r\ .$$

The function $N(r, a; f)$ $(\overline{N}(r, a; f))$ are called the counting function of a-points (distinct a-points) of f. In many occasions $N(r, \infty; f)$ and $\overline{N}(r, \infty; f)$ are denoted respectively by $N(r, f)$ and $\overline{N}(r, f)$.

We also put

$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})|\, d\theta\ ,$$

where $\log^+ x = \log x$ if $x \geq 1$ and $\log^+ x = 0$ if $0 \leq x < 1$.

For $a \in \mathbb{C}$ we denote $m(r, \frac{1}{f-a})$ by $m(r, a; f)$ and we mean by $m(r, \infty; f)$ the function $m(r, f)$, which is called the proximity function of f.

The function $T(r, f) = m(r, f) + N(r, f)$ is called the characteristic function of f. If $a \in \mathbb{C} \cup \{\infty\}$, the quantity

$$\delta(a; f) = 1 - \limsup_{r\to\infty} \frac{N(r, a; f)}{T(r, f)} = \liminf_{r\to\infty} \frac{m(r, a; f)}{T(r, f)}$$

is called Nevanlinna deficiency of the value a.

From the second fundamental theorem it follows that the set of values of $a \in \mathbb{C} \cup \{\infty\}$ for which $\delta(a; f) > 0$ is countable and $\sum_{a\neq\infty} \delta(a; f) + \delta(\infty; f) \leq 2$ (cf. [7, p.43]). If, in particular, $\sum_{a\neq\infty} \delta(a; f) + \delta(\infty; f) = 2$, we say that f has the maximum deficiency sum.

The numbers $\rho = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}$ and $\lambda = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log r}$ are called respectively the order and the lower order of the function f.

We denote by $S(r, f)$ any function of r such that $S(r, f) = o\{T(r, f)\}$ as $r \to \infty$ through all values of r if f is of finite order and except for a set of values of r of finite linear measure if f is of infinite order.

Definition 1. A meromorphic function $a = a(z)$ is called small with respect to f if $T(r, a) = S(r, f)$. $\square$

Definition 2. Let $a_1, a_2, ..., a_k$ be linearly independent meromorphic functions and small with respect to f. We denote by $L(f) = W(a_1, a_2, ..., a_k, f)$ the Wronskian determinant of $a_1, a_2, ..., a_k, f$ i.e.,

$$L(f) = \begin{vmatrix} a_1 & a_2 & \cdots & a_k & f \\ a_1' & a_2' & \cdots & a_k' & f' \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_1^{(k)} & a_2^{(k)} & \cdots & a_k^{(k)} & f^{(k)} \end{vmatrix} . \square$$

Definition 3 ([1])**.** We denote by $\sigma(f)$ the collection of all meromorphic functions $a = a(z)$ satisfying $T(cr, a) = o\{T(r, f)\}$ as $r \to \infty$ for some $c > 1$, where c may depend on $a(z)$. $\square$

Definition 4 ([8] (see also [14]))**.** We put for $a \in \mathbb{C} \cup \{\infty\}$

$$T_o(r, f) = \int_1^r \frac{T(t, f)}{t}\, dt\,, \qquad N_o(r, a; f) = \int_1^r \frac{N(t, a; f)}{t}\, dt\ ,$$

$$m_o(r, a; f) = \int_1^r \frac{m(t, a; f)}{t}\, dt\,, \qquad S_o(r, f) = \int_1^r \frac{S(t, f)}{t}\, dt\,, \qquad \text{etc.}$$

Also we put for a transcendental meromorphic function f (cf. [14])

$$\delta_o(a; f) \;=\; 1 - \limsup_{r\to\infty} \frac{N_o(r, a; f)}{T_o(r, f)} \;=\; \liminf_{r\to\infty} \frac{m_o(r, a; f)}{T_o(r, f)}$$

for $a \in \mathbb{C} \cup \{\infty\}$. $\square$

Shah–Singh [11, 12], Singh–Kulkarni [13], Wang–Dai [15], Yang [16] and Yi [18] studied the comparative growth of a meromorphic function having maximum deficiency sum and its derivatives. Lahiri–Sharma [9] studied the comparative growth and value distribution of a linear differential polynomial generated by a meromorphic function having maximum deficiency sum.

