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FROM A SINGLE CHAIN TO A LARGE FAMILY OF SUBMODULES

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Abstract: By making use of an ingenious idea of Paul Hill [8], we prove a general lemma showing how to obtain from a continuous well-ordered ascending chain of submodules of certain types a large collection of submodules of the same type. Several applications are exhibited.

Introduction

Suppose that a module M (over any ring) has a continuous well-ordered ascending chain $0 = M_0 < M_1 < \cdots < M_\alpha < \cdots < \bigcup_{\alpha<\tau} M_\alpha = M$ $(\alpha < \tau)$ of submodules M_α which are assumed to have prescribed properties. Can we find a large collection of submodules in M with the same properties? In some arguments the existence of a chain is insufficient, it is necessary to have a large supply of submodules (e.g. in the application of Shelah's Singular Compactness Theorem).

The answer to the above question is that in certain cases we can find a large collection of submodules with the same properties — as was demonstrated by P. Hill [8]. He gave an interesting construction showing how to create from a chain of 'nice' subgroups in an abelian torsion group a large collection of 'nice' subgroups (here 'nice' has a well-defined meaning, see Section 4). This idea has been applied to several other situations; see e.g. [1], [6], [11]. In the present note,

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we will imitate Hill's method in a fairly general setting and, starting from a given continuous chain, we will construct a large collection of submodules such that the submodules in the collection will admit the same kind of continuous chains.

We will list several (old and new) applications of our construction to various modules and abelian groups.

1 – Preliminaries

In this note all groups are abelian, and all modules are unital over a commutative ring R with 1. $\operatorname{gen} M$ stands for the minimal cardinality of generating systems of the module M.

Recall Hill's definition of a *totally projective* p-group A [8]:

A has a family $\mathcal{F}$ of 'nice' subgroups (for the definition of 'nice' subgroup see item 8 in Section 4 below) such that

H1. $0, A \in \mathcal{F}$;

H2. $\mathcal{F}$ is closed under arbitrary sums;

H3. if $B \in \mathcal{F}$, and if the subset $X \subset M$ has cardinality $\leq \aleph_0$, then there is a $C \in \mathcal{F}$ such that $B \cup X \subset C$ and $\operatorname{gen} C/B \leq \aleph_0$.

Hill calls this the *3rd Axiom of Countability*. Griffith [7] showed that H2 can be replaced by the weaker condition:

G2. $\mathcal{F}$ is closed under unions of chains.

The following condition used by Fuchs [4] is even weaker than G2:

F2. There is a continuous chain of subgroups

$$0 = N_0 < N_1 < \cdots < N_\alpha < \cdots < N_\tau = A\ ,$$

where the links N_α are nice subgroups and factor groups $N_{\alpha+1}/N_\alpha$ are countable. (Here *continuous* means that for every limit ordinal $\beta < \tau$, $N_\beta = \bigcup_{\alpha<\beta} N_\alpha$ holds.)

The implications H2 ⇒ G2 ⇒ F2 are obvious. Hill [8] found an ingenious direct proof of the implication F2 ⇒ H2 (assuming H1 and H3). It has various versions, depending on the hypotheses imposed on the links of the original chain; see e.g. [1], [6], [11]. Theorem 2.1 gives a more general version for modules.

We will need the general definition of an $H(\kappa)$-family, where κ denotes an infinite cardinal. By an $H(\kappa)$-*family* in the module M is meant a collection $\mathcal{F}$ of submodules of M such that

H1. $0, M \in \mathcal{F}$;

H2. $\mathcal{F}$ is closed under arbitrary sums;

H3. if $B \in \mathcal{F}$ and the subset $X \subset M$ has cardinality $\leq \kappa$, then there is a $C \in \mathcal{F}$ such that $B \cup X \subset C$ and $\operatorname{gen} C/B \leq \kappa$.

2 – The general lemma

We are now ready to formulate and prove the main result of this note. Recall that a ring R is said to be κ-*coherent* if $\operatorname{gen} I \leq \kappa$ for each of its ideals I.