In order to study the value distribution of the derivative of a meromorphic function having maximum deficiency sum Singh–Kulkarni [13] proved the following result.

Theorem A. *Suppose that f is a transcendental meromorphic function of finite order and f has the maximum deficiency sum. Then*

$$\frac{1-\delta(\infty;f)}{2-\delta(\infty;f)} \le K(f') \le \frac{2(1-\delta(\infty;f))}{2-\delta(\infty;f)} ,$$

where $K(f') = \limsup_{r\to\infty} \frac{N(r,0;f')+N(r,\infty;f')}{T(r,f')}$. ∎

Improving Theorem A recently Fang [4] proved the following result.

Theorem B. *Suppose that f is a transcendental meromorphic function of finite order and f has the maximum deficiency sum. Then for any positive integer k,*

$$K(f^{(k)}) = \frac{2\,k(1-\delta(\infty;f))}{1+k-k\,\delta(\infty;f)} ,$$

where $K(f^{(k)}) = \lim_{r\to\infty} \frac{N(r,0;f^{(k)})+N(r,\infty;f^{(k)})}{T(r,f^{(k)})}$. ∎

If $f = \exp(z)(1-\exp(z))$ then $\sum_{a\neq\infty} \delta(a;f)+\delta(\infty;f) = 3/2$, $\delta(\infty;f) = 1$ but $K(f') = 1/2$. So it appears that the condition $\sum_{a\neq\infty} \delta(a;f)+\delta(\infty;f) = 2$ is necessary for Theorems A and B. Although it remains an open question for which minimum value of the total deficiency of f, Theorem B is valid.

If $f = \exp(\exp(z))$ then f is of infinite order with the maximum deficiency sum. Also we see that $\delta(\infty;f) = 1$ and $K(f') = 0$. So one may expect that the order restriction of f for the validity of Theorem B is unnecessary. In fact, in the paper we show that the finiteness restriction on the order of f in Theorem B is redundant. Further using a result of Frank and Weissenborn [5] on the estimation of the zeros of a Wronskian we extend Theorem B to a Wronskian. Finally as an application of the main theorem we solve Drasin's problem (see Section 4) for meromorphic functions of unrestricted order.

2 – Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 1 ([5]). *Let f be a transcendental meromorphic function and let $a_1, a_2, ..., a_k$ be meromorphic functions which are linearly independent and small with respect to f. Then for every $\varepsilon(>0)$*

$$k\overline{N}(r,f) \le N(r,0;L(f)) + (1+\varepsilon)\,N(r,f) + S(r,f) . \blacksquare$$

Lemma 2 ([3]). *Let f be a meromorphic function with more than one deficient value. Then the lower order of f is a positive number or infinity.* ∎

Lemma 3 ([1]). *Let f be a meromorphic function and $\{a_i\}_{i=1}^{k}$ be k linearly independent functions of the class $\sigma(f)$. Then $L(f) = W(a_1, a_2, ..., a_k, f)$ and f have the same order and lower order.* ∎

Henceforth we shall denote by $L(f)$ the Wronskian $W(a_1, a_2, ..., a_k, f)$ where $a_1, a_2, ..., a_k$ are linearly independent elements of $\sigma(f)$ and $a_1 = 1$.

Lemma 4 ([18]). *Let f be a meromorphic function. If f has the maximum deficiency sum then $\sum\limits_{b\neq\infty} \delta(b; f) = \sum\limits_{b\neq\infty} \delta_o(b; f)$ and $\delta(\infty; f) = \delta_o(\infty; f)$.* ∎

Lemma 5 ([6]). *Suppose that g is an increasing real valued function of $x \geq 0$ and that*

$$\lim_{x\to\infty} \frac{g(x)}{x^{\alpha}} = \infty$$

for every $\alpha > 0$. Then

$$\lim_{x\to\infty} \frac{G(x)g'(x)}{\{g(x)\}^2} = 1$$

where $G(x) = \int\limits_0^x g(t)\,dt$. ∎

Lemma 6 ([10]). *Suppose that F and G be two real valued functions of $x \geq 0$ and satisfy the following conditions for all sufficiently large x:*

(**i**) *F and G have continuous derivatives,*

(**ii**) *F is an increasing convex functions,*

(**iii**) *$\dfrac{1}{G}$ is a convex function,*

(**iv**) $\lim\limits_{x\to\infty} \frac{F(x)}{G(x)} = A(> 0)$.

Then $\lim\limits_{x\to\infty} \frac{F'(x)}{G'(x)} = A$. ∎

Lemma 7. *Let f be a meromorphic function having positive or infinite lower order. If $\lim\limits_{r\to\infty} \frac{T_o(r,L(f))}{T_o(r,f)} = A(> 0)$ then $\lim\limits_{r\to\infty} \frac{T(r,L(f))}{T(r,f)} = A$.* ∎

Lemma 8. *Let f be a meromorphic function having positive or infinite lower order. If for $a \in \mathbb{C} \cup \{\infty\}$ $\lim\limits_{r\to\infty} \frac{N_o(r,a;L(f))}{T_o(r,L(f))} = A(>0)$ then $\lim\limits_{r\to\infty} \frac{N(r,a;L(f))}{T(r,L(f))} = A$.* ∎

Using Lemma 3, Lemma 5 and Lemma 6 we can prove Lemma 7 and Lemma 8 in the line of Lemma 6 [18].

Lemma 9. *Let f be a meromorphic function having positive or infinite lower order. Let a and b be two distinct complex numbers, finite or infinite. If*

$$\lim_{r\to\infty} \frac{N_o(r,a;L(f)) + N_o(r,b;L(f))}{T_o(r,L(f))} = A(>0)$$

then

$$\lim_{r\to\infty} \frac{N(r,a;L(f)) + N(r,b;L(f))}{T(r,L(f))} = A\ .$$

Proof: Let $n(r,a,b;f)$ denote the number of a-points and b-points of f in $|z| \le r$, counted with proper multiplicities. Also let $N(r,a,b;f)$ be defined in terms of $n(r,a,b;f)$ in the usual manner. Then clearly $N(r,a,b;f) = N(r,a;f) + N(r,b;f)$.

Now we follow some techniques of [18]. Let $F(x) = N_o(e^x,a,b;L(f))$ and $G(x) = T_o(e^x,L(f))$. By the given condition we get

$$\lim_{x\to\infty} \frac{F(x)}{G(x)} = A > 0\ .$$

Also we note that

$$F(x) = \int_0^x N(e^t,a,b;L(f))\,dt \quad \text{and} \quad G(x) = \int_0^x T(e^t,L(f))\,dt\ .$$

We can easily verify that $F(x)$, $G(x)$ have continuous derivatives and $F(x)$ is an increasing convex function of x.

Since f has a non-zero lower order and so by Lemma 3 $L(f)$ also has a non-zero lower order, there exists a $\mu(>0)$ such that

$$T(r,L(f)) > r^{\mu}$$

for all large values of r.

Let $g(x) = T(e^x,L(f))$. Then for all large values of x we get $g(x) > e^{\mu x}$ and so $\lim\limits_{x\to\infty} \frac{g(x)}{x^{\alpha}} = \infty$ for every $\alpha(>0)$. So by Lemma 5 we get $\lim\limits_{x\to\infty} \frac{G(x)g'(x)}{\{g(x)\}^2} = 1$. Since

$$\frac{d^2}{dx^2}\left(\frac{1}{G(x)}\right) = \frac{2\,(g(x))^2 - G(x)\,g'(x)}{\{G(x)\}^3}\ ,$$

it follows that $\frac{d^2}{dx^2}\left(\frac{1}{G(x)}\right) > 0$ for all large values of x and so $\frac{1}{G(x)}$ is a convex function of x for all large values of x. Therefore by Lemma 6 we obtain

$$\lim_{x\to\infty} \frac{F'(x)}{G'(x)} = A ,$$

$$\text{i.e.,} \qquad \lim_{x\to\infty} \frac{N(e^x, a, b; L(f))}{T(e^x, L(f))} = A ,$$

$$\text{i.e.,} \qquad \lim_{r\to\infty} \frac{N(r, a, b; L(f))}{T(r, L(f))} = A .$$