Theorem 2.1 (Generalized Hill's Lemma)**.** *Suppose κ is an infinite cardinal and the ring R is κ-coherent. Let the R-module M be the union of a continuous well-ordered ascending chain of submodules*

$$0 = M_0 < M_1 < \cdots < M_\alpha < \cdots < M_\tau = M \qquad (\alpha < \tau) \tag{1}$$

for some ordinal τ such that for each $\alpha + 1 < \tau$,

$$M_{\alpha+1} = M_\alpha + A_\alpha$$

holds for some submodule A_α with $\operatorname{gen} A_\alpha \leq \kappa$. Then:

(**i**) *M admits an $H(\kappa)$-family $\mathcal{F}$ of submodules;*

(**ii**) *every $C \in \mathcal{F}$ has a continuous well-ordered ascending chain of submodules with union C such that*

(**a**) *every successor submodule in the chain is obtainable from its predecessor by adding some A_α;*

(**b**) *the factors are isomorphic to factors in the chain* (1)*.*

Proof: To start with, note that the hypothesis guarantees that $M_\beta = \sum_{\alpha<\beta} A_\alpha$ for all $\beta \leq \tau$. Thus every $a \in M$ is contained in the sum of a finite number of A_α's.

Call a subset S of τ *closed* if every $\beta \in S$ satisfies

$$M_\beta \cap A_\beta \;\leq \sum_{\alpha\in S,\ \alpha<\beta} A_\alpha \;.$$

For a closed subset S, we set $M(S) = \sum_{\alpha \in S} A_\alpha$. Our claim is that

$$\mathcal{F} = \Big\{ M(S) \mid S \text{ a closed subset in } \tau \Big\}$$

is a desired $H(\kappa)$-family in M. The following four steps will provide a proof.

Step 1. Unions of closed subsets of τ are again closed.

Suppose that S_i $(i \in I)$ are closed subsets of τ, and $\beta \in \bigcup_{i \in I} S_i$. Then $\beta \in S_j$ for some $j \in I$, and $M_\beta \cap A_\beta$ is evidently contained in the sum of the A_α for $\alpha < \beta$ with $\alpha \in S_j$, and *a fortiori* with $\alpha \in \bigcup_{i \in I} S_i$.

Step 2. Every subset of τ of cardinality $\leq \kappa$ is contained in a closed subset of τ of cardinality $\leq \kappa$.

By Step 1, it suffices to verify this for a finite subset X of τ. We induct on the largest ordinal β contained in X. If $\beta = 0$, then the claim is true, as $\{0\}$ is evidently a closed subset of τ. By the assumed κ-coherency, the submodule $M_\beta \cap A_\beta$ is at most κ-generated, so it contains a generating set $\{a_i \mid i \in I\}$ with $|I| \leq \kappa$. All the a_i's are in M_β, so each a_i is contained in a finite sum of the A_α's with $\alpha < \beta$. By induction, the finite index set of these A_α's is contained in a closed subset of τ, thus there is a closed subset $S' \subset \tau$ such that $|S'| \leq \kappa$ and all the a_i are contained in $M(S')$. There is no loss of generality in assuming that $S' \subset \beta$, since otherwise S' can be replaced by the closed subset $S' \cap \beta$. To show that $S = S' \cup \{\beta\}$ is a closed subset of τ, it suffices to check the definition for β. This is easy: $M(S')$ contains the a_i, hence it contains $M_\beta \cap A_\beta$.

Step 3. $\mathcal{F}$ is an $H(\kappa)$-family of submodules in M.

Obviously, both $\emptyset$ and τ are closed subsets of τ. Since by Step 1 the equality $\sum_{i \in I} M(S_i) = M(\bigcup_{i \in I} S_i)$ holds for closed subsets $S_i \subset \tau$, the collection $\mathcal{F}$ is closed under arbitrary unions. If T is a closed subset of τ and X is a subset of M of cardinality $\leq \kappa$, then there is a closed subset $S \subset \tau$ such that $|S| \leq \kappa$, $X \subset M(S)$, where $\operatorname{gen} M(S) \leq \kappa$. Since $\operatorname{gen}[M(S \cup T)/M(T)] \leq \kappa$, $\mathcal{F}$ satisfies condition H3 as well.

Step 4. Let S be a closed subset of τ, and $\beta \in S$. As $M_\beta \cap A_\beta \leq \sum_{\alpha \in S, \alpha < \beta} A_\alpha \leq M_\beta$ implies $(\sum_{\alpha \in S, \alpha < \beta} A_\alpha) \cap A_\beta = M_\beta \cap A_\beta$, we have the natural isomorphisms

$$\Big(\sum_{\alpha \in S, \alpha \leq \beta} A_\alpha \Big) \Big/ \Big(\sum_{\alpha \in S, \alpha < \beta} A_\alpha \Big) = \Big(\sum_{\alpha \in S, \alpha < \beta} A_\alpha + A_\beta \Big) \Big/ \Big(\sum_{\alpha \in S, \alpha < \beta} A_\alpha \Big)$$
$$\cong A_\beta \,/\, (M_\beta \cap A_\beta) \cong M_{\beta+1} \,/\, M_\beta \ .$$

Thus $M(S)$ has a chain of submodules $\sum_{\alpha\in S,\alpha<\beta} A_\alpha$ for $\beta \in S$ such that the factors in the chain are isomorphic to the corresponding factors in (1). This proves (b), while (a) is obvious from the construction. ∎

Remark 2.2. The proof above shows that if, for some α, the submodule A_α can be chosen as a complement of M_α in $M_{\alpha+1}$, then in the chain of $M(S)$, in the link corresponding to $\alpha \in S$, A_α will be a summand as well. □

Very often the rank version of Theorem 2.1 (see [6, XVI.8.11]) is more useful whenever we deal with torsion-free modules over integral domains R. In this case we need a modification in the definition of $H(\kappa)$-family: we will say $\mathcal{F}$ is an *$H^\star(\kappa)$-family* in the R-module M if M is torsion-free and satisfies H1, H2 and the rank version of H3:

H3*. If $B \in \mathcal{F}$ and the subset $X \subset M$ has cardinality $\leq \kappa$, then there is a $C \in \mathcal{F}$ such that $B \cup X \subset C$ and $\operatorname{rk} C/B \leq \kappa$.

The above proof applies to verify:

Theorem 2.3. *Suppose R is a domain. Let the torsion-free R-module M be the*
union of a continuous well-ordered ascending chain of submodules

$$0 = M_0 < M_1 < \cdots < M_\alpha < \cdots \qquad (\alpha < \tau) \tag{2}$$

such that for each $\alpha+1 < \tau$, $M_{\alpha+1} = M_\alpha + A_\alpha$ holds for some submodule A_α of rank $\leq \kappa$. Then:

- (**i**) *M admits an $H^\star(\kappa)$-family $\mathcal{F}$ of submodules;*
- (**ii**) *every $C \in \mathcal{F}$ has a continuous well-ordered ascending chain of submodules with union C such that*
  - (**a**) *every successor submodule is obtainable from its predecessor by adding some A_α;*
  - (**b**) *its factors are isomorphic to factors in the chain* (2). ∎

3 – Additional hypotheses on the links

We now turn our attention to the following question: If the submodules in (1) are assumed to have additional properties, are then these properties inherited by the $M(S)$?

We concentrate on the following two properties. A submodule N of an R-module M is called an *RD-submodule* if $rN = N \cap rM$ for all $r \in R$, and a *pure* submodule if the solvability of any finite system of equations

$$\sum_{k=1}^{m} r_{ik} x_k = a_i \in N \qquad (r_{ik} \in R, \quad i = 1, ..., n)$$

(m, n are positive integers, and x_k are unknowns) in M implies that the system also has a solution in N.

Proposition 3.1. *Suppose that all the submodules M_α in the chain (1) are pure (respectively, RD-)submodules of M. Then the same holds for the submodules in $\mathcal{F}$.*

Moreover, the links in the chains of the modules $C \in \mathcal{F}$ (see (2.1) (ii)) are pure (resp. RD-)submodules of C.

Proof: We prove only the pure version, the other is a special case for $n = 1 = m$.

So we assume that all the M_α in (1) are pure in M. To verify the purity of $M(S)$ for a closed S, assume that

$$(3) \qquad \sum_{k=1}^{m} r_{ik}\, x_k = c_i \in M(S) \qquad (r_{ik} \in R; \quad i = 1, ..., n)$$

is a system of equations in the unknowns $x_1, ..., x_m$ that is solvable in M. Write

$$c_i = a_{i\alpha_1} + \cdots + a_{i\alpha_h} \qquad (a_{i\alpha_j} \in A_{\alpha_j}) ,$$

where $\alpha_1 < \alpha_2 < \cdots < \alpha_h$ in S; to simplify the notation, we allow some of the $a_{i\alpha_j}$ to vanish. In addition, assume that the largest index α_h occurring in the c_i has been chosen minimal, i.e. $c_i \in M_{\alpha_h+1}$ for each i, and $\alpha_h + 1$ is the minimal index with this property. We induct on α_h, and assume that systems like the one above are solvable in $M(S)$ whenever the right hand sides are contained in M_{α_h}.