This proves the lemma. ∎

Lemma 10 ([8])**.** *Let f be a meromorphic function. Then*

$$\lim_{r\to\infty} \frac{S_o(r, f)}{T_o(r, f)} = 0$$

through all values of r. ∎

Lemma 11. *Let f be a transcendental meromorphic function having the maximum deficiency sum. Then*

$$\lim_{r\to\infty} \frac{T(r, L(f))}{T(r, f)} = \lim_{r\to\infty} \frac{T_o(r, L(f))}{T_o(r, f)} = 1 + k - k\,\delta(\infty; f) .$$

Proof: Let $b_1, b_2, ..., b_p$ be distinct finite complex numbers. Then on integration we get from Littlewood's inequality

$$\begin{aligned} \sum_{\nu=1}^{p} m_o(r, b_\nu; f) &\le m_o(r, 0; f') + S_o(r, f) \\ &\le T_o(r, f) + \overline{N}_o(r, f) + S_o(r, f) . \end{aligned}$$

By Lemma 10 this implies

$$\begin{aligned} \text{(1)} \qquad \sum_{\nu=1}^{p} \delta_o(b_\nu; f) &\le 1 + \liminf_{r\to\infty} \frac{\overline{N}_o(r, f)}{T_o(r, f)} \\ &\le 1 + \limsup_{r\to\infty} \frac{N_o(r, f)}{T_o(r, f)} \\ &= 2 - \delta_o(\infty; f) . \end{aligned}$$

Since p is arbitrary and f has maximum deficiency sum, by Lemma 4 we get from (1)

$$\lim_{r\to\infty} \frac{\overline{N}_o(r,f)}{T_o(r,f)} = \lim_{r\to\infty} \frac{N_o(r,f)}{T_o(r,f)} = 1 - \delta(\infty; f) . \tag{2}$$

Again for finite complex numbers $b_1, b_2, ..., b_p$ we get (cf. [9])

$$\begin{aligned}\sum_{\nu=1}^{p} m(r, b_\nu; f) &\le m(r, 0; L(f)) + S(r,f)\\ &= T(r, L(f)) - N(r, 0; L(f)) + S(r,f) .\end{aligned}$$

By Lemma 1 we obtain

$$\sum_{\nu=1}^{p} m(r, b_\nu; f) \le T(r, L(f)) + (1+\varepsilon)\, N(r,f) - k\,\overline{N}(r,f) + S(r,f) ,$$

which on integration gives

$$\sum_{\nu=1}^{p} m_o(r, b_\nu; f) \le T_o(r, L(f)) + (1+\varepsilon)\, N_o(r,f) - k\,\overline{N}_o(r,f) + S_o(r,f) .$$

Hence by Lemma 10 we get in view of (2)

$$\sum_{\nu=1}^{p} \delta_o(b_\nu; f) \le \liminf_{r\to\infty} \frac{T_o(r, L(f))}{T_o(r,f)} + (1+\varepsilon)\,(1 - \delta(\infty; f)) - k(1 - \delta(\infty; f)) .$$

Since p and $\varepsilon(> 0)$ are arbitrary, we obtain by Lemma 4 and the given condition

$$\liminf_{r\to\infty} \frac{T_o(r, L(f))}{T_o(r,f)} \ge 1 + k - k\,\delta(\infty; f) . \tag{3}$$

On the other hand (cf. [9])

$$T(r, L(f)) \le T(r,f) + k\,\overline{N}(r,f) + S(r,f)$$

which on integration gives

$$T_o(r, L(f)) \le T_o(r,f) + k\,\overline{N}_o(r,f) + S_o(r,f) .$$

So by (2) and Lemma 10 we obtain

$$\limsup_{r\to\infty} \frac{T_o(r, L(f))}{T_o(r,f)} \le 1 + k - k\,\delta(\infty; f) . \tag{4}$$

Since f has the maximum deficiency sum and $1+k-k\,\delta(\infty;f)>0$, it follows from (3) and (4) by Lemma 2 and Lemma 7 that

$$\lim_{r\to\infty}\frac{T(r,L(f))}{T(r,f)} \;=\; \lim_{r\to\infty}\frac{T_o(r,L(f))}{T_o(r,f)} \;=\; 1+k-k\,\delta(\infty;f)\;.$$

This proves the lemma. ∎

3 – The Main Result

In this section we present the main theorem of the paper.