As $c_i \in M_{\alpha_h+1}$, the system has a solution $y_1, ..., y_m \in M_{\alpha_h+1}$ in view of the purity hypothesis. We can write $y_k = z_k + b_k$ with $z_k \in M_{\alpha_h}$, $b_k \in A_{\alpha_h}$. Thus

$$\sum_{k=1}^{m} r_{ik}\, z_k = \sum_{k=1}^{m} r_{ik}\, y_k - \sum_{k=1}^{m} r_{ik}\, b_k = a_{i\alpha_1} + \cdots + a_{i\alpha_h} - \sum_{k=1}^{m} r_{ik}\, b_k ,$$

whence

$$a_{i\alpha_h} - \sum_{k=1}^{m} r_{ik}\, b_k \;\in\; M_{\alpha_h} \cap A_{\alpha_h} \;\leq \sum_{\alpha\in S,\ \alpha<\alpha_h} A_\alpha \ .$$

We obtain

$$\sum_{k=1}^{m} r_{ik}\, z_k \;\in \sum_{\alpha\in S,\ \alpha<\alpha_h} A_\alpha \;\leq\; M_{\alpha_h} \ .$$

By induction hypothesis, we can find $t_1, ..., t_m \in M(S)$ such that $\sum_{k=1}^{m} r_{ik}\, z_k = \sum_{k=1}^{m} r_{ik}\, t_k$ for each i. Hence $x_k = t_k + b_k$ $(k = 1, ..., m)$ is a solution in $M(S)$. This proves the existence of a solution of system (3) in $M(S)$.

For the purity of the links in the chain of $M(S)$ it suffices to observe that they are of the form $M(S \cap \beta)$ for $\beta \in S$ (see the proof of Theorem 2.1), and, along with S, the intersection $S \cap \beta$ is also a closed subset of τ. ∎

4 – Applications

We wish to consider various applications of the results proved above.

1. Pure Submodules

It is well known (and easy to see) that if R is a ring of infinite cardinality κ, then in any R-module M, every element embeds in a κ-generated pure submodule. Hence M admits a continuous well-ordered ascending chain of pure submodules with κ-generated factors.

Corollary 4.1. *Let R be a ring of cardinality κ. Every R-module has an $H(\kappa)$-family of pure submodules.*

Proof: Starting from a chain of pure submodules with at most κ-generated factor modules, Theorem 2.1 yields such a family. The submodules A_α are chosen arbitrarily satisfying only the two requirements: $\operatorname{gen} A_\alpha \leq \kappa$ and $M_{\alpha+1} = M_\alpha + A_\alpha$. ∎

The following corollary was proved by Hill [8] for abelian groups; it is valid over any domain.

Corollary 4.2. *A torsion-free module M over a domain R admits an $H^\star(\aleph_0)$-family of RD-submodules.*

Proof: It is straightforward to prove that M has a continuous well-ordered ascending chain of RD-submodules with rank 1 factors. Apply the rank version Theorem 2.3 along with Proposition 3.1. ∎

2. Projective Dimension

Let R be any domain. It is shown in [6, p. 216] that each R-module M of projective dimension ≤ 1 admits a continuous well-ordered ascending chain of submodules with countably generated factors of projective dimension ≤ 1. In view of Theorem 2.1 we can hence conclude:

Corollary 4.3. *If a module M over a domain R has projective dimension ≤ 1, then it admits an $H(\aleph_0)$-family of submodules of projective dimension ≤ 1.* ∎

If R is not a domain, then for an R-module of projective dimension 1 we cannot establish the existence of a chain like in the domain case, but we can still state something relevant. It says less for projective dimension ≤ 1, but it holds for arbitrary projective dimensions.

Corollary 4.4. *If an R-module M has a chain (1) with factors of projective dimension $\leq k$, then it admits an $H(\aleph_0)$-family of submodules of projective dimension $\leq k$. Every submodule in this family admits a chain (1) with factors of projective dimension $\leq k$.*

Proof: This follows from Theorem 2.1 at once. ∎

Because of Proposition 3.1, similar statement can be established for the pure-projective dimension:

Corollary 4.5. *If an R-module M has a chain* (1) *of pure submodules with factors of pure-projective dimension $\leq k$, then it admits an $H(\aleph_0)$-family of pure submodules of pure-projective dimension $\leq k$. Every submodule in this family admits a chain* (1) *with pure submodules and factors of pure-projective dimension $\leq k$.* ∎