Theorem 1. *Let f be a transcendental meromorphic function having the maximum deficiency sum. Let $a_1=1,\ a_2,a_3,...,a_k$ be linearly independent meromorphic functions of the class $\sigma(f)$ and $L(f)=W(a_1,a_2,...,a_k,f)$. Then*

$$\lim_{r\to\infty}\frac{N(r,0;L(f))+N(r,\infty;L(f))}{T(r,L(f))} \;=\; \frac{2\,k(1-\delta(\infty;f))}{1+k-k\,\delta(\infty;f)}\;.$$

Proof: For finite complex numbers $b_1,b_2,...,b_p$ we get (cf. [9])

$$\sum_{\nu=1}^{p} m(r,b_\nu;f) \;\le\; m(r,0;L(f))+S(r,f)$$

which on integration gives

$$\sum_{\nu=1}^{p} m_o(r,b_\nu;f) \;\le\; m_o(r,0;L(f))+S_o(r,f)$$

and so by Lemma 10 we get

$$\liminf_{r\to\infty}\frac{m_o(r,0;L(f))}{T_o(r,f)} \;\ge \sum_{\nu=1}^{p}\delta_o(b_\nu;f)\;.$$

Since p is arbitrary, we get by the given condition and Lemma 4

$$\liminf_{r\to\infty}\frac{m_o(r,0;L(f))}{T_o(r,f)} \;\ge\; 2-\delta(\infty;f)\;. \tag{5}$$

Again since $T(r, L(f)) \leq T(r, f)+k\,\overline{N}(r, f)+S(r, f)$, by the first fundamental theorem and Lemma 1 we get

$$\begin{aligned} m(r,0;L(f)) &= T(r,L(f)) - N(r,0;L(f)) + S(r,L(f)) \\ &\leq T(r,f) + k\,\overline{N}(r,f) - N(r,0;L(f)) + S(r,f) \\ &\leq T(r,f) + (1+\varepsilon)\,N(r,f) + S(r,f)\ , \end{aligned}$$

which gives on integration

$$m_o(r,0;L(f)) \leq T_o(r,f) + (1+\varepsilon)\,N_o(r,f) + S_o(r,f)\ .$$

Hence by Lemma 4 and Lemma 10 we get because $\varepsilon(>0)$ is arbitrary

$$\limsup_{r\to\infty} \frac{m_o(r,0;L(f))}{T_o(r,f)} \leq 2 - \delta(\infty;f)\ . \tag{6}$$

From (5) and (6) we obtain

$$\lim_{r\to\infty} \frac{m_o(r,0;L(f))}{T_o(r,f)} = 2 - \delta(\infty;f)\ . \tag{7}$$

Now by Lemma 11 and (7) we get in view of the modified first fundamental theorem (cf. [14])

$$\begin{aligned} \lim_{r\to\infty} \frac{N_o(r,0;L(f))}{T_o(r,L(f))} &= 1 - \lim_{r\to\infty} \frac{m_o(r,0;L(f))}{T_o(r,f)} \lim_{r\to\infty} \frac{T_o(r,f)}{T_o(r,L(f))} \\ &= 1 - \frac{2-\delta(\infty;f)}{1+k-k\,\delta(\infty;f)} \\ &= \frac{(k-1)\,(1-\delta(\infty;f))}{1+k-k\,\delta(\infty;f)}\ . \end{aligned} \tag{8}$$

Since $N_o(r,\infty;L(f)) = N_o(r,f) + k\,\overline{N}_o(r,f) + S_o(r,f)$, it follows by (2) and Lemma 10, Lemma 11 that

$$\begin{aligned} \lim_{r\to\infty} \frac{N_o(r,\infty;L(f))}{T_o(r,L(f))} &= \lim_{r\to\infty} \frac{N_o(r,f) + k\,\overline{N}_o(r,f) + S_o(r,f)}{T_o(r,f)} \lim_{r\to\infty} \frac{T_o(r,f)}{T_o(r,L(f))} \\ &= \frac{(k+1)\,(1-\delta(\infty;f))}{1+k-k\,\delta(\infty;f)}\ . \end{aligned} \tag{9}$$

Now we consider two cases.

Case I Let $\delta(\infty; f) < 1$.

From (8) and (9) we get

$$\lim_{r\to\infty} \frac{N_o(r,0;L(f)) + N_o(r,\infty;L(f))}{T_o(r,L(f))} = \frac{2\,k(1-\delta(\infty;f))}{1+k-k\,\delta(\infty;f)}$$

and so by Lemma 9 we obtain

$$\lim_{r\to\infty} \frac{N(r,0;L(f)) + N(r,\infty;L(f))}{T(r,L(f))} = \frac{2\,k(1-\delta(\infty;f))}{1+k-k\,\delta(\infty;f)}\ .$$

Case II Let $\delta(\infty; f) = 1$.

From (8) and (9) we get

$$\lim_{r\to\infty} \frac{N_o(r,0;L(f))}{T_o(r,L(f))} = \lim_{r\to\infty} \frac{N_o(r,\infty;L(f))}{T_o(r,L(f))} = 0\ . \tag{10}$$

For $a \neq 0, \infty$ we get on integration from the second fundamental theorem

$$T_o(r,L(f)) \le N_o(r,a;L(f)) + N_o(r,0;L(f)) + N_o(r,\infty;L(f)) + S_o(r,L(f))\ .$$

So by (10) and Lemma 10 we get

$$\lim_{r\to\infty} \frac{N_o(r,a;L(f))}{T_o(r,L(f))} = 1\ . \tag{11}$$

Hence by Lemma 8 we get from (11)

$$\lim_{r\to\infty} \frac{N(r,a;L(f))}{T(r,L(f))} = 1\ . \tag{12}$$

Now from (10) and (11) we obtain

$$\lim_{r\to\infty} \frac{N_o(r,0;L(f)) + N_o(r,a;L(f))}{T_o(r,L(f))} = 1$$

and

$$\lim_{r\to\infty} \frac{N_o(r,\infty;L(f)) + N_o(r,a;L(f))}{T_o(r,L(f))} = 1\ .$$

Therefore by Lemma 9 we get

$$\begin{aligned}\lim_{r\to\infty} \frac{N(r,0;L(f)) + N(r,a;L(f))}{T(r,L(f))} &= \lim_{r\to\infty} \frac{N(r,\infty;L(f)) + N(r,a;L(f))}{T(r,L(f))} \\ &= 1\ . \end{aligned}\tag{13}$$

From (12) and (13) we obtain

$$\lim_{r\to\infty} \frac{N(r,0;L(f))}{T(r,L(f))} = \lim_{r\to\infty} \frac{N(r,\infty;L(f))}{T(r,L(f))} = 0 .$$

Therefore

$$\lim_{r\to\infty} \frac{N(r,0;L(f)) + N(r,\infty;L(f))}{T(r,L(f))} = 0 .$$

This proves the theorem. ∎

4 – Application

In this section we discuss an application of the main result.

In 1976 D. Drasin [2] posed the following problem:

If the order of f is finite and $\sum_{b\neq\infty} \delta(b;f) = 2$

then must we have $\sum_{b\neq\infty} \delta(b;f') = \delta(0;f') = 1$ *?*

Generalizing this problem to the k^{th} derivative Yang and Wang [17] solved it affirmatively and proved the following theorem.

Theorem C. *Let f be a transcendental meromorphic function of finite order. If $\sum_{b\neq\infty} \delta(b;f) = 2$ then for any positive integer k*

$$\sum_{b\neq\infty} \delta(b;f^{(k)}) = \delta(0;f^{(k)}) = \frac{2}{k+1} . \ \blacksquare$$

As an application of Theorem 1 we show that Theorem C remains valid even if the finiteness restriction on the order of f is withdrawn. Thus we solve Drasin's problem for functions of unrestricted order.