3. Baer Modules

Let R be a domain. By a *Baer module* is meant an R-module B such that

$$\mathrm{Ext}^1_R(B,T)=0 \quad \text{for all torsion modules } T\ .$$

It is shown by Eklof–Fuchs–Shelah [3] that a Baer module B admits, for some ordinal τ, a continuous well-ordered ascending chain

$$0 = B_0 < B_1 < \cdots < B_\alpha < \cdots < B_\tau = B \qquad (\alpha < \tau)$$

of submodules such that, for each $\alpha<\tau$, $B_{\alpha+1}/B_\alpha$ is a countably generated Baer module.

From Theorem 2.1 we conclude:

Corollary 4.6 ([6, XVI.8.12]). *A Baer module B admits an $H(\aleph_0)$-family of Baer submodules.* ∎

If we specialize to Prüfer domains, then we obtain a stronger result. Observing that a countably generated Baer module over a Prüfer domain is projective (see [6, XVI.8.10]), we are led to the conclusion that a module over a Prüfer domain is a Baer module exactly if it is projective.

A generalization of the Baer property was introduced by Lee [9]. A *semi-Baer* module B over a domain R is defined by the property

$$\mathrm{Ext}^1_R(B,D)=0 \quad \text{for all divisible modules } D\ .$$

It is shown that such a B admits a similar chain as a Baer module, but this time the factors are countably generated semi-Baer modules. An application of Theorem 2.1 yields

Corollary 4.7. *Every semi-Baer module B admits an $H(\aleph_0)$-family of semi-Baer submodules.* ∎

4. Whitehead Modules

An R-module W is said to be a *Whitehead module* if it satisfies

$$\mathrm{Ext}^1_R(W,R)=0\ .$$

In addition to the standard axioms ZFC of set theory, assume Gödel's Axiom of Constructibility. The result by Becker–Fuchs–Shelah [2] is slightly modified in Fuchs–Salce [6, XVI.10.6] as follows.

Let μ be a regular cardinal, and R a domain of cardinality $\leq \mu$. An R-module W of projective dimension ≤ 1 is a Whitehead module if and only if it is the union of a continuous well-ordered ascending chain

$$0 = W_0 < W_1 < \cdots < W_\alpha < \cdots \qquad (\alpha < \kappa)$$

of submodules such that the factors $W_{\alpha+1}/W_\alpha$ are μ-generated Whitehead modules of projective dimension ≤ 1 for $\alpha + 1 < \kappa$.

Applying Theorem 2.1 to this situation, we conclude:

Corollary 4.8. *Let μ be a regular cardinal, and R a domain of cardinality $\leq \mu$. A Whitehead R-module W of projective dimension ≤ 1 has an $H(\mu)$-family of Whitehead submodules of projective dimension ≤ 1.* ∎

5. B_2-Groups

A torsion-free abelian group G is called a *B_2-group* if it is the union of a continuous well-ordered ascending chain $0 = G_0 < G_1 < \cdots < G_\alpha < \cdots$ $(\alpha < \tau)$ of pure subgroups G_α such that $G_{\alpha+1} = G_\alpha + B_\alpha$ for each α, where B_α is a Butler group of finite rank (i.e. B_α is a pure subgroup of a finite direct sum of rank 1 torsion-free groups).

Furthermore, Albrecht and Hill [1] call G a *B_3-group* if it admits an $H(\aleph_0)$-family of decent subgroups. Recall that a pure subgroup A in a torsion-free group G is said to be *decent* if for any finite subset S of G, there exists a finite rank Butler group B such that $A + B$ contains S and is pure in G.

Since in the definition of B_2-groups, the subgroups in the chain are decent, it is clear that B_3-groups are B_2-groups. In order to show the converse, we can apply Theorem 2.1:

Corollary 4.9 (Albrecht–Hill [1]). *A B_2-group admits an $H(\aleph_0)$-family of decent subgroups, all of these are B_2-groups.*

Proof: The existence of an $H(\aleph_0)$-family is clear, and so is the claim that its members admit a chain required of B_2-groups (cp. the argument in Step 4 in the proof of Theorem 2.1). It remains to verify that they are themselves decent subgroups. We refer to Lemma 5.7 in [1] to complete the proof. ∎

6. Butler Modules over Valuation Domains

By a *Butler module* B we mean a module B over a domain R satisfying

$$\mathrm{Bext}_R^1(B,T)=0 \quad \text{for all torsion modules } T\ ,$$

where $\mathrm{Bext}_R^1(B,T)$ stands for the subgroup of $\mathrm{Ext}_R^1(B,T)$ consisting of the balanced extensions of T by B.

A pure submodule A of B is *balanced* if A is a summand in every pure submodule C of B for which C/A is of rank 1. Rangaswamy [11] calls A *pseudo-balanced* in B if for every C of the mentioned kind either A is a summand in C or else C/A is countably generated.

Rangaswamy [11] shows that every module M of balanced-projective dimension ≤ 1 admits a continuous well-ordered ascending chain (1) where each link is pseudo-balanced in its successor. If we choose A_α to be a complement of M_α in $M_{\alpha+1}$, or just countably generated, then the proof of Theorem 2.1 (along with Remark 1) yields:

Corollary 4.10. *Let R be a valuation domain, and suppose M is an R-module of balanced-projective dimension ≤ 1. Then M admits an $H(\aleph_0)$-family of submodules each of which has a continuous well-ordered ascending chain where each link is pseudo-balanced in its successor. The members of this family are pseudo-balanced in M.*

Proof: The last claim follows from [11, Theorem 3.1]. ∎

7. n-Flat Modules

A module M over any ring R is called by Lee [10] *n-flat* if it satisfies

$$\mathrm{Tor}_1^R(M,N)=0 \quad \text{for all finitely presented modules } N$$

with $\mathrm{p.d.}_R\, N\leq n$.

Of course flatness of M means that the same holds for all finitely presented N without the projective dimension restriction.

Corollary 4.11. *Let R be a ring of infinite cardinality κ. Every flat (n-flat) R-module M has an $H(\kappa)$-family of flat (n-flat) submodules.*

Proof: In view of Theorem 2.1, it suffices to establish a continuous well-ordered ascending chain of submodules with flat (n-flat) factors. In order to construct such a chain in M, start with a pure submodule M_1 of cardinality $\leq \kappa$. By the argument of [6, VI.9.8], both M_1 and M/M_1 are flat (n-flat). Hence we can repeat this argument to obtain a submodule M_2 such that M_2/M_1 is of cardinality $\leq \kappa$ and pure in M/M_1. An obvious transfinite induction (taking unions at limit ordinals) leads to a continuous well-ordered ascending chain of pure submodules with at most κ-generated factors. All the factors are flat (n-flat), so this is a chain as desired. ∎

8. Nice Composition Chains

The original proof by Hill in [8] was concerned with totally projective p-groups. We are going to consider a generalization to arbitrary valuation domains, as given in Fuchs [5]. The definition given there was based on the property of simple presentation, but we are using now a different definition that corresponds to Hill's definition for abelian groups. Define a module M to admit a *nice composition chain* if there is a continuous well-ordered ascending chain

$$0 = N_0 < N_1 < \cdots < N_\alpha < \cdots < N_\kappa = M$$

of submodules in M, where each $N_{\alpha+1}$ is of the form $N_{\alpha+1} = N_\alpha + R\,a_\alpha$ for some $a_\alpha \in N_{\alpha+1}$ which is perfect with respect to N_α. Here a_α *perfect* with respect to N_α means that 1) the annihilator ideal J_α of $a_\alpha + N_\alpha$ in the factor module $N_{\alpha+1}/N_\alpha$ is a principal ideal; and 2) for each $s \in R \backslash J_\alpha$, the element $s\,a_\alpha$ has the largest height in its coset $s\,a_\alpha + N_\alpha$. (For abelian p-groups, conditions 1)–2) can be replaced by the requirements that the factors are of order p and a_α has the largest height in its coset $a_\alpha + N_\alpha$.)

The following corollary yields a direct generalization of Hill's result [8, Theorem 1] from abelian groups.

Corollary 4.12. *Every module M over a valuation domain R with a nice composition chain admits an $H(\aleph_0)$-family of nice submodules with nice composition chains.*

Proof: We wish to apply Theorem 2.1 with the choice $A_\alpha = R\,a_\alpha$, where a_α is a perfect element with respect to N_α. We show that the modules $M(S)$ are nice in M. Again, we refer to Step 4 in the proof of Theorem 2.1, and observe that it is readily seen that a_α is a perfect element with respect to $\sum_{\beta \in S, \beta < \alpha} A_\beta$ as well. ∎

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