Theorem 2. *Let f be a transcendental meromorphic function having the maximum deficiency sum. If $\delta(\infty;f) = 0$ then for any positive integer k*

$$\sum_{b\neq\infty} \delta(b;f^{(k)}) = \delta(0;f^{(k)}) = \frac{2}{k+1} .$$

Proof: From (9) we get by Lemma 8 for $L(f) = f^{(k)}$ that

$$\lim_{r\to\infty} \frac{N(r, f^{(k)})}{T(r, f^{(k)})} = 1 \ . \tag{14}$$

So by Theorem 1 we get for $L(f) = f^{(k)}$

$$\lim_{r\to\infty} \frac{N(r, 0; f^{(k)})}{T(r, f^{(k)})} = \frac{k-1}{k+1}$$

and hence

$$\delta(0; f^{(k)}) = \frac{2}{k+1} \ . \tag{15}$$

By Littlewood's inequality we get for distinct finite complex numbers $b_1, b_2, ..., b_q$

$$\begin{aligned}\sum_{\nu=1}^{q} m(r, b_\nu; f) &\le m(r, 0; f^{(k+1)}) + S(r, f^{(k)})\\ &\le T(r, f^{(k+1)}) - N(r, 0; f^{(k+1)}) + S(r, f^{(k)})\\ &\le T(r, f^{(k)}) + \overline{N}(r, f) - N(r, 0; f^{(k+1)}) + S(r, f^{(k)}) \ .\end{aligned}$$

So by Lemma 1 and Lemma 11 we get for $L(f) = f^{(k+1)}$ and $L(f) = f^{(k)}$ respectively

$$\sum_{\nu=1}^{q} m(r, b_\nu; f^{(k)}) + m(r, f^{(k)}) \ \le$$

$$\begin{aligned}&\le\ 2\,T(r, f^{(k)}) + \overline{N}(r, f) - (k+1)\,\overline{N}(r, f) + (1+\varepsilon)\,N(r, f) - N(r, f^{(k)}) + S(r, f^{(k)})\\ &\le\ 2\,T(r, f^{(k)}) - 2k\overline{N}(r, f) + \varepsilon T(r, f^{(k)}) + S(r, f^{(k)}) \ ,\end{aligned}$$

which gives on integration

$$\begin{aligned}\sum_{\nu=1}^{p} m_o(r, b_\nu; f^{(k)}) + m_o(r, f^{(k)}) \ &\le\\ &\le\ 2\,T_o(r, f^{(k)}) - 2\,k\,\overline{N}_o(r, f) + \varepsilon\,T_o(r, f^{(k)}) + S_o(r, f^{(k)}) \ .\end{aligned}$$

Now by Lemma 10 and Lemma 11 for $L(f) = f^{(k)}$ and by (2) we get because $\delta(\infty; f) = 0$

$$\sum_{\nu=1}^{q} \delta_o(b_\nu; f^{(k)}) + \delta_o(\infty; f^{(k)}) \ \le\ 2 - \frac{2\,k}{1+k} + \varepsilon \ .$$

Since $\delta(a; f) \le \delta_o(a; f)$ for any $a \in \mathbb{C} \cup \{\infty\}$ (cf. [14]), it follows that

$$\sum_{\nu=1}^{q} \delta(b_\nu; f^{(k)}) + \delta(\infty; f^{(k)}) \;\le\; 2 - \frac{2\,k}{1+k} + \varepsilon \;.$$

Again since q and $\varepsilon(>0)$ are arbitrary, by (14) we get from above

$$\sum_{b\neq\infty} \delta(b; f^{(k)}) \;\le\; \frac{2}{1+k} \;. \tag{16}$$

The theorem follows from (15) and (16). This proves the theorem. ∎

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Indrajit Lahiri,
Department of Mathematics, University of Kalyani,
West Bengal 741235 – INDIA
E-mail: `indrajit@cal2.vsnl.net.in`

and

Abhijit Banerjee,
Department of Mathematics, Kalyani Government Engineering College,
West Bengal 741235 – INDIA
E-mail: `abanerjee@movemail.com`